Non-archimedean connected Julia sets

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Complex dynamics

Let $\phi \in \mathbb{C}(z)$ be a rational function of degree $d \geq 2$.

[$\deg \phi := \max\{\deg f, \deg g\}$, where $\phi = f/g$ in lowest terms.]

Then $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Write $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}$.

ϕ has an associated **Fatou set**

$$\begin{aligned}\mathcal{F} &= \text{region where small errors stay small} \\ &= \{x \in \mathbb{P}^1(\mathbb{C}) : \text{for } y \text{ near } x,\ \phi^n(y) \text{ is near } \phi^n(x)\} \\ &= \{x \in \mathbb{P}^1(\mathbb{C}) : \{\phi^n\}_{n \geq 0} \text{ equicontinuous on a nbhrd of } x\}\end{aligned}$$

and **Julia set**

$$\mathcal{J} = \mathbb{P}^1(\mathbb{C}) \setminus \mathcal{F} = \text{region where small errors may become large.}$$

(with respect to spherical metric)

Non-archimedean dynamics

Throughout this talk, K will denote a complete and algebraically closed non-archimedean field, with absolute value $|\cdot|$.

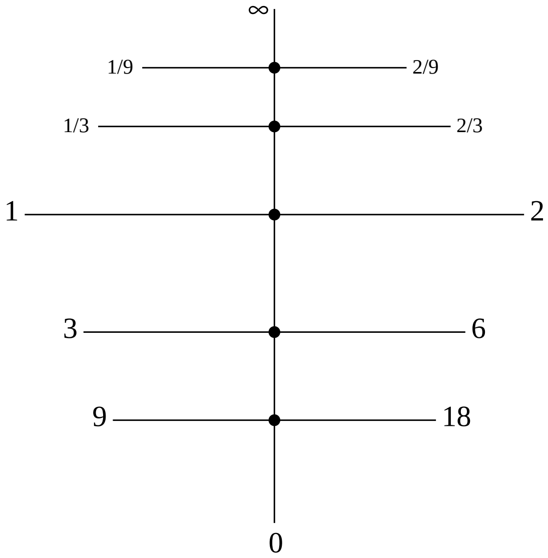
Reminder: "non-archimedean" $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Goal: Do dynamics, including Fatou sets, Julia sets, and even ergodic theory, over our non-archimedean field K.

Problem: The naive analog of the Riemann sphere, $\mathbb{P}^1(K) = K \cup \{\infty\}$, is not a good setting for dynamics or ergodic theory. (E.g.: not compact.)

Idea: Make a bigger space $\mathbb{P}^1_{\text{Ber}}$ containing $\mathbb{P}^1(K)$, but **compact and path-connected**.

Building the Berkovich Projective Line (over $\mathbb{C}_3$)



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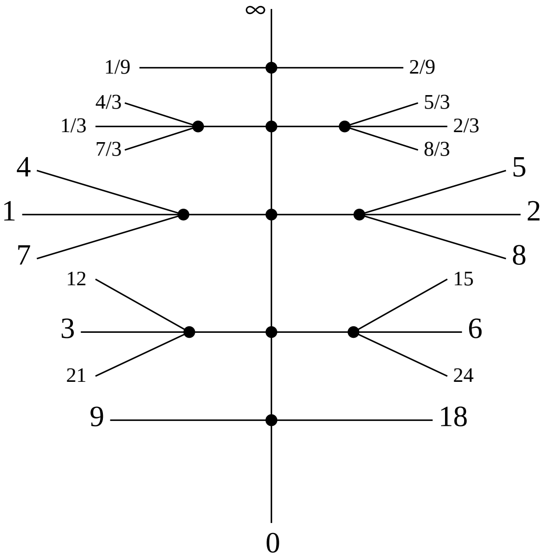

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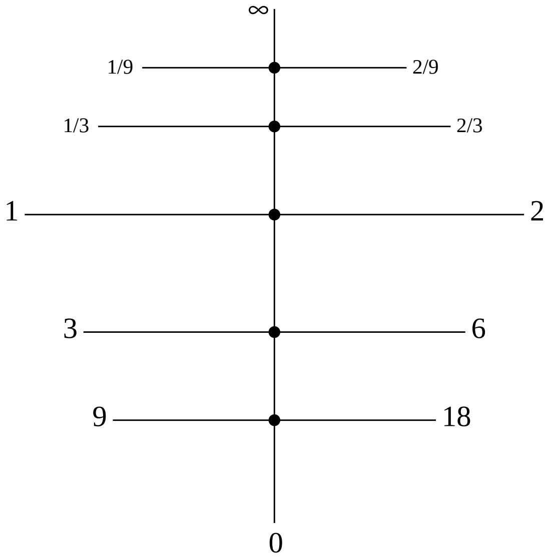

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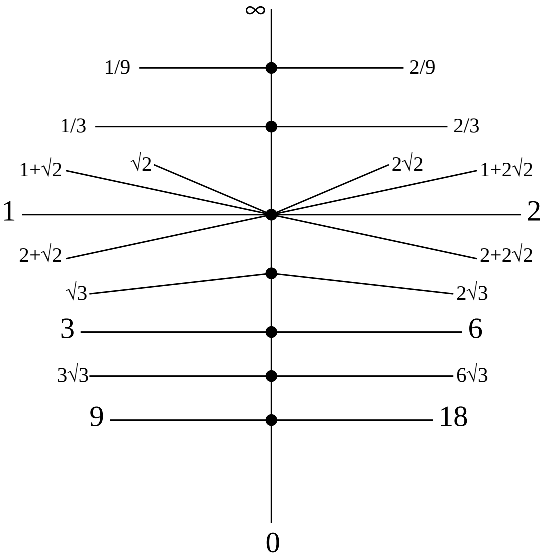
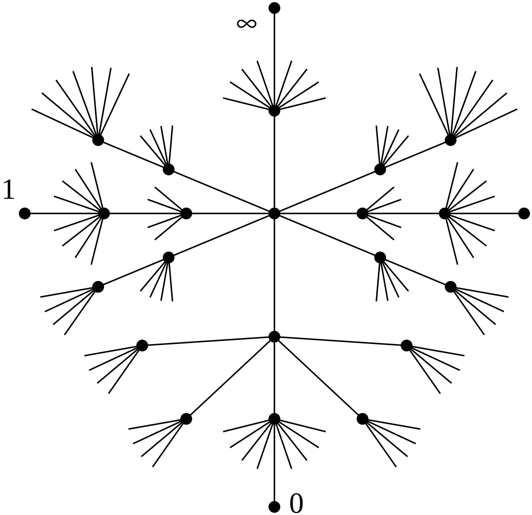


Building the Berkovich Projective Line (over any K)



Rational Functions on the Berkovich Projective Line

Let $\phi(z) \in K(z)$ be a rational function. Then $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$, and this action extends continuously to $\phi : \mathbb{P}^1_{\text{Ber}} \to \mathbb{P}^1_{\text{Ber}}$.

Idea: A Berkovich point $\zeta(a, r)$ corresponding to a closed disk $\overline{D}(a, r) \subseteq K$ behaves like a "generic" element of $\overline{D}(a, r)$.

Example: $\phi(z) = \dfrac{z^3 + z}{z^3 + c^3}$ for $0 < |c| < 1$.

Then for $0 < r < \infty$, we have:

$$\phi(\zeta(0, r)) = \begin{cases} \zeta(0, r/|c|^3) & \text{if } 0 < r \leq |c| \\ \zeta(0, 1/r^2) = \zeta(0, 1/r^2) & \text{if } |c| \leq r \leq 1 \\ \zeta(1, 1/r^2) & \text{if } r \geq 1 \end{cases}$$

Details of the Example

$$\phi(z) = \frac{z^3 + z}{z^3 + c^3} \text{ for } 0 < |c| < 1.$$

- $0 < r < |c|$: For $|z| = r$, we have $\phi(z) \approx z/c^3$, so $\phi(\zeta(0, r)) = \zeta(0, r/|c|^3)$
- $|c| < r < 1$: For $|z| = r$, we have $\phi(z) \approx z/z^3 = 1/z^2$. Hence, $\phi(\zeta(0, r)) = \zeta(0, 1/r^2)$.
- $r > 1$: For $|z| = r$, we have $\phi(z) \approx z^3/z^3 = 1$, so look at $\phi(z) - 1 = (z - c^3)/(z^3 + c^3) \approx 1/z^2$. Hence, $\phi(\zeta(0, r)) = \zeta(1, 1/r^2)$.

When $r = |c|$ or 1, we get the answer by continuity or by direct computation. E.g., for $|z| = |c|$, we have

$$\phi(z) = \frac{z^3 + z}{z^3 + c^3} \approx \frac{z}{z^3 + c^3},$$

so $\phi(\zeta(0, |c|)) = \zeta(0, 1/|c|^2)$ with multiplicity 3.

Fatou and Julia sets in Berkovich Space

For $\phi \in K(z)$, define the (Berkovich) **Fatou set** of ϕ to be

$$\mathcal{F} = \{x \in \mathbb{P}^1_{\mathrm{Ber}} : x \text{ has a neighborhood } U \text{ s.t. } \mathbb{P}^1_{\mathrm{Ber}} \setminus \bigcup_{n \geq 0} \phi^n(U) \text{ is infinite}\},$$

and the (Berkovich) **Julia set** of ϕ to be $\mathcal{J} = \mathbb{P}^1_{\mathrm{Ber}} \setminus \mathcal{F}$.

Facts:

- $\mathcal{J}$ is closed and hence compact.
- $\mathcal{J}$ is invariant under ϕ, i.e., $\phi^{-1}(\mathcal{J}) = \mathcal{J}$.
- There is a natural Borel probability measure $\mu = \mu_\phi$ such that
  - $\mathrm{supp}(\mu) = \mathcal{J}$.
  - μ is invariant under ϕ, i.e., $\mu(\phi^{-1}(E)) = \mu(E)$.

Example 1: Good Reduction

Roughly speaking, $\phi \in K(z)$ has *good reduction* if it still has the same degree after reducing all the coefficients modulo the maximal ideal $\mathcal{M}_K$.

In particular, a polynomial $\phi(z) = a_d z^d + \cdots + a_0$ has good reduction if and only if $\phi(z) \in \mathcal{O}[z]$ with $|a_d| = 1$.

Fact: A rational function $\phi \in K(z)$ has good reduction if and only if its Julia set is $\mathcal{J} = \{\zeta(0,1)\}$.

More generally, ϕ has potential good reduction — i.e., $h \circ \phi \circ h^{-1}$ has good reduction for some $h \in PGL(2,K)$ — if and only if $\mathcal{J}$ is a single point.

Example 2: Quadratic Polynomials of Bad Reduction

Let $\phi(z) = z^2 - az$ with $|a| > 1$.

Let $W = \overline{D}(0, |a|)$.

Then $\phi^{-1}(W) = \overline{D}(0, 1) \cup \overline{D}(a, 1) \subsetneq W$.

More generally, $\phi^{-n}(W)$ is a union of 2^n disks,
and $\mathcal{J} = \bigcap_{n \geq 0} \phi^{-n}(W)$ is subset of $\mathbb{P}^1(K)$
homeomorphic to the Cantor set.

Example 3: A Cubic Polynomial

Let $\phi(z) = bz^3 - z^2$ with $|b| < 1$.

Let $W = \overline{D}(0, |b|^{-1})$.

Again, $\mathcal{J} = \partial\Big[\bigcap_{n\geq 0} \phi^{-n}(W)\Big]$ is a Cantor set,

but this time including points of $\mathbb{P}^1(K)$ as well as points like $\zeta(0, 1)$.

Example 4: Lattès Maps

Let E be the elliptic curve $y^2 + xy = x^3 + c$, with $0 < |c| < 1$.

[N.B.: E has multiplicative reduction.]

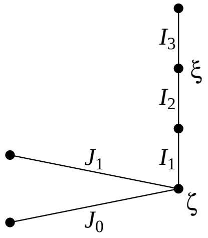
Then $[2] : E \to E$ induces a *Lattès* map $\phi \in K(z)$ with $x([2]P) = \phi(x[P])$.

Specifically, $\phi(z) = \dfrac{z^4 - 8cz - c}{4z^3 + z^2 + 4c}$.

Then $\mathcal{J} = [\zeta(0,1), \zeta(0,|c|^{1/2})]$,
i.e., the line segment in $\mathbb{P}^1_{\text{Ber}}$ from $\zeta(0,1)$ to $\zeta(0,|c|^{1/2})$.

Towards Other Connected Julia Sets

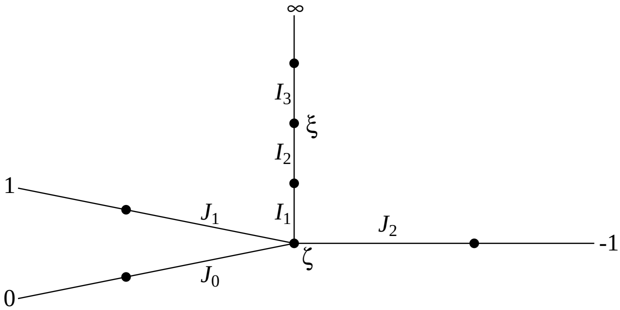
Can we rig up the following dynamics?



- The finite tree $\Gamma := J_0 \cup J_1 \cup I_1 \cup I_2 \cup I_3$ maps into itself.
- $\zeta \mapsto \zeta$, and $J_0, J_1 \to I := I_1 \cup I_2 \cup I_3$.
- $\xi \mapsto \zeta$, and $I_1, I_2 \to J_0$, and $I_3 \to J_1$.

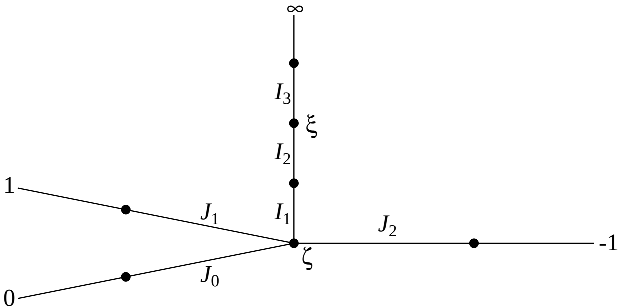
Yes, e.g.: $\phi(z) = \dfrac{az^6 + 1}{az^6 + z^3 - z} = 1 + \dfrac{-z^3 + z + 1}{az^6 + z^3 - z}$
(for $0 < |a| < 1$).

Dynamics of $\phi(z) = \frac{az^6+1}{az^6+z^3-z} = 1 + \frac{-z^3+z+1}{az^6+z^3-z}$



- $\zeta \mapsto \zeta$, and $J_0, J_1, J_2 \to I := I_1 \cup I_2 \cup I_3$.
- $\xi \mapsto \zeta$, and $I_1, I_2 \to J_0$, and $I_3 \to J_1$.
- The (countably many) other branches off ξ and ζ map to the various branches off ζ (including J_1 and J_2).

Dynamics of $\phi(z) = \dfrac{az^6+1}{az^6+z^3-z} = 1 + \dfrac{-z^3+z+1}{az^6+z^3-z}$



Theorem (Bajpai, RB, Chen, Kim, Marschall, Onul, Xiao)

The Julia set $\mathcal{J}$ of ϕ is path-connected and contains the tree Γ. $\mathcal{J}$ has infinitely many branch points, with infinite branching at each branch point.

[From 2013 REU; finally written up 2015.]

Measure-theoretic Entropy (a.k.a. Metric Entropy)

Let X be a topological space and $f : X \to X$ a continuous map. Let μ be an f-invariant Borel probability measure on X.

(**Recall**: f-invariant means $\mu(f^{-1}(E)) = \mu(E)$.)

Vague Question: For a finite partition $\mathcal{P}$ of X into Borel subsets, and for $x \in X$, if we know what partition element each of $x, f(x), \ldots, f^n(x)$ belong to, how well (or badly) can we predict $f^{n+1}(x)$, on average?

Measure-theoretic Entropy (a.k.a. Metric Entropy)

Let X be a topological space and $f : X \to X$ a continuous map. Let μ be an f-invariant Borel probability measure on X.

Definition. The *measure-theoretic entropy* of (f, μ) is

$$h_\mu(f) = \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P} \vee f^{-1}\mathcal{P} \vee \cdots \vee f^{-n}\mathcal{P}),$$

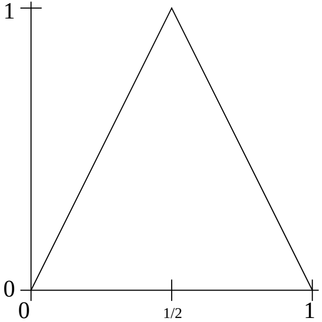
where

- $f^{-j}\{U_1, \ldots, U_m\} = \{f^{-j}(U_1), \ldots, f^{-j}(U_m)\}$,
- $\mathcal{P} \vee \mathcal{P}' = \{U \cap U' : U \in \mathcal{P}, U' \in \mathcal{P}'\}$,
- $H_\mu(\mathcal{P}) = \sum_{U\in\mathcal{P}} -\mu(U) \log(\mu(U))$.

and the supremum is over all finite Borel partitions of X.

Example: the Tent Map

Let $X = [0,1]$, $\lambda =$Lebesgue measure, and $f : X \to X$ with graph



Using the partition $\mathcal{P} = \{[0,1/2],(1/2,1]\}$, one can show that $h_\lambda(f) = \log 2$.

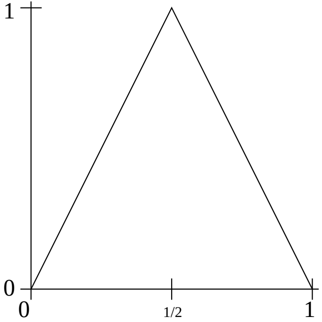
Similarly, the d-to-1 version of the tent map, with d zigs, has metric entropy $\log d$.

Lattès Maps Revisited

Let $\phi \in K(z)$ be the Lattès map $\phi(z) = \dfrac{z^4 - 8cz - c}{4z^3 + z^2 + 4c}$ from before, with $0 < |c| < 1$.

Recall $\mathcal{J} = [\zeta(0,1), \zeta(0, |c|^{1/2})]$.

Then $(\mathcal{J}, \mu)$ is homeomorphic to $([0,1], \lambda)$, with ϕ given by:



So $h_\mu(\phi) = \log 2$.

Some Other Non-archimedean Dynamical Systems

Example. If $\phi \in K(z)$ has (potential) good reduction, then $h_\mu(\phi) = 0$.

[Recall μ is supported on $\mathcal{J}$, which in this case is one point.]

Example. $\phi(z) = z^2 - az \in K[z]$ with $|a| > 1$.

Using $\mathcal{P} = \{\overline{D}(0,1) \cap \mathcal{J}, \overline{D}(a,1) \cap \mathcal{J}\}$, one can show that $h_\mu(\phi) = \log 2$.

Topological Entropy

Let X be a *compact* topological space and $f : X \to X$ a continuous map.

Definition. The *topological entropy* of f is

$$h_{\text{top}}(f) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee f^{-1}\mathcal{U} \vee \cdots \vee f^{-n}\mathcal{U}),$$

where

- the supremum is over all finite open covers $\mathcal{U}$ of X,
- $f^{-j}\{U_1, \dots, U_m\} = \{f^{-j}(U_1), \dots, f^{-j}(U_m)\}$,
- $\mathcal{U} \vee \mathcal{U}' = \{U \cap U' : U \in \mathcal{U}, U' \in \mathcal{U}'\}$,
- $N(\mathcal{U})$ = min number of elements of $\mathcal{U}$ needed to cover X.

The Variational Principle: If X is compact and metrizable, then

$$h_{\text{top}}(f) = \sup_{\mu} h_{\mu}(f),$$

where the sup is over all f-invariant Borel probability measures.

Entropy: Complex vs. Non-archimedean Dynamics

Fact: Let $\phi \in \mathbb{C}(z)$ be a rational function of degree $d \geq 2$, with associated Julia set $\mathcal{J} \subseteq \mathbb{P}^1(\mathbb{C})$ and invariant measure μ. Then

$$h_\mu(\phi) = h_{\text{top}}(\phi) = \log d.$$

Theorem (Favre & Rivera-Letelier, 2010)

Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$, with associated Julia set $\mathcal{J} \subseteq \mathbb{P}^1_{\text{Ber}}$ and invariant measure μ. Then

$$0 \leq h_\mu(\phi) \leq h_{\text{top}}(\phi) \leq \log d.$$

But both equalities of the $\mathbb{C}$ theorem can fail (or not) for K.

Recalling Some Non-archimedean Entropies

$$0 \le h_\mu(\phi) \le h_{\mathrm{top}}(\phi) \le \log d.$$

Example. $\phi(z) = z^2 - az$ with $|a| > 1$. Then

$$h_\mu(\phi) = h_{\mathrm{top}}(\phi) = \log 2.$$

Example. $\phi(z) \in K(z)$ has potentially good reduction, degree d. Then $\mathcal{J}$ is a single point. So

$$0 = h_\mu(\phi) = h_{\mathrm{top}}(\phi) < \log d.$$

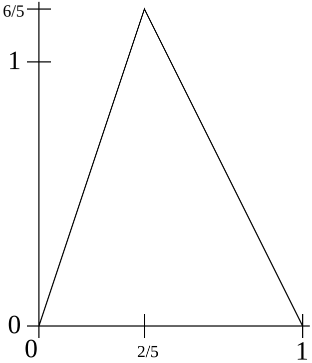
Example. $\phi(z) \in K(z)$ is the $[m]$-Lattès map for E/K of multiplicative reduction; note $\deg \phi = m^2$. Then

$$0 < \log m = h_\mu(\phi) = h_{\mathrm{top}}(\phi) < \log(m^2).$$

Non-Maximal Entropy

Favre and Rivera-Letelier gave examples where $h_\mu(\phi) < h_{\text{top}}(\phi)$.

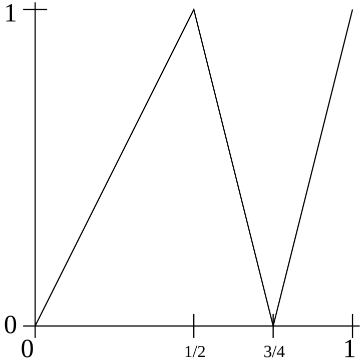
Example. Fix $a \in K$ with $0 < |a| < 1$. Let $\phi(z) = \dfrac{z^3}{1 + az^5}$.
Then $\mathcal{J}$ is homeomorphic to a Cantor set in $[0, 1]$ with ϕ given by:



So $0 < h_\mu(\phi) = \log 5 - \frac{2}{5}\log 2 - \frac{3}{5}\log 3 < \log 2 = h_{\text{top}}(\phi) < \log 5$.

Non-Maximal Entropy and Connected Julia Set

Example. Fix $a \in K$ with $0 < |a| < 1$. Let $\phi(z) = \dfrac{z^2(1 + a^2 z^8)}{1 + az^6}$.
Then $(\mathcal{J}, \mu)$ is homeomorphic to $([0,1], \lambda)$, with ϕ given by:



So $0 < h_\mu(\phi) = \log 5 - \frac{4}{5} \log 2 < \log 3 = h_{\text{top}}(\phi) < \log 10$.

Lower Degree?

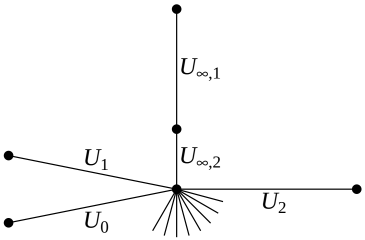
Motivated by these examples, Favre and Rivera-Letelier ask:

Question: Is there a rational function ϕ of degree ≤ 9 with connected Julia set $\mathcal{J}$ and with $h_\mu(\phi) < h_{\text{top}}(\phi)$?

Our Answer: For residue characteristic 3, the answer is **yes**, using the same sextic.

Fix $a \in K^\times$ with $|3| \leq |a| < 1$. Let $\phi(z) = \dfrac{az^6 + 1}{az^6 + z^3 - z}$.

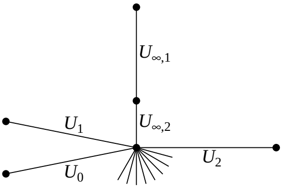
A Countable Partition for ϕ



Partition $\mathcal{J}$ into $U_{\infty,1}$, $U_{\infty,2}$, U_0, U_1, U_2,. . . .
It can be shown that $\mathcal{P} = \{U_{\infty,1}, U_{\infty,2}, U_0, U_1, U_2, \ldots\}$ is a countable generator for ϕ of finite entropy.

i.e., "We can use $\mathcal{P}$ to compute the entropy."

Entropy of ϕ



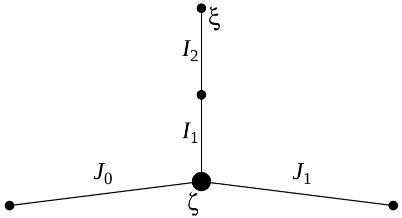
$$\mathcal{P} = \{U_{\infty,1}, U_{\infty,2}, U_0, U_1, U_2, \ldots\}$$

$$h_\mu(\phi) = (\log 2)\mu(U_{\infty,1} \cup U_{\infty,2}) + (\log 6)\mu(U_0 \cup U_1 \cup U_2 \cup \cdots)$$
$$= \frac{6}{11}\log 2 + \frac{5}{11}\log 6 \approx \log 3.2954$$

and $h_{\text{top}}(\phi) = \log \beta$, where $\beta \approx 3.8558$ is the largest real root of $t^3 - 4t^2 - t + 6$.

So $0 < h_\mu(\phi) < h_{\text{top}}(\phi) < \log 6$.

Dynamics of $\phi(z) = \dfrac{az^4+1}{z^2-z}$



We can show:

- $\zeta, \xi \mapsto \zeta$, and $I_1, I_2 \mapsto J_0$, and $J_0, J_1 \mapsto I_1 \cup I_2$.
- The Julia set $\mathcal{J}$ of ϕ is path-connected and contains the above tree.
- $\mathcal{J}$ has infinitely many branch points, with infinite branching at each branch point.
- If $p = 2$, then $h_\mu(\phi) = \dfrac{19}{14} \log 2 \approx \log 2.5618$.
- If $p = 2$, then $h_{\mathrm{top}}(\phi) = \log \gamma$, where $\gamma \approx 2.8136$ is the largest real root of $t^3 - 2t^2 - 3t + 2 = 0$.