Polyhedra: Plato, Archimedes, Euler

Robert L. Benedetto
Department of Mathematics and Statistics

Amherst College

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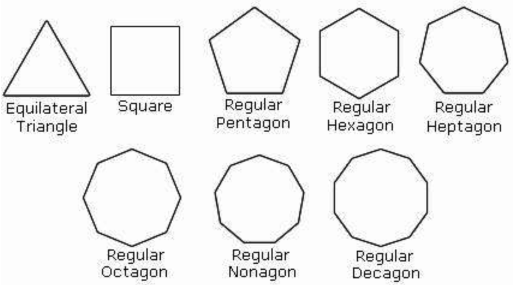
Regular Polygons

Definition

A **polygon** is a planar region R bounded by a finite number of straight line segments that together form a loop.

If all the line segments are congruent, and all the angles between the line segments are also congruent, we say R is a **regular polygon**.

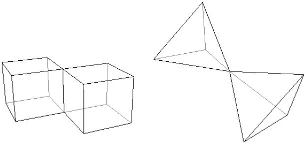


Polyhedra

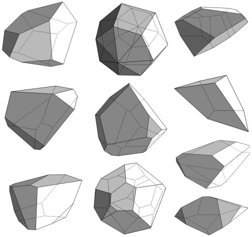
Definition

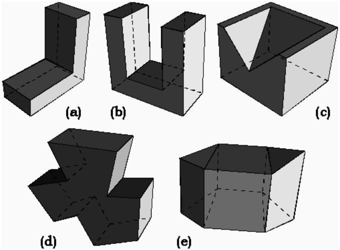
A **polyhedron** is a spatial region S bounded by a finite number of polygons meeting along their edges, and so that the interior is a single connected piece.

So these are not polyhedra:



But these are:





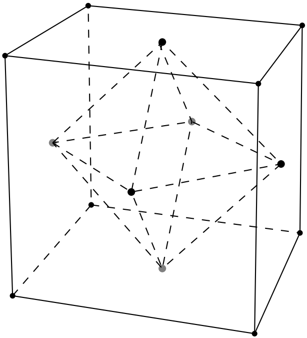
Regular Polyhedra

Definition

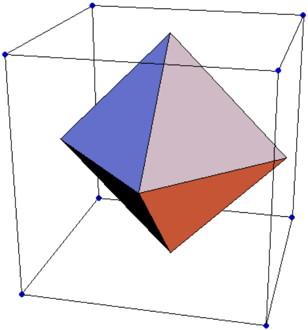
Let S be a polyhedron. If

- all the faces of S are regular polygons,
- all the faces of S are congruent to each other, and
- all the vertices of S are congruent to each other,

we say S is a **regular polyhedron**.

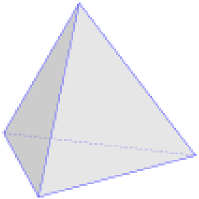


Cube (Regular Hexahedron)

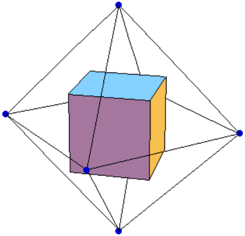


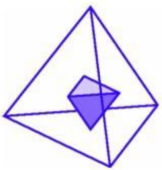
Regular Octahedron

Another Regular Polyhedron

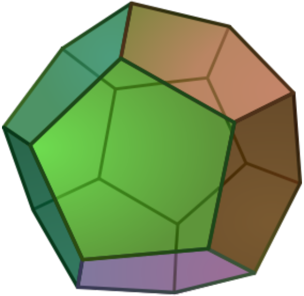


Regular Tetrahedron

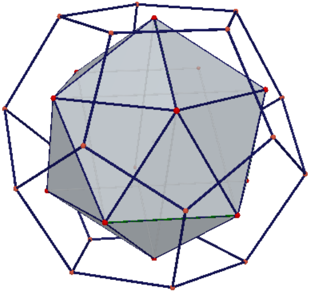




Two More Regular Polyhedra

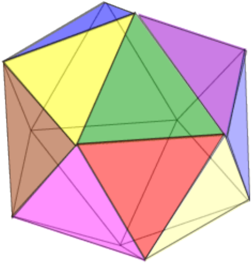


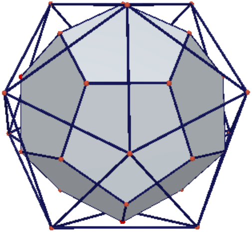
Regular Dodecahedron



Regular Icosahedron

And They're Dual:





Some Ancient History (of Regular Polyhedra)

- **Pythagoras of Samos**, c.570–c.495 BCE: Knew about at least three, and possibly all five, of these regular polyhedra.
- **Theaetetus** (Athens), 417–369 BCE: Proved that there are exactly five regular polyhedra.
- **Plato** (Athens), c.426–c.347 BCE: Theorizes four of the solids correspond to the four elements, and the fifth (dodecahedron) to the universe/ether.
- **Euclid** (Alexandria), 3xx–2xx BCE: Book XIII of *The Elements* discusses the five regular polyhedra, and gives a proof (presumably from Theaetetus) that they are the only five.

The five regular polyhedra are often called the **Platonic solids**.

There Can Be Only Five: Setup

Let S be a regular polyhedron. Define integers n, k by:

- $\boxed{\text{the faces of } S \text{ are regular } n\text{-gons}}$
- $\boxed{k \text{ faces meet at each vertex}}$

---

1. $n \geq 3$ and $k \geq 3$.
2. S is completely determined by the numbers n and k.
3. S must be **convex**: For any two points in S, the whole line segment between the two points is contained in S.
4. Since S is convex, the total of angles meeting at a vertex of S is less than 360 degrees.
5. Each angle of a regular n-gon has measure $\left(\frac{180(n-2)}{n}\right)^{\circ}$.

   [Triangle: 60°, Square: 90°, Pentagon: 108°, Hexagon: 120°.]

There Can Be Only Five: Payoff Part 1

Faces of S are regular n-gons and k faces meet at each vertex

$n, k \geq 3$, **but** total angles at each vertex must be **less** than $360°$.

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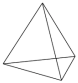
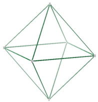
n = 3 (triangles)

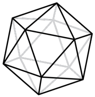
$k = 3$ triangles per vertex: **Tetrahedron**



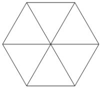
$k = 4$ triangles per vertex: **Octahedron**



$k = 5$ triangles per vertex: **Icosahedron**



$k \geq 6$ triangles per vertex: **NO! Angles** $\geq 360°$



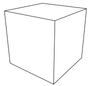
There Can Be Only Five: Payoff Part 2

$\boxed{\text{Faces of } S \text{ are regular } n\text{-gons}}$ and $\boxed{k \text{ faces meet at each vertex}}$

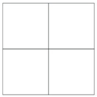
$n, k \geq 3$, **but** total angles at each vertex must be **less** than $360°$.

$\boxed{\mathbf{n = 4 \ (squares)}}$

$k = 3$ squares per vertex: **Cube**



$k \geq 4$ squares per vertex: **NO! Angles** $\geq 360°$

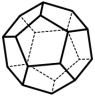


There Can Be Only Five: Payoff Part 3

$\boxed{\text{Faces of } S \text{ are regular } n\text{-gons}}$ and $\boxed{k \text{ faces meet at each vertex}}$

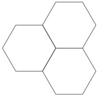
$n, k \geq 3$, **but** total angles at each vertex must be **less** than $360°$.

$\boxed{\mathbf{n = 5 \ (pentagons)}}$

$k = 3$ pentagons per vertex: **Dodecahedron** 

$k \geq 4$ pentagons per vertex: **NO! Angles** $> 360°$

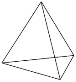
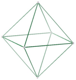
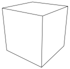
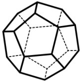
$\boxed{\mathbf{n \geq 6 \ (hexagons\ and\ more)}}$

$k \geq 3$ n-gons per vertex: **NO! Angles** $\geq 360°$ 

There Can Be Only Five: Summary of the Proof

Faces of S are regular n-gons and k faces meet at each vertex

$n, k \geq 3$, **but** total angles at each vertex must be **less** than 360°.

n / k	3	4	5	≥ 6
3				—
4		—	—	—
5		—	—	—
≥ 6	—	—	—	—

The Five Regular Polyhedra

Faces of S are regular n-gons and k faces meet at each vertex

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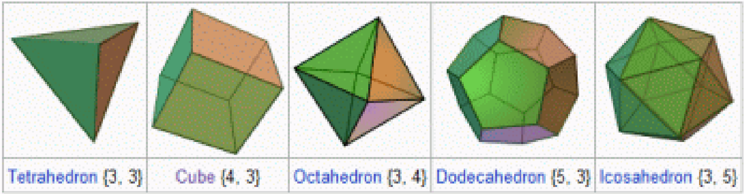
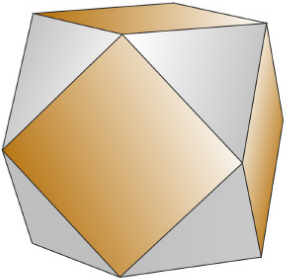
Possible choices for (n, k):



But What About This Solid?



Cuboctahedron

- Known to Plato
- Definitely not regular (squares **and** triangles), but:
- All faces are regular polygons, and
- All the vertices are congruent to each other.

Semiregular Polyhedra

Definition

Let S be a polyhedron. If

- all the faces of S are regular polygons, and
- all the vertices of S are congruent to each other,

we say S is a **semiregular polyhedron**.

[Same as definition of regular polyhedron, except the faces don't need to be congruent to each other.]

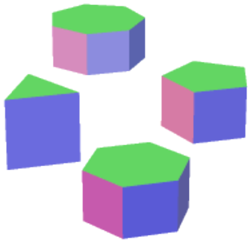
Question: How many (non-regular) semiregular polyhedra are there?

Answer: Infinitely many.

Prisms

Definition

Let $n \geq 3$. An n-**gonal prism** is the solid obtained by connecting two congruent regular n-gons by a belt of n squares.



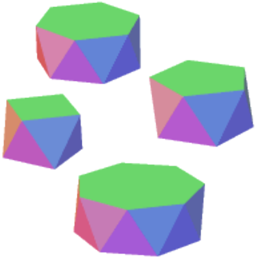
Modified Question: How many (non-regular) semiregular polyhedra are there **besides** the prisms?

Answer: Still infinitely many.

Antiprisms

Definition

Let $n \geq 3$. An *n*-**gonal antiprism** is the solid obtained by connecting two congruent regular *n*-gons by a belt of $2n$ equilateral triangles.



Re-Modified Question: How many semiregular polyhedra are there besides the regular polyhedra, prisms, **and** antiprisms?

Answer: FINALLY that's the right question. Archimedes says: 13.

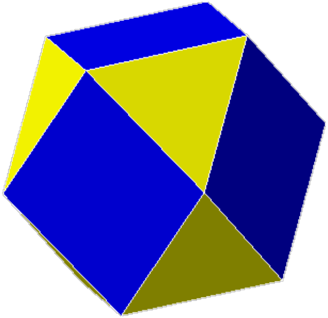
Archimedean Solids

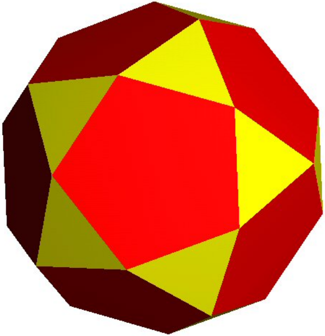
Definition

An **Archimedean solid** is a semiregular polyhedron that is not regular, a prism, or an antiprism.

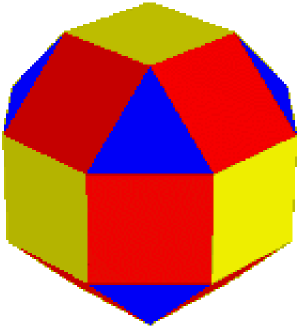
- **Archimedes of Syracuse**, 287–212 BCE: Among his *many* mathematical contributions, described the 13 Archimedean solids. But this work is lost. We know of it only through:
- **Pappus of Alexandria**, c.290–c.350 CE: One of the last ancient "Greek" mathematicians. Describes the 13 Archimedean solids in Book V of his *Collections*. **But he gives no proof!**
-  **Johannes Kepler** (Germany/Austria), 1571–1630 CE: Rediscovered the 13 Archimedean solids. In his book *Harmonices Mundi* (1619), he gave the first surviving proof that there are only 13.

Cuboctahedron and Icosidodecahedron

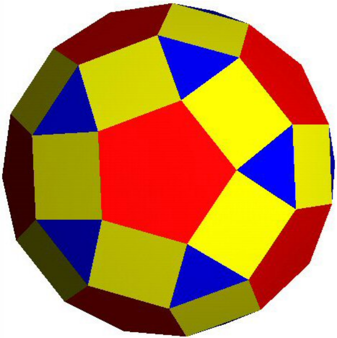




Adding Some Squares



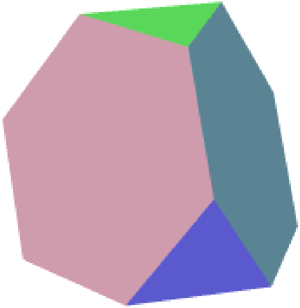
Rhombicuboctahedron



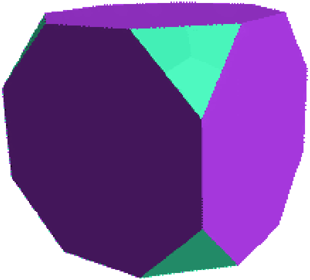
Rhombicosidodecahedron

Truncating Regular Polyhedra

To **truncate** a polyhedron means to slice off its corners.

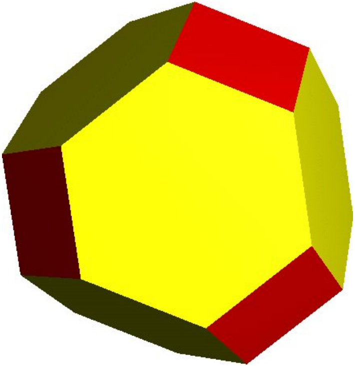


Truncated Tetrahedron

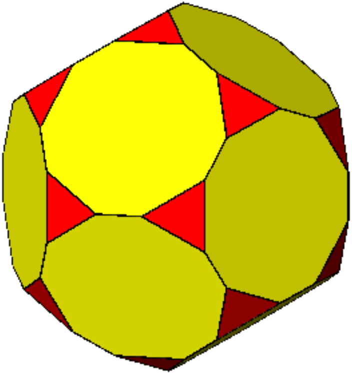


Truncated Cube

More Truncated Regular Polyhedra

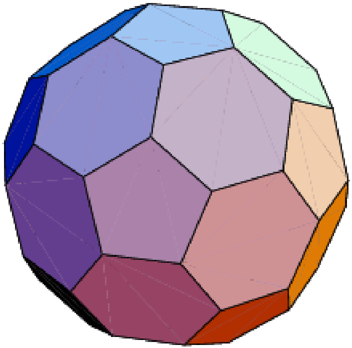


Truncated Octahedron



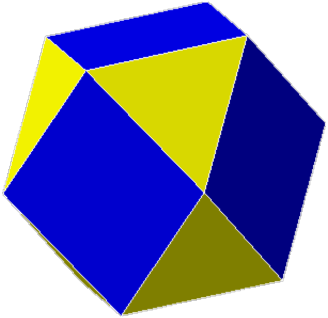
Truncated Dodecahedron

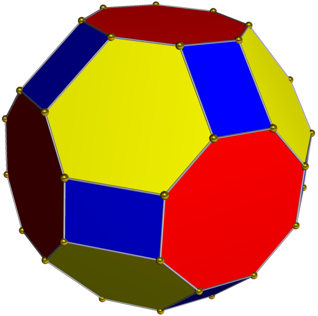
Truncated Icosahedron



Truncation FAIL

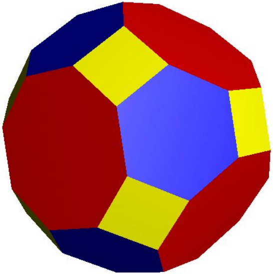
If you truncate, say, the cuboctahedron, you don't quite get regular polygons — the sliced corners give **non-square** rectangles.

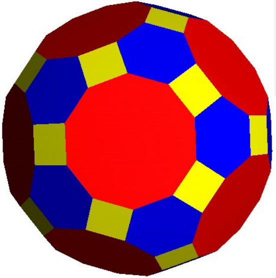




Fixing a Failed Truncation

But if you squish the rectangles into squares, you can get regular polygons all around. Same deal for truncating the icosidodecahedron:





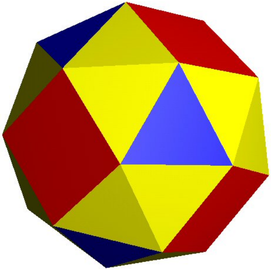
Truncated Cuboctahedron **Truncated Icosidodecahedron**

a.k.a.

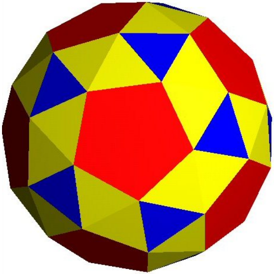
Great Rhombicuboctahedron, Great Rhombicosidodecahedron

And Two More

Adding more triangles to the cuboctahedron (or cube) and to the icosidodecahedron (or dodecahedron) gives:



Snub Cube



Snub Dodecahedron

Note: Unlike all the others, these two are **not** mirror-symmetric.

Why Are There Only Thirteen?

Kepler's proof for Archimedean solids is similar in spirit to Theaetetus' proof for Platonic solids, but of course it's longer and more complicated.

I'll give a sketch based on the description of Kepler's proof in Chapter 4 of *Polyhedra*, by Peter Cromwell. (Cambridge U Press, 1997).

Goal: Given a semiregular polyhedron S, we want to show the arrangement of polygons around each vertex agrees with one of the specific examples we already know about.

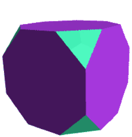
Some Notation

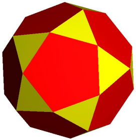
To describe the arrangement of polygons at a vertex, let's write

$$[a, b, c, \ldots]$$

to indicate that there's an a-gon, then a b-gon, then a c-gon, etc., as we go around the vertex.

Example: The truncated cube is $[8, 8, 3]$, and the icosidodecahedron is $[3, 5, 3, 5]$.





Warning: the order matters, but only up to rotating/reflecting. So

$$[8, 8, 3] = [8, 3, 8] = [3, 8, 8]$$

and

$$[3, 5, 3, 5] = [5, 3, 5, 3] \neq [3, 3, 5, 5].$$

Three Lemmas

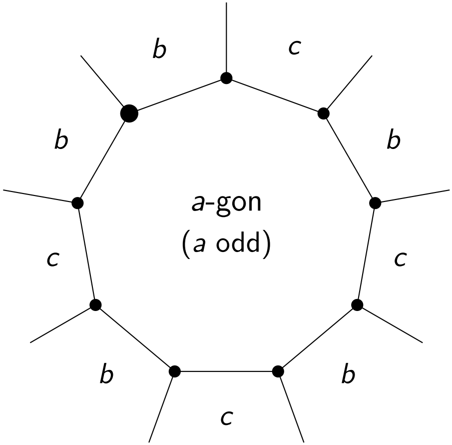
Lemma 1. Suppose $[a, b, c]$ is an arrangement for a semiregular polyhedron. If a is odd, then $b = c$.

Lemma 2. Suppose $[3, 3, a, b]$ is an arrangement for a semiregular polyhedron. Then either $a = 3$ or $b = 3$. (Antiprism.)

Lemma 3. Suppose $[3, a, b, c]$ is an arrangement for a semiregular polyhedron. If $a, c \neq 3$, then $a = c$.

Proof of Lemma 1

Lemma 1. Suppose $[a, b, c]$ is an arrangement for a semiregular polyhedron. If a is odd, then $b = c$.



Reminder of the Three Lemmas

Lemma 1. Suppose $[a, b, c]$ is an arrangement for a semiregular polyhedron. If a is odd, then $b = c$.

Lemma 2. Suppose $[3, 3, a, b]$ is an arrangement for a semiregular polyhedron. Then either $a = 3$ or $b = 3$. (Antiprism.)

Lemma 3. Suppose $[3, a, b, c]$ is an arrangement for a semiregular polyhedron. If $a, c \neq 3$, then $a = c$.

Sketch of the Proof

There are now a whole lot of cases to consider, depending on what sorts of polygonal faces the solid S has:

1. 2 sorts: Triangles and Squares.
2. 2 sorts: Triangles and Pentagons.
3. 2 sorts: Triangles and Hexagons.
4. 2 sorts: Triangles and n-gons, with $n \geq 7$.
5. 2 sorts: Squares and n-gons, with $n \geq 5$.
6. 2 sorts: Pentagons and n-gons, with $n \geq 6$.
7. 3 sorts: Triangles, Squares, and n-gons, with $n \geq 5$.
8. 3 sorts: Triangles, m-gons, and n-gons, with $n > m \geq 5$.
9. 3 sorts: ℓ-gons, m-gons, and n-gons, with $n > m > \ell \geq 4$.
10. 2 or 3 sorts, all with ≥ 6 sides each.
11. ≥ 4 sorts of polygons.

And most of these cases have multiple sub-cases.

Example: Case 2: Triangles and Pentagons as faces

One pentagon at each vertex:

- $[3, 3, 5]$: Impossible by Lemma 1: 3 odd, $3 \neq 5$.
- $[3, 3, 3, 5]$: Pentagonal Antiprism
- $[3, 3, 3, 3, 5]$: Snub Dodecahedron
- ≥ 5 triangles and 1 pentagon: **NO**; angles total $> 360°$.

Two pentagons at each vertex:

- $[3, 5, 5] = [5, 5, 3]$: Impossible by Lemma 1: 5 odd, $5 \neq 3$.
- $[3, 3, 5, 5]$: Impossible by Lemma 2: $5, 5 \neq 3$
- $[3, 5, 3, 5]$: Isocidodecahedron
- ≥ 3 triangles and 2 pentagons: **NO**; angles total $> 360°$.

≥ 3 **pentagons at each vertex**:

- ≥ 1 triangle(s) and ≥ 3 pentagons: **NO**; angles total $> 360°$.

Reminder: The Outline of the Proof

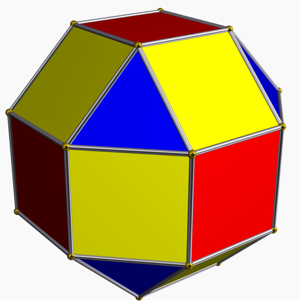
Sorts of polygonal faces the solid S has:

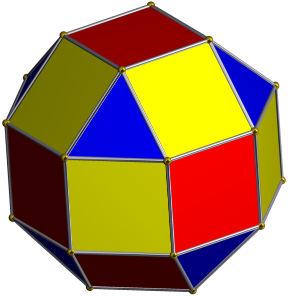
1. 2 sorts: Triangles and Squares.
2. 2 sorts: Triangles and Pentagons.
3. 2 sorts: Triangles and Hexagons.
4. 2 sorts: Triangles and n-gons, with $n \geq 7$.
5. 2 sorts: Squares and n-gons, with $n \geq 5$.
6. 2 sorts: Pentagons and n-gons, with $n \geq 6$.
7. 3 sorts: Triangles, Squares, and n-gons, with $n \geq 5$.
8. 3 sorts: Triangles, m-gons, and n-gons, with $n > m \geq 5$.
9. 3 sorts: ℓ-gons, m-gons, and n-gons, with $n > m > \ell \geq 4$.
10. 2 or 3 sorts, all with ≥ 6 sides each.
11. ≥ 4 sorts of polygons.

Each case is of the same general flavor as "Case 2" that we just did.

A Twist

OK, so the proof is done. But what about this:

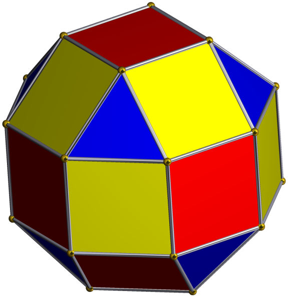




This new solid has only squares and triangles for faces, **and** each vertex has 3 squares and 1 triangle, with the same set of angles between them.

It is called the **Elongated Square Gyrobicupola** or **Pseudorhombicuboctahedron**.

The Pseudorhombicuboctahedron



First known appearance in print: Duncan Sommerville, 1905.

Rediscovered by J.C.P. Miller, 1930. (Sometimes called "Miller's solid").

But it's **not** usually considered an Archimedean solid.

Two Questions

1. Why did Kepler's proof miss the PRCOH?

A: Unlike regular polyhedra, a convex polyhedron with the same arrangement of regular polygons at each vertex is **not** completely determined by the arrangement of polygons around each vertex.

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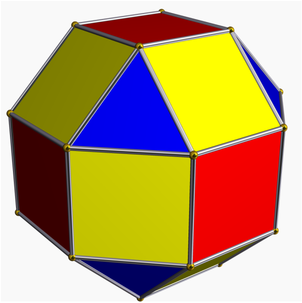
2. So why isn't the PRCOH considered semiregular?

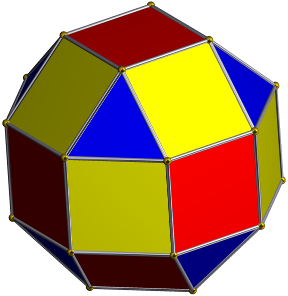
A: Because we (like the ancients) were vague about what "all vertices are congruent" means.

- Is it **just** the faces meeting at the vertex that look the same?
- Or is that the **whole solid** looks the same if you move one vertex to where another one was?

Although terminology varies, there is general agreement that the nice class (usually called either "semiregular" or "uniform") should use the "whole solid" definition. So the PRCOH is **not** an Archimedean solid.

Another look at the RCOH and PRCOH





Counting Vertices, Edges, and Faces

Let's count how many vertices, edges, and faces these solids have:

	V	E	F			V	E	F
TH	4	6	4		T.TH	12	18	8
Cu	8	12	6		T.Cu	24	36	14
OH	6	12	8		T.OH	24	36	14
DH	20	30	12		T.DH	60	90	32
IH	12	30	20		T.IH	60	90	32
COH	12	24	14		IDH	30	60	32
RCOH	24	48	26		RIDH	60	120	62
T.COH	48	72	26		T.IDH	120	180	62
S.Cu	24	60	38		S.DH	60	150	92

	V	E	F			V	E	F
n-Pr	$2n$	$3n$	$n+2$		n-APr	$2n$	$4n$	$2n+2$

Euler observes: $V - E + F = 2$.

Euler's Polyhedron Formula

Leonhard Euler (Switzerland/Russia/Germany), 1707–1783 CE: Among many, many, **MANY** other things, showed (1750):

Theorem

Let S be a convex polyhedron, with V vertices, E edges, and F faces. Then $V - E + F = 2$.

Key idea of proof: If you change a polyhedron by:

- adding a vertex somewhere in the middle of an edge,
- cutting a face in two by connecting two nonadjacent vertices with a new edge,
- reversing either of the above two kinds of operations, or
- bending or stretching it,

the quantity $V - E + F$ remains unchanged.

Regular Polyhedra Revisited

Let S be a regular polyhedron with m faces, and with k regular n-gons meeting at each vertex. Then:

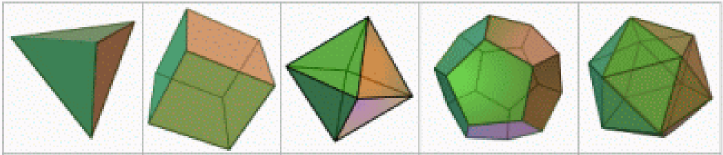
$$V = \frac{mn}{k}, \quad E = \frac{mn}{2}, \quad F = m.$$

So

$$2 = V - E + F = \frac{mn}{k} - \frac{mn}{2} + m = mn\left(\frac{1}{k} - \frac{1}{2} + \frac{1}{n}\right).$$

In particular, $\frac{1}{k} - \frac{1}{2} + \frac{1}{n} > 0$, i.e., $\frac{1}{n} + \frac{1}{k} > \frac{1}{2}$.

And it's not hard to show that the only pairs of integers (n, k) with $\boxed{n, k \geq 3}$ and $\boxed{\frac{1}{n} + \frac{1}{k} > \frac{1}{2}}$ are:



Tetrahedron {3, 3} | Cube {4, 3} | Octahedron {3, 4} | Dodecahedron {5, 3} | Icosahedron {3, 5}