Towards Dynamical Uniform Boundedness for Rational Functions

Robert L. Benedetto
Amherst College

AMS/MAA Joint Meetings, San Francisco

Wednesday, January 13, 2010

Dynamics on $\mathbb{P}^1$

Let K be a field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

[$\deg \phi := \max\{\deg f_1, \deg f_2\}$, where $\phi = f_1/f_2$ in lowest terms.]

Definition

A point $z \in \mathbb{P}^1(\overline{K}) = \overline{K} \cup \{\infty\}$ is called **preperiodic** if

$$\phi^n(z) = \phi^m(z) \quad \text{for some } n > m \geq 0.$$

Write $\text{Preper}(\phi, K) := \{z \in \mathbb{P}^1(K) : z \text{ is preperiodic under } \phi\}$.

The Dynamical Uniform Boundedness Conjecture

From now on, K **is a global field**.

Theorem (Northcott, 1950)

Let $\phi \in K(z)$ of degree $d \geq 2$. Then

$$\#Preper(\phi, K) < \infty.$$

Conjecture (Morton & Silverman, 1994)

For any integer $d \geq 2$, there is a constant $C = C(d, K)$ such that for any $\phi \in K(z)$ of degree d,

$$\#Preper(\phi, K) \leq C(d, K).$$

Quadratic Polynomial Record Holders Over $\mathbb{Q}$

$$\phi(z) = z^2 - \frac{133}{144}. \qquad \infty \to \infty$$

$$\frac{7}{12} \to -\frac{7}{12} \to -\frac{7}{12} \qquad -\frac{19}{12} \to \frac{19}{12} \to \frac{19}{12}$$

$$\frac{1}{12} \to -\frac{11}{12} \leftrightarrows -\frac{1}{12} \leftarrow \frac{11}{12}$$

$$\phi(z) = z^2 - \frac{29}{16}. \qquad \infty \to \infty$$

$$\begin{array}{ccccccc} & & -\frac{1}{4} & \longrightarrow & -\frac{7}{4} & \longrightarrow & \frac{5}{4} & \longrightarrow & -\frac{1}{4} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \pm\frac{3}{4} & \longrightarrow & -\frac{5}{4} & & \frac{1}{4} & & \frac{7}{4} & & \end{array}$$

Cubic Polynomial Record Holders Over $\mathbb{Q}$

$$\phi(z) = -\frac{3}{2}z^3 + \frac{19}{6}z. \qquad \infty \to \infty \qquad 0 \to 0$$

$$\begin{array}{ccccccccccccc}
\frac{1}{3} & \to & 1 & \to & \frac{5}{3} & \leftrightarrows & -\frac{5}{3} & \leftarrow & -1 & \leftarrow & -\frac{1}{3} \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & \frac{4}{3} & \to & \frac{2}{3} & & -\frac{2}{3} & \leftarrow & -\frac{4}{3} & &
\end{array}$$

$$\phi(z) = -\frac{3}{2}z^3 + \frac{73}{24}z. \qquad \infty \to \infty \qquad 0 \to 0$$

$$\frac{7}{6} \to \frac{7}{6} \qquad\qquad -\frac{7}{6} \to -\frac{7}{6}$$

$$\begin{array}{ccccccccccc}
-\frac{3}{2} & \to & \frac{1}{2} & \leftrightarrows & \frac{4}{3} & \qquad & \frac{3}{2} & \to & -\frac{1}{2} & \leftrightarrows & -\frac{4}{3} \\
 & & \uparrow & & & & & & \uparrow & & \\
 & & \frac{1}{6} & & & & & & -\frac{1}{6} & &
\end{array}$$

Good Reduction

Definition

Let $v \in M_K$ be a non-archimedean place of K, and let $\phi \in K(z)$ a rational function.
We say ϕ has **good reduction** at v if ϕ may be written as

$$\phi\left(\frac{x}{y}\right) = \frac{F_1(x,y)}{F_2(x,y)}$$

with $F_1, F_2 \in \mathcal{O}_v[x,y]$ homogeneous of the same degree such that **the reductions $\overline{F}_1$ and $\overline{F}_2$ modulo v have no common zeros in $\overline{k}_v \times \overline{k}_v$ besides $(0,0)$.**

Definition

ϕ has **potentially good reduction** at v if $\theta \circ \phi \circ \theta^{-1}$ has good reduction at some place $w|v$, for some $\theta \in \mathrm{PGL}(2, \overline{K})$.
Otherwise, ϕ has **bad reduction** at v.

Strong Non-Uniform Bounds for Polynomials

Theorem (RB, 2004)

Let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$. Let s be the number of bad places of ϕ (including archimedean places). Then

$$\#\mathit{Preper}(\phi, K) \leq O\left(\frac{d^2}{\log d} \cdot s \log s\right).$$

For number fields, the big-Oh constant depends only on $[K : \mathbb{Q}]$. For function fields, the big-Oh constant is essentially 1.

Filled Julia Sets

Denote by $\mathbb{C}_v$ the completion of an algebraic closure of $\mathbb{Q}_v$.

Definition

The v-adic *filled Julia set* of a polynomial $\phi \in \mathbb{C}_v[z]$ of degree $d \geq 2$ is

$$\mathcal{K}_{\phi,v} = \{z \in \mathbb{C}_v : \{|\phi^n(z)|_v\}_{n\geq 0} \text{ is bounded}\}.$$

Note:

- All preperiodic points (besides ∞) lie in $\mathcal{K}_{\phi,v}$.
- If ϕ is monic, then the diameter $r_{\phi,v}$ of $\mathcal{K}_{\phi,v}$ satisfies $r_{\phi,v} \geq 1$, with equality **iff** ϕ is potentially good at v.

Sketch of Proof of Theorem

(For ease, we omit archimedean fudge factors and also assume ϕ is monic.)

- At each $v \in M_K$, given $z_1, \ldots, z_N \in \mathcal{K}_{\phi,v}$, prove **Lemma 1**:

$$-\sum_{i \neq j} \log |z_i - z_j|_v \geq -(d-1)(N \log_d N) \log r_{\phi,v}.$$

- At a place $w \in M_K$ maximizing $r_{w,\phi}$, partition $\mathcal{K}_{\phi,v} = U \sqcup V$. Given $z_1, \ldots, z_N \in U$, prove **Lemma 2**, that

$$-\sum_{i \neq j} \log |z_i - z_j|_w \geq \left(\frac{N^2}{d-1} - (d-1)(N \log_d N) \right) \log r_{\phi,w},$$

  (And similarly for $z_1, \ldots, z_N \in V$.)

Sum the bounds across all $v \in M_K$ and get $0 > 0$ if N is too big.

Proving the Lemmas: Lower Bounds for $-\sum_{i,j} \log |z_i - z_j|_v$

Key idea: if $\overline{D}(a, r_{\phi,v})$ is the smallest disk containing $\mathcal{K}_{\phi,v}$, then

$$|\phi^i(z) - a|_v \leq r_{\phi,v}$$

for all $i \geq 0$ and all $z \in \mathcal{K}_{\phi,v}$.

So we can write down monic polynomials

$$f_j(z) := \prod_{i=0}^{M} [\phi^i(z) - a]^{c_i}$$

of arbitrary degree $j \geq 0$ so that

$$\sup\{|f_j(z)|_v : z \in \mathcal{K}_{\phi,v}\}$$

is surprisingly small.

Completing the Proofs of the Lemmas

$\prod_{i \neq j}(z_i - z_j) = \pm(\det V)^2$, where V is the Vandermonde matrix

$$V = \begin{bmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{N-1} \\ 1 & z_2 & z_2^2 & \dots & z_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_N & z_N^2 & \dots & z_N^{N-1} \end{bmatrix}.$$

Since each f_j is monic, by column operations, $\det V = \det A$, where

$$A = \begin{bmatrix} 1 & f_1(z_1) & f_2(z_1) & \dots & f_{N-1}(z_1) \\ 1 & f_1(z_2) & f_2(z_2) & \dots & f_{N-1}(z_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_1(z_N) & f_2(z_N) & \dots & f_{N-1}(z_N) \end{bmatrix}.$$

Hadamard's inequality bounds $|\det A|_v$ above by the product of the norms of the columns.

Example: A Rational Function

$$\phi(z) = \frac{7z}{24} - \frac{7}{6z} = \frac{7(z^2-4)}{24z}. \qquad \text{Bad places: } v = 2, 3, 7, \infty.$$

---

12 $\mathbb{Q}$-rational preperiodic points:

$$\pm 2 \to 0 \to \infty \to \infty$$

$$\begin{array}{ccccccccc}
\frac{14}{5} & \longrightarrow & \frac{2}{5} & \longrightarrow & -\frac{14}{5} & \longrightarrow & -\frac{2}{5} & \longrightarrow & \frac{14}{5} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
10 & & -\frac{10}{7} & & -10 & & \frac{10}{7} & &
\end{array}$$

Some Notation for Homogeneous Coordinates

- Given $(x, y) \in \mathbb{C}_v^2$, define

$$\|(x,y)\|_v := \begin{cases} \max\{|x|_v, |y|_v\}, & \text{if } v \text{ is non-archimedean,} \\ (|x|_v^2 + |y|_v^2)^{1/2}, & \text{if } v \text{ is archimedean.} \end{cases}$$

- Given $(x_1, y_1), (x_2, y_2) \in \mathbb{C}_v^2$, recall their exterior product is

$$(x_1, y_1) \wedge (x_2, y_2) := x_1 y_2 - x_2 y_1.$$

Note that $\dfrac{x_1}{y_1} - \dfrac{x_2}{y_2} = \dfrac{(x_1, y_1) \wedge (x_2, y_2)}{y_1 y_2}$.

Baker and Rumely's Arakelov-Green Function

Definition

Given $\phi(z) \in \mathbb{C}_v(z)$ of degree $d \geq 2$, write $\phi\left(\frac{x}{y}\right) = \frac{F_1(x,y)}{F_2(x,y)}$, where $F_1, F_2 \in \mathbb{C}_v[x,y]$ are homogeneous polynomials of degree d.
Define $\Phi : \mathbb{C}_v^2 \to \mathbb{C}_v^2$ by $\Phi(x,y) := (F_1(x,y), F_2(x,y))$.
Given $z_1 = [x_1, y_1], z_2 = [x_2, y_2] \in \mathbb{P}^1(\mathbb{C}_v)$, lift them to $P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in \mathbb{C}_v^2$. Define

$$g_{\phi,v}(z_1, z_2) := \\ -\log|P_1 \wedge P_2|_v + \hat{\Lambda}_{\Phi,v}(P_1) + \hat{\Lambda}_{\Phi,v}(P_2) - \frac{\log|\mathrm{Res}(F_1, F_2)|_v}{d(d-1)},$$

where

$$\hat{\Lambda}_{\Phi,v}(x,y) := \lim_{n\to\infty} d^{-n} \log\|\Phi^n(x,y)\|_v.$$

Properties of $g_{\phi,v}$

$$\frac{-\log|P_1 \wedge P_2|_v + \hat{\Lambda}_{\Phi,v}(P_1) + \hat{\Lambda}_{\Phi,v}(P_2) - \frac{\log|\mathrm{Res}(F_1,F_2)|_v}{d(d-1)}}{}$$

- $g_{\phi,v}(z_1,z_2)$ is well-defined (i.e., independent of choices of Φ, P_1, P_2.)
- $g_{\phi,v}(z_1,z_2)$ is coordinate-independent: if $\psi = \theta \circ \phi \circ \theta^{-1}$ for some $\theta \in \mathrm{PGL}(2,\mathbb{C}_v)$, then $g_{\psi,v}(\theta(z_1),\theta(z_2)) = g_{\phi,v}(z_1,z_2)$.
- If ϕ has good reduction, then $g_{\phi,v} = -\log \Delta_v$, where $\Delta_v(z_1,z_2)$ is the spherical distance on $\mathbb{P}^1(\mathbb{C}_v)$.
- If ϕ is a monic polynomial and $z_1, z_2 \in \mathcal{K}_{\phi,v}$, then $g_{\phi,v}(z_1,z_2) = -\log|z_1 - z_2|_v$.
- $g_{\phi,v}$ extends continuously to the Berkovich space $\mathbb{P}^1_{\mathrm{Berk}}$.

Lemma 1 for Rational Functions

Given $\phi \in \mathbb{C}_v(z)$, define

$$\rho_{\phi,v} := \inf\{g_{\phi,v}(z_1, z_2) : z_1, z_2 \in \mathbb{P}^1(\mathbb{C}_v)\}.$$

(For monic polynomials, $\rho_{\phi,v} = -\log r_{\phi,v}$.)

Proposition (Baker, 2005)

Given $\phi \in \mathbb{C}_v(z)$, we have $\rho_{\phi,v} \leq 0$, with equality **iff** *ϕ has potentially good reduction.*

Theorem (Baker, 2005)

Given $d \geq 2$, there is a constant C_d so that for any $\phi \in \mathbb{C}_v(z)$ of degree d and $z_1, \dots, z_N \in \mathbb{P}^1(\mathbb{C}_v)$,

$$\sum_{i \neq j} g_{\phi,v}(z_i, z_j) \geq (C_d N \log N)\rho_{\phi,v}.$$

Idea of Baker's Proof

Essentially the same as before, only homogeneous:

Start with the homogeneous Vandermonde matrix

$$V = \begin{bmatrix} y_1^{N-1} & x_1 y_1^{N-2} & x_1^2 y_1^{N-3} & \cdots & x_1^{N-1} \\ y_2^{N-1} & x_2 y_2^{N-2} & x_2^2 y_2^{N-3} & \cdots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_N^{N-1} & x_N y_N^{N-2} & x_N^2 y_N^{N-3} & \cdots & x_N^{N-1} \end{bmatrix}$$

and change the (i, j)-entry to $\prod_k \left(\Phi^k(P_i) \wedge A\right)^{a_k} \left(\Phi^k(P_i) \wedge B\right)^{b_k}$.

(Choose A, B, a_k, b_k appropriately, and compute the determinant of the column-reduction matrix carefully.)

What about Lemma 2?

It looks feasible to partition $\mathbb{P}^1$ into two pieces in a manner similar to the polynomial strategy.

For example, choose $A, B \in \mathbb{P}^1(\mathbb{C}_v)$ carefully (e.g., to minimize $g_{\phi,v}(A, B)$). Then define

$$U = \{z \in \mathbb{P}^1(\mathbb{C}_v) : g_{\phi,v}(z, A) \geq g_{\phi,v}(z, B)\},$$

and $V = \mathbb{P}^1(\mathbb{C}_v) \smallsetminus U$.

BUT there are still obstacles to finishing the proof.

A Taste of Berkovich Space

Recall $\rho_{\phi,v} := \inf\{g_{\phi,v}(z_1, z_2) : z_1, z_2 \in \mathbb{P}^1(\mathbb{C}_v)\}$.

It is also useful to define:

$$\sigma_{\phi,v} := \inf\{g_{\phi,v}(\zeta,\zeta) : \zeta \in \mathbb{P}^1_{\mathrm{Berk}}\}.$$

Proposition (Baker, 2005— continued)

Given v non-archimedean and $\phi \in \mathbb{C}_v(z)$, we have

$$\rho_{\phi,v} \le 0, \quad \textit{and} \quad \sigma_{\phi,v} \ge 0.$$

Moreover, the following are equivalent:

1. *ϕ has potentially good reduction.*
2. $\rho_{\phi,v} = 0$.
3. $\sigma_{\phi,v} = 0$.

The Major Obstacle

(Recall

$$\rho_{\phi,v} = \inf\{g_{\phi,v}(z_1, z_2) : z_1, z_2 \in \mathbb{P}^1(\mathbb{C}_v)\} \leq 0$$
$$\sigma_{\phi,v} = \inf\{g_{\phi,v}(\zeta, \zeta) : \zeta \in \mathbb{P}^1_{\mathrm{Berk}}\} \geq 0,$$

and $\rho_{\phi,v} = 0$ **iff** $\sigma_{\phi,v} = 0$.)

To prove Lemma 2, we seem to need:

Conjecture

Given $d \geq 2$, there is a constant $C_d > 0$ with the following property.
Given v non-archimedean and $\phi \in \mathbb{C}_v(z)$ of degree d,

$$\sigma_{\phi,v} \geq -C_d \cdot \rho_{\phi,v}.$$

(Guess: $C_d = \frac{1}{d}$.)