Weak Néron Models for Lattès Maps

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Notation

Throughout this talk, we set the following notation:

- K is a field of characteristic zero,
- v is a discrete valuation on K,
- $|\cdot|_v$ is an absolute value associated with v,
- $\pi = \pi_v \in K$ is a uniformizer for K,
- $\mathcal{O}_v = \{x \in K : |x|_v \leq 1\} \subseteq K$ is the ring of integers,
- $\mathcal{M}_v = \pi_v \mathcal{O}_v = \{x \in K : |x|_v < 1\} \subseteq K$ is the maximal ideal,
- $k = \mathcal{O}_v / \mathcal{M}_v$ is the residue field,
- $p = \operatorname{char} k \geq 0$ is the residue characteristic.

For example, $K = \mathbb{Q}_p$, $\mathcal{O}_v = \mathbb{Z}_p$, $k = \mathbb{F}_p$, $\pi = p$,

or $K = \mathbb{Q}_p^{\mathrm{ur}}$, $\mathcal{O}_v = \mathbb{Z}_p^{\mathrm{ur}}$, $k = \overline{\mathbb{F}}_p$, $\pi = p$.

Néron Models of Elliptic Curves: Quick Summary

Definition

Let E/K be an elliptic curve, and let $\mathcal{E}$ be a smooth, separated $\operatorname{Spec}\mathcal{O}_v$-scheme. Suppose that

- The generic fiber $\mathcal{E}_K$ of $\mathcal{E}$ is isomorphic to E over K.
- ("Néron Mapping Property") For any smooth $\operatorname{Spec}\mathcal{O}_v$-scheme $\mathcal{X}$ and rational map $\psi_K : \mathcal{X}_K \to \mathcal{E}_K$, there is a unique $\operatorname{Spec}\mathcal{O}_v$-morphism $\psi : \mathcal{X} \to \mathcal{E}$ extending ψ_K.

Then we say $\mathcal{E}$ is a *Néron model* for E over K.

In particular, the Néron model $\mathcal{E}$ has the following properties:

- $\mathcal{E}$ is of finite type over $\operatorname{Spec}\mathcal{O}_v$,
- $\mathcal{E}(\mathcal{O}_v) \cong E(K)$, i.e., every point $x \in E(K)$ extends to a section $s_x : \operatorname{Spec}\mathcal{O}_v \to \mathcal{E}$,
- For any endomorphism $\psi : E \to E$ there is a $\operatorname{Spec}\mathcal{O}_v$-morphism $\Psi : \mathcal{E} \to \mathcal{E}$ such that $\Psi|_E = \psi$.

Dynamics on $\mathbb{P}^1$, and Good Reduction

Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

$\deg \phi := \max\{\deg f_1, \deg f_2\}$, where $\phi = f_1/f_2$ in lowest terms.

Setting $\phi^n := \phi \circ \cdots \circ \phi$, consider the dynamical system $\{\phi^n : n \geq 0\}$ acting on $\mathbb{P}^1$.

Definition

Given $\phi = f_1/f_2 \in K(z)$ as above, we may assume without loss that $f_1, f_2 \in \mathcal{O}_v[z]$ have no common factors, and that at least one coefficient a of f_1 or f_2 has $|a|_v = 1$.

Let $\overline{f}_1, \overline{f}_2 \in k[z]$ be the reductions of f_1 and f_2 mod $\mathcal{M}_v$. If $\deg(\overline{f}_1/\overline{f}_2) = \deg(\phi)$, then we say ϕ has *good reduction*.

Motivating Weak Néron Models

In scheme-theoretic language:

Consider the $\operatorname{Spec}\mathcal{O}_v$-scheme $\mathbb{P}^1_{\mathcal{O}_v}$, which has generic fiber $\mathbb{P}^1_K$.

Then $\phi \in K(z)$ has good reduction if and only if $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ extends from a morphism of the generic fiber to a morphism $\Phi : \mathbb{P}^1_{\mathcal{O}_v} \to \mathbb{P}^1_{\mathcal{O}_v}$ of $\operatorname{Spec}\mathcal{O}_v$-schemes.

Recall: A Néron model $\mathcal{E}$ for E is a finite type, smooth, separated $\operatorname{Spec}\mathcal{O}_v$-scheme with the properties that (among other things):

- The generic fiber $\mathcal{E}_K$ of $\mathcal{E}$ is isomorphic to E over K.
- $\mathcal{E}(\mathcal{O}_v) \cong E(K)$, i.e., every point $x \in E(K)$ extends to a section $s_x : \operatorname{Spec}\mathcal{O}_v \to \mathcal{E}$,
- Every endomorphism $\psi : E \to E$ extends to a $\operatorname{Spec}\mathcal{O}_v$-morphism $\Psi : \mathcal{E} \to \mathcal{E}$

Weak Néron Models

Definition

Let V/K be a smooth variety, and let $\phi : V \to V$ be a finite morphism defined over K. Meanwhile, let $\mathcal{V}$ be a finite type, smooth, separated $\operatorname{Spec}\mathcal{O}_v$-scheme, and let $\Phi : \mathcal{V} \to \mathcal{V}$ be a morphism of $\operatorname{Spec}\mathcal{O}_v$-schemes. Suppose that

- The generic fiber $\mathcal{V}_K$ of $\mathcal{V}$ is isomorphic to V over K.
- $\mathcal{V}(\mathcal{O}_v) \cong V(K)$, i.e., every point $x \in V(K)$ extends to a section $s_x : \operatorname{Spec}\mathcal{O}_v \to \mathcal{V}$,
- $\Phi|_V = \phi$.

Then we say $(\mathcal{V}, \Phi)$ is a *weak Néron model* for (V, ϕ) over K.

Example. If $\phi \in K(z)$ has good reduction, define $\Phi : \mathbb{P}^1_{\mathcal{O}_v} \to \mathbb{P}^1_{\mathcal{O}_v}$

by $$\Phi(x) = \begin{cases} \phi(x) & \text{for } x \in \mathbb{P}^1(K), \\ \overline{\phi}(x) & \text{for } x \in \mathbb{P}^1(k). \end{cases}$$

Then $(\mathbb{P}^1_{\mathcal{O}_v}, \Phi)$ is a weak Néron model for $(\mathbb{P}^1_K, \phi)$ over K.

Lattès Maps

Definition

A morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ over K is a *Lattès map* if there exist an elliptic curve E/K, a morphism $\psi : E \to E$ over K, and a finite separable morphism $\alpha : E \to \mathbb{P}^1$ over K so that

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ \downarrow \alpha & & \downarrow \alpha \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

commutes.

Fact. Given a Lattès map ϕ over a field of characteristic zero, we can always assume that

- $\psi \in \mathrm{End}(E)$, i.e., $\psi(O) = O$; and
- there is a nontrivial finite subgroup $\Gamma \subseteq \mathrm{Aut}(E)$ such that α is the projection map $\alpha : E \to E/\Gamma \cong \mathbb{P}^1$.

Examples of Lattès Maps

Example. $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ has quotient map $\alpha : E \to \mathbb{P}^1$ by $\alpha(x, y) = x$. If $\psi = [2] \in \mathrm{End}(E)$, then

$$\phi(x) = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6}$$

is Lattès map for ψ, making the diagram

$$\begin{array}{ccc} E & \xrightarrow{[2]} & E \\ \big\downarrow{\scriptstyle \alpha=x} & & \big\downarrow{\scriptstyle \alpha=x} \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

commute.

Example. If E has good reduction, then ϕ also has good reduction.

Multiplicative Reduction, Type I_1

Example. $E : y^2 + xy = x^3 + \pi$ has type I_1 reduction,

i.e., the special fiber of the Néron model $\mathcal{E}$ is a nodal cubic with the singularity $(\overline{x}, \overline{y}) = (\overline{0}, \overline{0})$ removed.

Taking the quotient by ± 1, the Lattès map for multiplication-by-2 is

$$\phi(x) = \frac{x^4 - 8\pi x - \pi}{4x^3 + x^2 + 4\pi}.$$

But $\mathcal{E}/\{\pm 1\}$ is **not** a weak Néron model, because $\overline{\phi}$ is bad at $\overline{0}$.

E had no K-rational points with $|x|_v \leq |\pi|_v$, but $\mathbb{P}^1$ does!

Fixing the I_1 model

For $\phi(x) = \dfrac{x^4 - 8\pi x - \pi}{4x^3 + x^2 + 4\pi}$, make a model with two components:

$$x = x_0, \qquad x_0 = \pi x_1.$$

with $X_0 \to X_0$ by

$$x_0' = \phi(x_0) = \frac{x_0^4 - 8\pi x_0 - \pi}{4x_0^3 + x_0^2 + 4\pi} \qquad \equiv \frac{\bar{x}_0^2}{4\bar{x}_0 + 1},$$

and $X_1 \to X_0$ by

$$x_0' = \phi(\pi x_1) = \frac{\pi^3 x_1^4 - 8\pi x_1 - 1}{4\pi^2 x_1^3 + \pi x_1^2 + 4} \qquad \equiv \frac{-1}{4}.$$

NOTE: This is **no longer** a finite morphism.

More General Multiplicative Reduction

Example. $E : y^2 + xy = x^3 + \pi^n$ has type I_n reduction:

$$E_0 : y_0^2 + x_0 y_0 = x_0^3 + \pi^n$$

$x_0 = \pi x_1, \; y_0 = \pi y_1$

$$E_{\pm 1} : y_1^2 + x_1 y_1 = \pi x_1^3 + \pi^{n-2}$$

$x_1 = \pi x_2, \; y_1 = \pi y_2$

$$E_{\pm 2} : y_2^2 + x_2 y_2 = \pi^2 x_2^3 + \pi^{n-4}$$

$$\vdots$$

$$\begin{cases} E_{\pm m} : y_m^2 + x_m y_m = \pi^m x_m^3 + \pi & \text{if } n = 2m+1, \\ E_m : y_m^2 + x_m y_m = \pi^m x_m^3 + 1 & \text{if } n = 2m. \end{cases}$$

The Néron model $\mathcal{E}$ consists of the union of the n components E_j with singularities on the special fiber removed.

Multiplicative Reduction: Quotient of the Néron model

Multiplication-by-2 on $E : y^2 + xy = x^3 + \pi^n$,
under quotient by ± 1, induces $\phi(x) = \dfrac{x^4 - 8\pi^n x - \pi^n}{4x^3 + x^2 + 4\pi^n}$.

Taking the quotient of the Néron model by ± 1 gives

$$x_0 = \pi x_1, \qquad x_1 = \pi x_2, \qquad \cdots, \qquad x_{m-1} = \pi x_m$$

with

$$X_0 \to X_0 : \qquad x_0' = \phi(x_0) = \frac{x_0^4 - 8\pi^n x_0 - \pi^n}{4x_0^3 + x_0^2 + 4\pi^n} \quad \equiv \frac{\overline{x}_0^2}{4\overline{x}_0 + 1},$$

$X_j \to X_{2j}$, for $1 \le j \le n/4$:

$$x_{2j}' = \pi^{-2j}\phi(\pi^j x_j) = \frac{x_j^4 - 8\pi^{n-3j} x_j - \pi^{n-4j}}{4\pi^j x_j^3 + x_j^2 + 4\pi^{n-2j}} \quad \equiv \overline{x}_j^2 - (0 \text{ or } \frac{1}{\overline{x}_j^2})$$

$X_j \to X_{n-2j}$, for $n/4 < j \le n/2$:

$$x_{n-2j}' = \pi^{2j-n}\phi(\pi^j x_j) = \frac{\pi^{4j-n} x_j^4 - 8\pi^j x_j - 1}{4\pi^j x_j^3 + x_j^2 + 4\pi^{n-2j}} \equiv \frac{-1}{\overline{x}_j^2 + (0 \text{ or } 4)}$$

Multiplicative Reduction: The Quotient, Continued

The Upshot: For type I_n reduction, with $n = 2m$ or $n = 2m + 1$, the quotient $\mathcal{V}$ of the Néron model $\mathcal{E}$ by ± 1 consists of the $m + 1$ components given by variables

$$x_0, \quad x_1, \quad \ldots, \quad x_m$$

If $m = 2n$ is even: Then $\mathcal{V}$, with the maps on the previous slide, is a weak Néron model for ϕ.

But if $n = 2m + 1$ is odd: Then $\mathcal{V}$ has a problem for $x \in \mathbb{P}^1(K)$ with $|x| < |\pi^m|$, as we'll now see...

Fixing the Odd I_n case

For $\phi(x) = \dfrac{x^4 - 8\pi^n x - \pi^n}{4x^3 + x^2 + 4\pi^n}$ and $n = 2m+1$, we have

$$x_0 = \pi x_1, \qquad x_1 = \pi x_2, \qquad \cdots, \qquad x_{m-1} = \pi x_m$$

with $X_m \to X_1$ by

$$x_1' = \pi^{-1}\phi(\pi^m x_m) = \frac{\pi^{2m-1}x_m^4 - 8\pi^m x_m - 1}{4\pi^m x_m^3 + x_m^2 + 4\pi} \equiv \frac{-1}{\overline{x}_m^2}$$

which fails to be a morphism at $\overline{x}_m \equiv 0$, because $X_1(k)$ has no point at ∞.

So let's blow up again at $\overline{x}_m = 0$, i.e., add a new component: X_{m+1}, with variable x_{m+1} satisfying $x_m = \pi x_{m+1}$, and with $X_{m+1} \to X_0$ by $x_{m+1} \mapsto \phi(\pi^{m+1}x_{m+1})$, i.e.,

$$x_{m+1} \mapsto \frac{\pi^{2m+3}x_{m+1}^4 - 8\pi^{m+1}x_{m+1} - 1}{4\pi^{m+2}x_{m+1}^3 + \pi x_{m+1}^2 + 4} \equiv -\frac{1}{4}$$

NOTE: Again, this is **not** a finite morphism.

Additive Reduction: Quotient of the Néron model

Example. $p \neq 2$, $E : y^2 = x^3 - \pi^2 x$ has type I_0^* reduction, with four components

$$y_0^2 = x_0^3 - \pi^2 x_0, \qquad y_1^2 = \pi^2 x_1^3 - x_1,$$
$$y_2^2 = \pi^2 x_2^3 + 3\pi x_2^2 + 2x_2, \qquad y_3^2 = \pi^2 x_3^3 - 3\pi x_3^2 + 2x_3$$

where

$$x_0 = \pi^2 x_1 = \pi + \pi^2 x_2 = -\pi + \pi^2 x_3.$$

The quotient $\mathcal{V}$ of the Néron model by ± 1 has the same four x_i-components but is **not** a weak Néron model for the Lattès map

$$\phi(x) = \frac{(x^2 + \pi^2)^2}{4x(x^2 - \pi^2)}.$$

After all, no points of $\mathbb{P}^1(K)$ with $|x|_v = |x - \pi|_v = |x + \pi|_v = |\pi|_v$ extend to the special fiber of $\mathcal{V}$.

Fixing the type I_0^* model

Instead, for $E : y^2 = x^3 - \pi^2 x$ and $\phi(x) = \dfrac{(x^2+\pi^2)^2}{4x(x^2-\pi^2)}$, make the coordinate change $x = \pi x_1$ to get

$$\psi(x_1) = \pi^{-1}\phi(\pi x_1) = \frac{(x^2+1)^2}{4x(x^2-1)},$$

which has good reduction.

Equivalently, we could work over the field

$$L = K(\sqrt{\pi}),$$

where E has good reduction, take the quotient of the (type I_0) Néron model by ± 1, and then base change back to K.

The Berkovich Projective Line

Informal Definition. Let $\mathbb{C}_v$ be the completion of an algebraic closure of K with respect to $|\cdot|_v$.

The *Berkovich projective line* $\mathbb{P}^1_{\mathrm{Ber}}$ is a compact, path-connected topological space containing $\mathbb{P}^1(\mathbb{C}_v)$.

(Most of) the extra points added correspond to closed disks $\overline{D}(a, r) \subseteq \mathbb{C}_v$.

More Formal Definition. $\mathbb{A}^1_{\mathrm{Ber}}$ is the set of multiplicative seminorms on $\mathbb{C}_v[z]$ extending $|\cdot|_v$ on $\mathbb{C}_v \subseteq \mathbb{C}_v[z]$, equipped with the *Gel'fand topology*, i.e., the weakest topology such that for all $f \in \mathbb{C}_v[z]$, the function $\mathbb{A}^1_{\mathrm{Ber}} \to [0, \infty)$ by $\|\cdot\| \mapsto \|f\|$ is continuous.

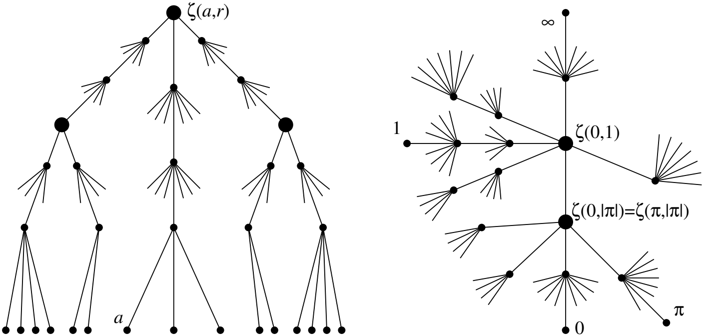
$\mathbb{P}^1_{\mathrm{Ber}}$ is then $\mathbb{A}^1_{\mathrm{Ber}}$ together with a point at infinity, i.e., the one-point compactification of $\mathbb{A}^1_{\mathrm{Ber}}$.

Berkovich Points

Points of $\mathbb{P}^1_{\mathrm{Ber}}$ come in four types:

- Type 1: "classical points," i.e., points of $\mathbb{P}^1(\mathbb{C}_v)$.
- Type 2: There is one Type 2 point $\zeta(a, r)$ for each rational closed disk $\overline{D}(a, r) \subseteq \mathbb{C}_v$.
  [Here, "rational" means $r \in |\mathbb{C}_v^\times|_v$.]
- Type 3: There is one Type 3 point $\zeta(a, r)$ for each irrational disk $\overline{D}(a, r) \subseteq \mathbb{C}_v$.
- Type 4: These points correspond to chains of disks $D_1 \supset D_2 \supset \cdots$ with empty intersection.
  We well not be concerned with Type 4 points.

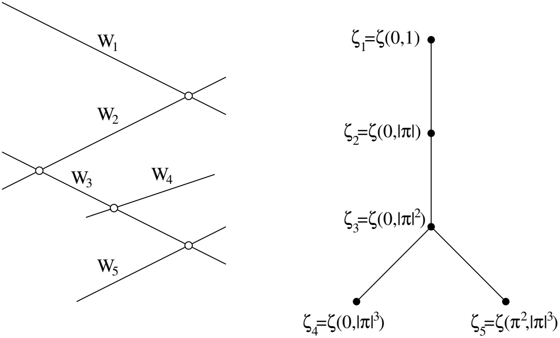
A Rough Sketch of $\mathbb{P}^1_{\text{Ber}}$



(a) (b)

Definition A type 2 point $\zeta = \zeta(a, r)$ is *K-rational* if at least three different components of $\mathbb{P}^1_{\text{Ber}} \setminus \{\zeta\}$ intersect $\mathbb{P}^1(K)$.
Equivalently, the disk $\overline{D}(a, r)$ contains a point of K and has $r \in |K^\times|_v$.

Models as sets of type 2 points



(a) (b)

A finite-type model $\mathcal{V}$ becomes a finite set of type 2 points.

- Each component in the model $\mathcal{V}$ corresponds to a K-rational type 2 point in $\mathbb{P}^1_{\text{Ber}}$.
- If $\mathcal{V}$ is to have $\mathcal{V}(\mathcal{O}_v) \cong \mathbb{P}^1(K)$, then for any two K-rational type 2 points in the Berkovich model, all K-rational type 2 points between them must also be in the Berkovich model.

Berkovich weak Néron models

Definition

Let $M \subseteq \mathbb{P}^1_{\text{Ber}}$ be a finite set of K-rational type 2 points.

- We say M is a *Berkovich model* for $\mathbb{P}^1$ over K if: for any two points $\zeta_1, \zeta_2 \in M$, all K-rational type 2 points between ζ_1 and ζ_2 also belong to M.
- Let $\phi \in K(z)$ be a non-constant rational function. We say M is a *Berkovich weak Néron model* for ϕ over K if: for every $\zeta \in M$ and every connected component U of $\mathbb{P}^1_{\text{Ber}} \smallsetminus \{\zeta\}$, if $U \cap M = \varnothing$, then $\phi(U) \cap M = \varnothing$.

Idea: In the second condition above,

- ζ corresponds to a component W_i of the special fiber of the model $\mathcal{V}$, and
- U, which is a disk, corresponds to a point on W_i.

Berkovich models under base change

Let $T_K \subseteq \mathbb{P}^1_{\text{Ber}}$ denote the set of K-rational type 2 points in $\mathbb{P}^1_{\text{Ber}}$.

Definition

Let L/K be a finite extension, and let $N \subseteq T_L$ be a Berkovich model for $\mathbb{P}^1$ over L.
Define

$$M = (N \cap T_K) \cup \bigcup_U \partial U,$$

where the union is over all connected components U of $\mathbb{P}^1_{\text{Ber}} \smallsetminus T_K$ that contain points of N.
Then we say M is the Berkovich model over K *induced* by N.

Idea: Each U is an annulus; ∂U consists of two K-rational type 2 points.

Berkovich weak Néron models under base change

Theorem

Let L be a finite extension of K, and let $\phi \in K(z)$ be a nonconstant rational function. Suppose that $N \subseteq T_L$ is a Berkovich weak Néron model for ϕ over L, and let M be the Berkovich model over K induced by N.
Then M is a Berkovich weak Néron model for ϕ over K.

Proof (sketch):
(1) (M is a Berkovich model over K): Given two points $\zeta_1, \zeta_2 \in M$, and a third $\zeta \in T_K$ lying between them,
we can show that ζ lies between two points of N.
Then $\zeta \in N \cap T_K \subseteq M$.

(2) (weak Néron property): Given $\zeta \in M$ and an open Berkovich disk U at ζ, disjoint from M,
U is contained in a corresponding disk V for $\zeta' \in N$, and hence $\phi(U) \cap N = \varnothing$.

Slightly more work then shows $\phi(U) \cap M = \varnothing$.

Constructing Berkovich weak Néron models

Corollary

Let ϕ be a Lattès map defined over K. Then ϕ has a weak Néron model defined over K.

Proof (Sketch). Let E be the corresponding elliptic curve, and let L/K be a finite extension over which E has reduction of type I_n for n even.

If $n = 0$, then E/L has good reduction, and so ϕ also has good reduction (after L-rational coordinate change) and hence a weak Néron model $\mathcal{V}'$ over L.

If $n \geq 2$, let $\mathcal{E}'$ be the Néron model over L, and let $\mathcal{V}'$ be $\mathcal{E}'/\{\pm 1\}$, which is a weak Néron model for ϕ over L.

Let N be the Berkovich weak Néron model over L associated to $\mathcal{V}'$.

Let M be the Berkovich model over K induced by N, which is a Berkovich weak Néron model by the previous theorem.

Then the associated model $\mathcal{V}$ is a weak Néron model over K.

Example

Let $K = \mathbb{Q}_2^{\mathrm{ur}}$, and $E : y^2 = x^3 - 4x$, which has type I_2^* reduction. Consider the Lattès map for multiplication-by-2,
$\phi(x) = \dfrac{(x^2+4)^2}{4x(x^2-4)}$.

Warning: Wild ramification: E.g. over $K' = K(\sqrt{2})$, E is isomorphic to $E' : y^2 = x^3 - x$, which has type I_3^* reduction.

Back to $K = \mathbb{Q}_2^{\mathrm{ur}}$: besides the 2-torsion points O, $(0,0)$, $(\pm 2, 0)$, all the 3- and 4-torsion points have x-coordinates in the disk $\overline{D}(2+2\sqrt{2}, |4|_2) \subseteq \overline{K}$.

So if E has semistable reduction over L/K, the algorithm says the associated Berkovich weak Néron model over K will have two points: $\zeta(0, |2|_2)$ and $\zeta(2, |4|_2)$.

Example, Continued

Recall: $K = \mathbb{Q}_2^{\mathrm{ur}}$, and $E : y^2 = x^3 - 4x$, and $\phi(x) = \dfrac{(x^2+4)^2}{4x(x^2-4)}$.

Our Berkovich weak Néron model over K has two points: $\zeta(0, |2|_2)$ and $\zeta(2, |4|_2)$.

That is, $x = 2x_0$ and $x_0 = 1 + 2x_1$, with $X_0 \to X_0$ by

$$x_0' = \frac{1}{2}\phi(2x_0) = \frac{(x_0^2+1)^2}{4x_0(x_0^2-1)} \qquad \equiv \infty$$

and $X_1 \to X_0$ by

$$x_0' = \frac{1}{2}\phi(4x_1+2) = \frac{(2x_1^2+2x_1+1)^2}{4x_1(x_1+1)(2x_1+1)} \qquad \equiv \infty$$