Non-archimedean connected Julia sets with branching

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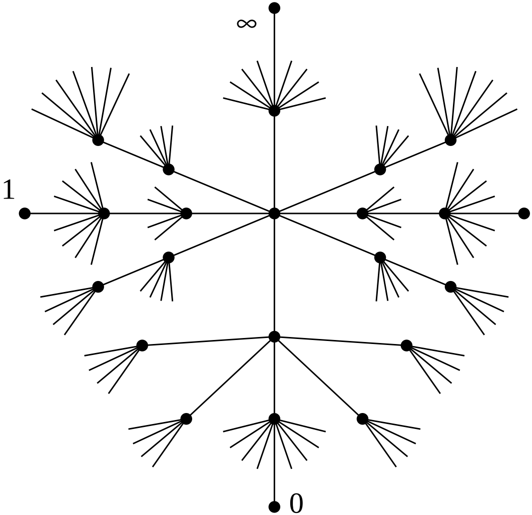
The Berkovich Projective Line



Rational Functions on the Berkovich Projective Line

Let K be a complete and algebraically closed non-archimedean field. (E.g. $K = \mathbb{C}_p$).

Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

[$\deg \phi := \max\{\deg f, \deg g\}$, where $\phi = f/g$ in lowest terms.]

Then $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$, and this action extends continuously to $\phi : \mathbb{P}^1_{\text{Ber}} \to \mathbb{P}^1_{\text{Ber}}$.

Write $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}$.

Fatou and Julia sets in Berkovich Space

For $\phi \in K(z)$, define the (Berkovich) **Fatou set** of ϕ to be

$$\mathcal{F} = \{x \in \mathbb{P}^1_{\mathrm{Ber}} : x \text{ has a neighborhood } U \text{ s.t. } \mathbb{P}^1_{\mathrm{Ber}} \setminus \bigcup_{n \geq 0} \phi^n(U) \text{ is infinite}\},$$

and the (Berkovich) **Julia set** of ϕ to be $\mathcal{J} = \mathbb{P}^1_{\mathrm{Ber}} \setminus \mathcal{F}$.

Facts:

- $\mathcal{J}$ is closed and hence compact.
- $\mathcal{J}$ is invariant under ϕ, i.e., $\phi^{-1}(\mathcal{J}) = \mathcal{J}$.
- There is a natural Borel probability measure $\mu = \mu_\phi$ such that
  - $\mathrm{supp}(\mu) = \mathcal{J}$.
  - μ is invariant under ϕ, i.e., $\mu(\phi^{-1}(E)) = \mu(E)$.

Measure-theoretic Entropy (a.k.a. Metric Entropy)

Let X be a topological space and $f : X \to X$ a continuous map. Let μ be an f-invariant Borel probability measure on X.

(**Recall**: f-invariant means $\mu(f^{-1}(E)) = \mu(E)$.)

Definition. The *measure-theoretic entropy* of (f, μ) is

$$h_\mu(f) = \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{P} \vee f^{-1}\mathcal{P} \vee \cdots \vee f^{-n}\mathcal{P}),$$

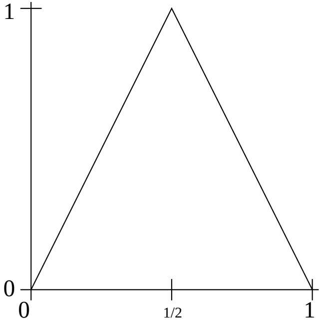
where

- $f^{-j}\{U_1, \ldots, U_m\} = \{f^{-j}(U_1), \ldots, f^{-j}(U_m)\}$,
- $\mathcal{P} \vee \mathcal{P}' = \{U \cap U' : U \in \mathcal{P}, U' \in \mathcal{P}'\}$,
- $H_\mu(\mathcal{P}) = \sum_{U\in\mathcal{P}} -\mu(U)\log(\mu(U))$.

and the supremum is over all finite Borel partitions of X.

Example: the Tent Map

Let $X = [0,1]$, $\lambda =$Lebesgue measure, and $f : X \to X$ with graph



Using the partition $\mathcal{P} = \{[0,1/2],(1/2,1]\}$, one can show that $h_\lambda(f) = \log 2$.

Similarly, the d-to-1 version of the tent map, with d zigs, has $h_\lambda(f) = \log d$.

Topological Entropy

Let X be a *compact* topological space and $f : X \to X$ a continuous map.

Definition. The *topological entropy* of f is

$$h_{\text{top}}(f) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee f^{-1}\mathcal{U} \vee \cdots \vee f^{-n}\mathcal{U}),$$

where

- the supremum is over all finite open covers $\mathcal{U}$ of X,
- $f^{-j}\{U_1, \ldots, U_m\} = \{f^{-j}(U_1), \ldots, f^{-j}(U_m)\}$,
- $\mathcal{U} \vee \mathcal{U}' = \{U \cap U' : U \in \mathcal{U}, U' \in \mathcal{U}'\}$,
- $N(\mathcal{U}) =$ min number of elements of $\mathcal{U}$ needed to cover X.

The Variational Principle: If X is compact and metrizable, then

$$h_{\text{top}}(f) = \sup_{\mu} h_{\mu}(f),$$

where the sup is over all f-invariant Borel probability measures.

Entropy: Complex vs. Non-archimedean Dynamics

Fact: Let $\phi \in \mathbb{C}(z)$ be a rational function of degree $d \geq 2$, with associated Julia set $\mathcal{J} \subseteq \mathbb{P}^1(\mathbb{C})$ and invariant measure μ. Then

$$h_\mu(\phi) = h_{\text{top}}(\phi) = \log d.$$

Theorem (Favre & Rivera-Letelier, 2010)

Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$, with associated Julia set $\mathcal{J} \subseteq \mathbb{P}^1_{\text{Ber}}$ and invariant measure μ. Then

$$0 \leq h_\mu(\phi) \leq h_{\text{top}}(\phi) \leq \log d.$$

But both equalities of the $\mathbb{C}$ theorem can fail (or not) for K.

Examples

$$0 \le h_\mu(\phi) \le h_{\text{top}}(\phi) \le \log d.$$

Example 1. $\phi \in K(z)$ has good reduction. Then $\mathcal{J} = \{\zeta(0,1)\}$ is a single point, and $0 = h_\mu(\phi) = h_{\text{top}}(\phi) < \log d$.

Example 2. $\phi(z) = z^2 - az$ with $|a| > 1$. Then $\mathcal{J}$ is a Cantor set contained in $\mathbb{P}^1(K)$, and $0 < h_\mu(\phi) = h_{\text{top}}(\phi) = \log 2$.

Example 3. Let E/K be an elliptic curve of multiplicative reduction, and $\phi \in K(z)$ the *Lattès* map with $x([m]P) = \phi(x[P])$.

Then $(\mathcal{J}, \mu) \cong ([0,1], \lambda)$, with ϕ acting as the m-zig tent map.

So $0 < \log m = h_\mu(\phi) = h_{\text{top}}(\phi) < \log(m^2)$.

Non-Maximal Entropy

Favre and Rivera-Letelier gave examples where $h_\mu(\phi) < h_{\text{top}}(\phi)$. These included:

- a degree 5 rational function with Julia set a Cantor set, and
- a degree 10 rational function with Julia set an interval.

In both cases, the Julia set was contained in an interval.

Motivated by these examples, Favre and Rivera-Letelier ask:

Question: Is there a rational function ϕ of degree ≤ 9 with connected Julia set $\mathcal{J}$ and with $h_\mu(\phi) < h_{\text{top}}(\phi)$?

Yes, there is, at least in small residue characteristic

Theorem (Bajpai, RB, Chen, Kim, Marschall, Onul, Xiao)

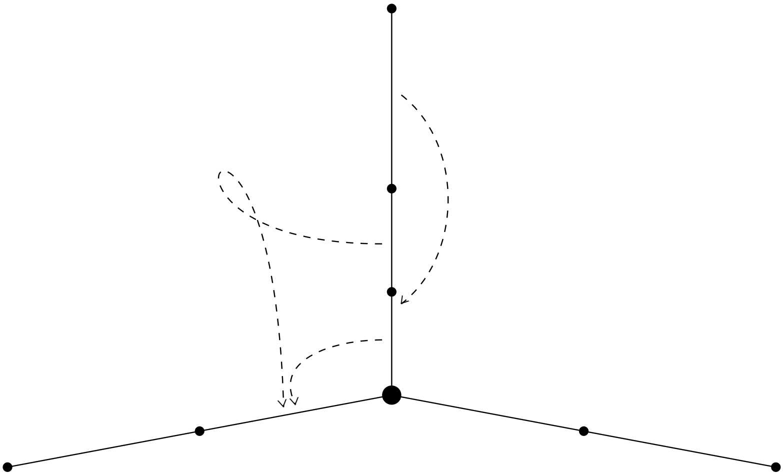
Let K have residue characteristic 3 (e.g. $K = \mathbb{C}_3$), fix $a \in K^\times$ with $|3| \leq |a| < 1$, and let $\phi(z) = \dfrac{az^6 + 1}{az^6 + z^3 - z}$.

Then

- *The Julia set $\mathcal{J}$ of ϕ is path-connected, with infinitely many branch points,*
- $h_\mu(\phi) = \dfrac{6}{11}\log 2 + \dfrac{5}{11}\log 6 \approx \log 3.2954$, *and*
- $h_{\mathrm{top}}(\phi) = \log\beta$, *where $\beta \approx 3.8558$ is the largest real root of* $t^3 - 4t^2 - t + 6$.

So $0 < h_\mu(\phi) < h_{\mathrm{top}}(\phi) < \log 6$.

Dynamics of $\phi(z) = \dfrac{az^4 + 1}{z^2 - z}$ in residue characteristic 2



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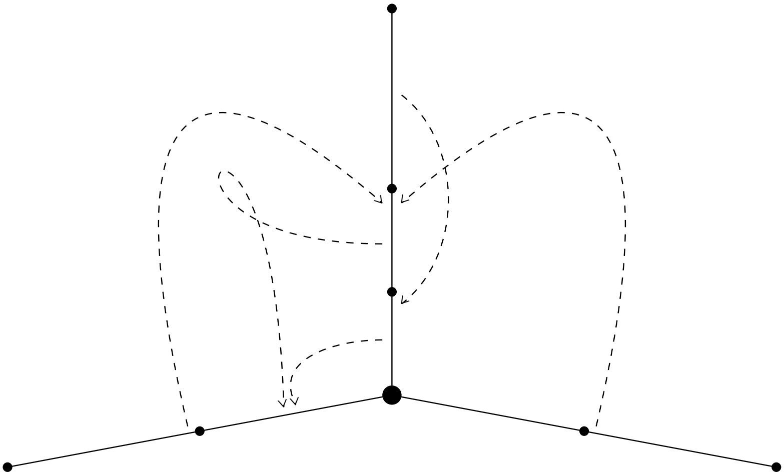

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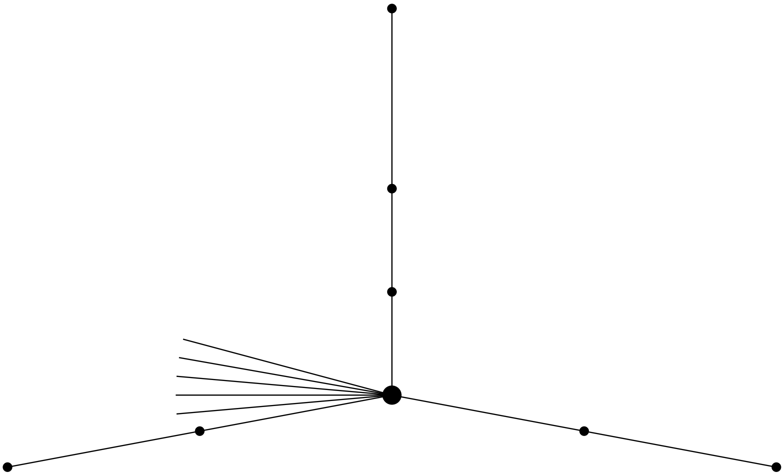

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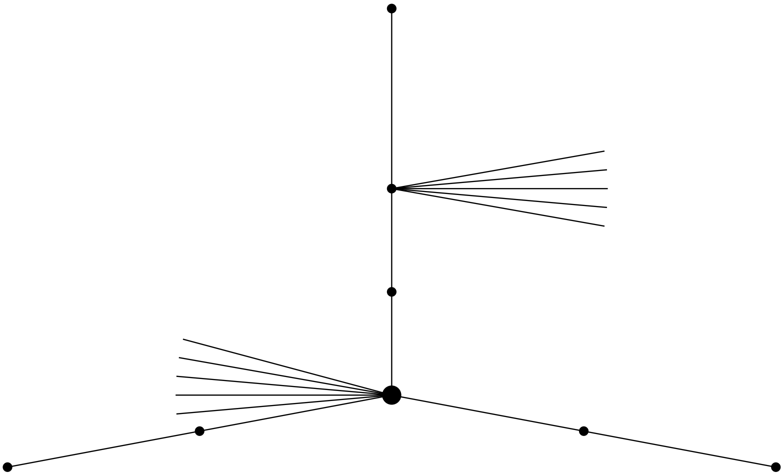

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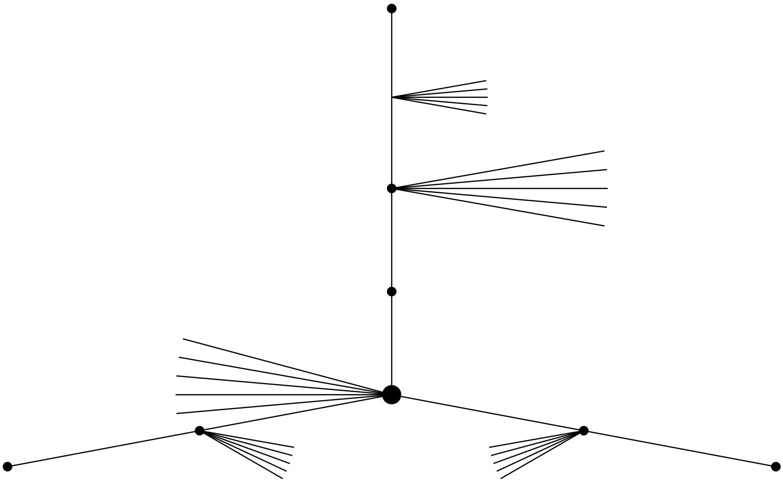

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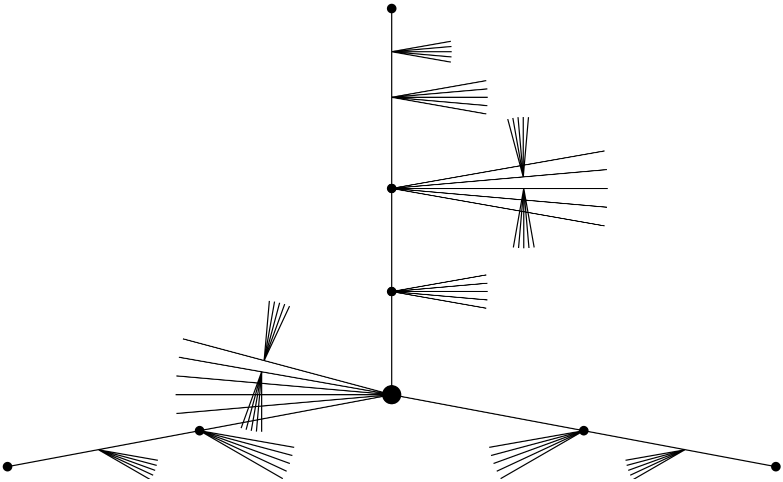


Dynamics of $\phi(z) = \dfrac{az^4 + 1}{z^2 - z}$ in residue characteristic 2

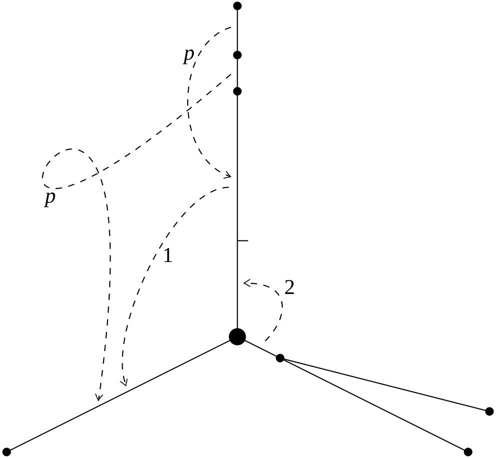
With $|2| \le |a| < 1$, we can show:

- The Julia set $\mathcal{J}$ of ϕ is path-connected and contains the above tree.
- $\mathcal{J}$ has infinitely many branch points, with infinite branching at each branch point.
- $h_\mu(\phi) = \dfrac{19}{14} \log 2 \approx \log 2.5618.$
- $h_{\mathrm{top}}(\phi) = \log \gamma$, where $\gamma \approx 2.8136$ is the largest real root of $t^3 - 2t^2 - 3t + 2 = 0$.

Another map, in residue characteristic $p \geq 3$

Fix $a, b \in K^\times$ with $|p|^{1/(p+1)} \leq |a| < 1 = |b| = |b-1| = |b+1|$,

and let $\phi(z) = \dfrac{(z-1)(z-b)(1+a^{p^2-1}z^p)}{z(z-a^2)}$. (deg $\phi = p+2$.)



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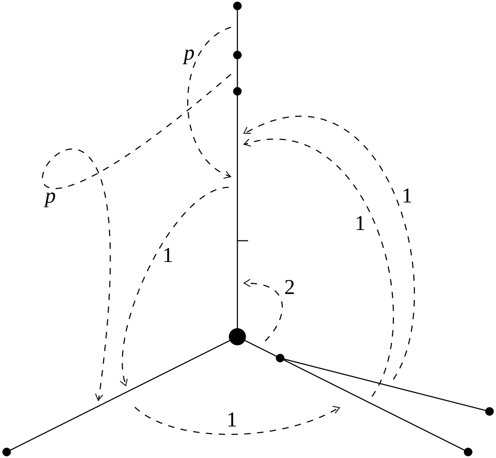

Dynamics of this new map

For the degree $p+2$ map of the previous slide,

- The Julia set $\mathcal{J}$ of ϕ is path-connected and contains the above tree.
- $\mathcal{J}$ has infinitely many branch points, with infinite branching at each branch point.
- $h_\mu(\phi) = \log(p+2) - \frac{p}{p+2} \log p < \log 3.$
- $h_{\text{top}}(\phi) = \log 3.$

Thus, $0 < h_\mu(\phi) < h_{\text{top}}(\phi) < \log(\deg \phi).$

Two Questions

Can we achieve $h_\mu(\phi) < h_{\mathrm{top}}(\phi)$ or $h_{\mathrm{top}}(\phi) < \log(\deg \phi)$ without either

(a) $\mathcal{J}$ contained in an interval, or

(b) exploiting wild ramification?

Is $h_{\mathrm{top}}(\phi)$ always the logarithm of an algebraic integer?