An N-point Version of the Mason-Stothers *abc* Theorem

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The Mason-Stothers *abc* Theorem

Theorem (Stothers, Mason, 1980s)

Given $a, b, c \in \mathbb{C}[t]$ *satisfying*

- $a + b = c$,
- $\gcd(a, b) = 1$, *and*
- *not all three of* a, b, c *are constant.*

Then

$$\max\{\deg a, \deg b, \deg c\} \leq -1 + S_0(abc)$$

where $S_0(f) :=$ *the number of distinct zeros of* f.

Note:

- If $\deg a = \deg b = \deg c$, then the inequality still holds with -1 replaced by -2.
- If a, b, c **are** all constant, then the Theorem obviously holds with -1 replaced by 0.

The Mason-Stothers *abc* Theorem for Function Fields

$\mathbb{F}$ = algebraically closed field of characteristic zero,
K = function field over $\mathbb{F}$ of genus g,
$M_K = \{\text{places of } K\}$.

Theorem

Given $a, b, c \in K^\times$ satisfying

- *$a + b = c$, and*
- *a/c is not constant.*

Then

$$\sum_{v \in M_K} \min\{v(a), v(b), v(c)\} \geq (2 - 2g) - \#S(a, b, c),$$

where $S(a, b, c) := \{v \in M_K : v(a), v(b), v(c) \text{ are not all equal}\}$.

Note: If a/c **is** constant, then $S = \varnothing$, and the Theorem holds with $2 - 2g$ replaced by 0.

Sketch of Proof of *abc*.

Let $u = \frac{a}{c}$ and $v = \frac{b}{c}$. Neither is constant, and $u + v = 1$.

Let $\omega = c^2 du = -c^2 dv \in \Omega^1(M_K)$.

(Local step): Bound the order of vanishing of ω at all points $v \in M_K$:

$$\mathrm{ord}_v(\omega) \geq (\max + \min)\{v(a), v(b), v(c)\} - \begin{cases} 1 & v \in S, \\ 0 & v \notin S. \end{cases}$$

(Global step): Since $\omega \neq 0$, Riemann-Roch says

$$\sum_v \mathrm{ord}_v(\omega) = 2g - 2.$$

Subtract $\sum_v v(abc) = 0$, rearrange, and the Theorem follows.
QED

Four Points in $\mathbb{P}^1(K)$

Given $f_1, f_2, f_3, f_4 \in \mathbb{P}^1(K)$,

$$\underbrace{(f_1 - f_4)(f_3 - f_2)}_{a} + \underbrace{(f_1 - f_3)(f_2 - f_4)}_{b} = \underbrace{(f_1 - f_2)(f_3 - f_4)}_{c}.$$

Moral: Think of *abc* as a statement about **four points in** $\mathbb{P}^1(K)$.

Goal: Make an *abc*-style statement about $N > 4$ **points in** $\mathbb{P}^1(K)$.

An Observation

Revisiting the *abc* proof for

$$\underbrace{(f_1 - f_4)(f_3 - f_2)}_{a} + \underbrace{(f_1 - f_3)(f_2 - f_4)}_{b} = \underbrace{(f_1 - f_2)(f_3 - f_4)}_{c},$$

we have $u = \dfrac{(f_1 - f_4)(f_3 - f_2)}{(f_1 - f_2)(f_3 - f_4)}$, and $\omega = c^2 du$, with

$$\mathrm{ord}_v(\omega) \geq (\max + \min) \begin{Bmatrix} v[(f_1 - f_4)(f_3 - f_2)], \\ v[(f_1 - f_2)(f_3 - f_4)], \\ v[(f_1 - f_3)(f_2 - f_4)] \end{Bmatrix} - \begin{Bmatrix} 0 \\ \text{or} \\ 1 \end{Bmatrix}.$$

One can check that $\omega = \det \begin{bmatrix} 1 & f_1 & f_1^2 & df_1 \\ 1 & f_2 & f_2^2 & df_2 \\ 1 & f_3 & f_3^2 & df_3 \\ 1 & f_4 & f_4^2 & df_4 \end{bmatrix}$.

Example: $6 = 2 \cdot 3$ points in $\mathbb{P}^1(K)$

$$\omega_{2,3} = \begin{vmatrix} 1 & f_1 & f_1^2 & f_1^3 & df_1 & f_1\,df_1 \\ 1 & f_2 & f_2^2 & f_2^3 & df_2 & f_2\,df_2 \\ 1 & f_3 & f_3^2 & f_3^3 & df_3 & f_3\,df_3 \\ 1 & f_4 & f_4^2 & f_4^3 & df_4 & f_4\,df_4 \\ 1 & f_5 & f_5^2 & f_5^3 & df_5 & f_5\,df_5 \\ 1 & f_6 & f_6^2 & f_6^3 & df_6 & f_6\,df_6 \end{vmatrix} \in \left(\Omega^1\right)^{\otimes 2}$$

Idea: Assign a "power" of 1 to f_i and 2 to df_i.

Then the i-th row consists of all monomials $f_i^r\,df_i^{\otimes s}$ with

- $s \leq 1$, and
- $r + 2s \leq 3$.

Example: $9 = 3 \cdot 3$ points in $\mathbb{P}^1(K)$

$$\omega_{3,3} = \begin{vmatrix} 1 & f_1 & f_1^2 & f_1^3 & f_1^4 & df_1 & f_1\,df_1 & f_1^2\,df_1 & df_1^{\otimes 2} \\ 1 & f_2 & f_2^2 & f_2^3 & f_2^4 & df_2 & f_2\,df_2 & f_2^2\,df_2 & df_2^{\otimes 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & f_9 & f_9^2 & f_9^3 & f_9^4 & df_9 & f_9\,df_9 & f_9^2\,df_9 & df_9^{\otimes 2} \end{vmatrix} \in \left(\Omega^1\right)^{\otimes 5}$$

Idea: Then the i-th row consists of all monomials $f_i^r\,df_i^{\otimes s}$ with

- $s \leq 2$, and
- $r + 2s \leq 4$.

General $N = k \cdot \ell$ points in $\mathbb{P}^1(K)$, with $\ell \geq k \geq 1$

Given $f_1, \dots, f_N \in \mathbb{P}^1(K)$, set $\omega_{k,\ell} = \det A \in (\Omega^1)^{\otimes k(k-1)(3\ell-k-1)/6}$ where A is the $N \times N$ matrix with i-th row consisting of all monomials $f_i^r df_i^{\otimes s}$ such that:

- $s \leq k-1$, and
- $r + 2s \leq m := k + \ell - 2$.

That is, the i-the row is

$$[\underbrace{\text{——}\ v_{i,0}\ \text{——}}_{m+1 \text{ entries}} \ \underbrace{\text{——}\ v_{i,1}\ \text{——}}_{m-1 \text{ entries}} \ \cdots \ \underbrace{\text{——}\ v_{i,k-1}\ \text{——}}_{m-2k+3 \text{ entries}}],$$

where $v_{i,0} = [1 \quad f_i \quad \cdots \quad f_i^m]$,

$v_{i,1} = [df_1 \quad f_i df_i \quad \cdots \quad f_i^{m-2} df_i]$,

and in general, $v_{i,j} = [df_i^{\otimes j} \quad f_i df_i^{\otimes j} \quad \cdots \quad f_i^{m+1-2j} df_i^{\otimes j}]$.

The Local Bound

Recall $N = k \cdot \ell,\quad m = k + \ell - 2,$
$M = k(k-1)(3\ell - k - 1)/6,\quad \omega_{k,\ell} \in (\Omega^1)^{\otimes M}$

The shape of the bound is

$$\operatorname{ord}_v(\omega_{k,\ell}) \geq B_{v,k,\ell}[f_1, \dots, f_N] := \sum_{(i,j)\in\mathcal{I}} v(f_i - f_j) - P,$$

where $\mathcal{I} \subseteq \{1, \dots, N\}^2$ is a set of $mN/2$ pairs such that each $i \in \{1, \dots, N\}$ appears exactly m times, and $0 \leq P \leq M$ is an integer.

E.g., recall

$$B_{v,2,2}[f_1, f_2, f_3, f_4] = (\max + \min) \left\{ \begin{matrix} v[(f_1 - f_4)(f_3 - f_2)], \\ v[(f_1 - f_2)(f_3 - f_4)], \\ v[(f_1 - f_3)(f_2 - f_4)] \end{matrix} \right\} - \left\{ \begin{matrix} 0 \\ \text{or} \\ 1 \end{matrix} \right\}$$

The Convex Hull of $\{f_1, \ldots, f_N\}$ in Berkovich Space

But the precise bound is complicated, involving certain maxima over all (recursive) partitions of the set $\{f_1, \ldots, f_N\}$.
It's easier to understand using graphs in Berkovich space.

Note. Fix $v \in M_K$. Given $a \in K$ and $u \in \mathbb{R}$, the closed disk

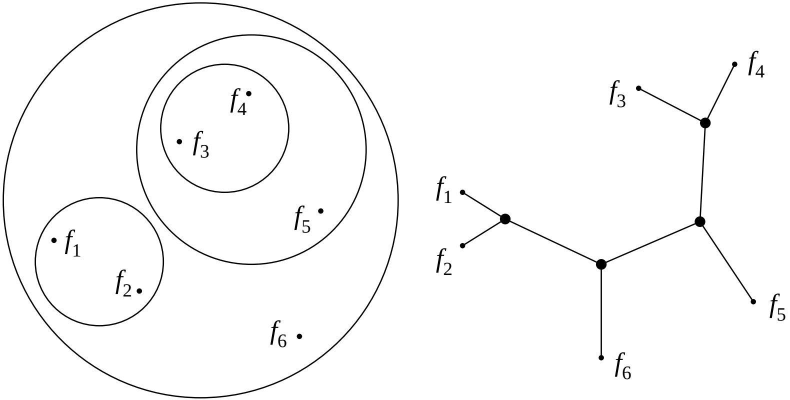
$$\overline{D}(a, u) := \{x \in K : v(a - x) \geq u\}$$

partitions $\mathbb{P}^1(K)$ into residue classes modulo v.

Idea: Given $f_1, \ldots, f_N \in K$, make a graph with:

- one vertex for each f_i
- one point (either a vertex or part of an edge) for each closed disk D for which $f_1 \ldots, f_N$ do **not** all lie in the same residue class.

The graph is a tree, with exterior vertices $f_1, \ldots, f_N$.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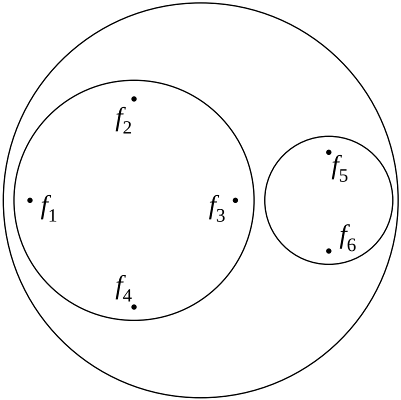
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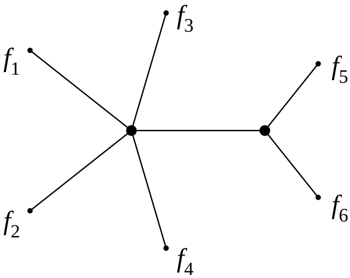
f_3
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f_4

The Bound $B_{v,2,2}[f_1, \ldots, f_4]$

Use a sum of terms of the form $v(f_i - f_j)$, where

- Each f_i appears exactly 2 times.
- No pair $v(f_i - f_j)$ is repeated.
- If there is an edge separating $\{f_i, f_j\}$ from the others, **make sure** $v(f_i - f_j)$ appears.

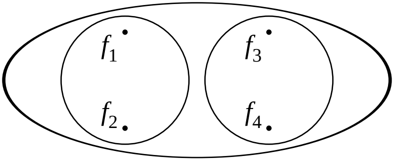
  Also, subtract 1 as a penalty.
- If the penalty subtracted is more than 1, make it only 1.

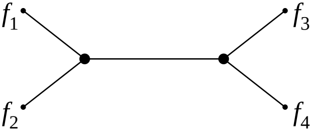
This gives

$$B_{v,2,2}[f_1, f_2, f_3, f_4] = (\max + \min) \left\{ \begin{array}{c} v[(f_1 - f_4)(f_3 - f_2)], \\ v[(f_1 - f_2)(f_3 - f_4)], \\ v[(f_1 - f_3)(f_2 - f_4)] \end{array} \right\} - \left\{ \begin{array}{c} 0 \\ \text{or} \\ 1 \end{array} \right\}$$

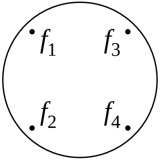
as before.

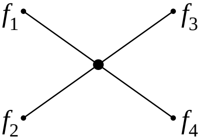
$B_{v,2,2}$ Examples





$$[v(f_1 - f_2) + v(f_3 - f_4) - 1] + v(f_1 - f_3) + v(f_2 - f_4)$$





$$v(f_1 - f_2) + v(f_1 - f_3) + v(f_2 - f_4) + v(f_3 - f_4)$$

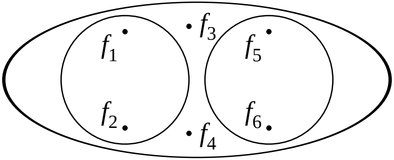
The Bound $B_{v,2,3}[f_1, \ldots, f_6]$

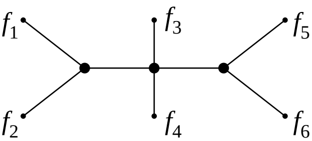
Use a sum of terms of the form $v(f_i - f_j)$, where

- Each f_i appears exactly 3 times.
- No pair $v(f_i - f_j)$ is repeated.
- If there is an edge separating $\{f_i, f_j\}$ from the others, **make sure** $v(f_i - f_j)$ appears. Also, subtract 1 as a penalty.
- If there is an edge separating $\{f_i, f_j, f_k\}$ from the others, **make sure exactly 2** of

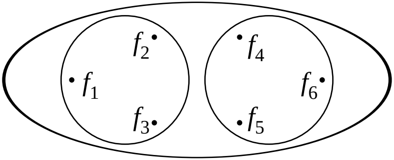
  $$v(f_i - f_j), \quad v(f_i - f_k), \quad v(f_j - f_k)$$

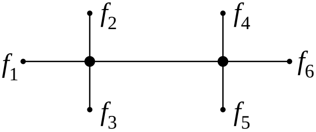
  appear, and don't subtract any penalty. (Well, usually.)
- If you subtracted more than 2 in total, change that to only 2.





$$[v(f_1 - f_2) - 1 + v(f_5 - f_6) - 1] + v(f_1 - f_3) + v(f_1 - f_5)$$
$$+ v(f_3 - f_4) + v(f_2 - f_4) + v(f_2 - f_6) + v(f_3 - f_5) + v(f_4 - f_6)$$





$$[v(f_1 - f_2) + v(f_1 - f_3) + v(f_4 - f_5) + v(f_4 - f_6)]$$
$$+ v(f_1 - f_4) + v(f_2 - f_5) + v(f_3 - f_5) + v(f_2 - f_6) + v(f_3 - f_6)$$

The Bound for General $N = k \cdot \ell$

Use a sum of terms of the form $v(f_i - f_j)$, where

- Each f_i appears exactly $m = k + \ell - 2$ times.
- No pair $v(f_i - f_j)$ is repeated.
- If there is an edge separating $2 \leq q \leq N/2$ of $f_1, \ldots, f_N$ from the others, make sure that **exactly** $Q(q)$ pairs $v(f_i - f_j)$ from those q appear, and subtract a penalty of $R(q)$, where:

q	1	2	3	4	5	6	7	8	9	10	$\cdots$
$Q(q)$	0	1	2	4	6	8	11	14	17	20	$\cdots$
$R(q)$	0	1	0	2	1	0	3	2	1	0	$\cdots$

- If the total penalty is more than M, make it only M.

Note: This bound gives more weight to large $v(f_i - f_j)$'s than to small $v(f_i - f_j)$'s.

Conjecture (N-Point *abc* for Function Fields)

Let $\ell \geq k \geq 1$ *be integers. Set* $N = k \cdot \ell$ *and*
$M = k(k-1)(3\ell - k - 1)/6$.
Then for all distinct $f_1, \ldots, f_N \in \mathbb{P}^1(K)$,

$$\sum_{v} B_{v,k,\ell}[f_1, \ldots, f_N] \leq \max\{M(2g-2), 0\}.$$

Why is this not a theorem?

Problem: What if $\omega_{k,\ell}[f_1, \ldots, f_N] = 0$?

(For $N = 4 = 2 \cdot 2$, that's the "a/c is constant" case of Mason-Stothers.)

Examples: $N = 6$ and $\omega_{2,3} = 0$

Recall the i-th row of the 6×6 matrix is $[1\ f_i\ f_i^2\ f_i^3\ \ df_i\ f_i df_i]$

Example 1. Suppose (possibly after a change of coordinates on $\mathbb{P}^1(K)$) that all of $f_1, \ldots, f_5$ are constant— i.e., $df_i = 0$.

Then the last two columns of the matrix force $\omega_{2,3} = 0$.

But I can still prove the conjecture in this case.

Examples: $N = 6$ and $\omega_{2,3} = 0$

Recall the i-th row of the 6×6 matrix is $[1\ f_i\ f_i^2\ f_i^3\ \ df_i\ f_i df_i]$

Example 2. For any $u \in K$, we have

$$\omega_{2,3}[0, \infty, -u, 1+u, u+u^2, 1+u+u^2] = 0,$$

even though no five of $\{0, \infty, -u, 1+u, u+u^2, 1+u+u^2\}$ are simultaneously constant, even after a change of coordinates on $\mathbb{P}^1(K)$.

I can still prove the conjecture in this particular case, but I don't (yet?) have a general proof for these sorts of examples.

A Possible Application.

Conjecture (Dynamical Uniform Boundedness for Function Fields)

Given $d \geq 2$, there is a constant $C = C(d, K)$ such that for any non-isotrivial morphism $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ of degree d,

$$\#\{K\text{-rational preperiodic points of } \phi\} \leq C.$$

It should be possible to use the N-point *abc*-Conjecture to imply uniform boundedness.

Example

If $g = 0$ and ϕ is a non-isotrivial polynomial of degree 2, and if

- $f_1, \ldots, f_{35} \in K$ are preperiodic,
- and $f_{36} = \infty$,

then it appears that

- $B_{v,6,6}[f_1, \ldots, f_{36}] = 0$ at all good primes, and
- $B_{v,6,6}[f_1, \ldots, f_{36}] \geq 5$ at all bad primes.