Equilibrium measures on the Berkovich projective line

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Weil heights over number fields

The **(standard) Weil height** on $\mathbb{Q}$ is

$$h\left(\frac{a}{b}\right) := \log\max\{|a|, |b|\} \quad \text{for } a, b \in \mathbb{Z} \text{ relatively prime.}$$

Equivalently, $h(x) = \sum_{v \in M_{\mathbb{Q}}} \log\max\{|x|_v, 1\}$, so we can generalize to

$$h(x) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log\max\{|x|_v, 1\} \quad \text{for } x \in \overline{\mathbb{Q}}$$

for $x \in K$ with $[K:\mathbb{Q}] < \infty$.

Note: For any $\phi \in \overline{\mathbb{Q}}(z)$ with $\deg(\phi) = d \geq 1$, we have

$$h(\phi(x)) = d \cdot h(x) + O_{\phi}(1).$$

Canonical heights

Let $\phi \in \overline{\mathbb{Q}}(z)$ with $\deg(\phi) = d \geq 2$. The **canonical height** for ϕ is

$$\hat{h}_\phi(x) := \lim_{n\to\infty} d^{-n} h\big(\phi^n(x)\big).$$

Facts: for all $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$:

- $\hat{h}_\phi(x) = h(x) + O_\phi(1)$
- $\hat{h}_\phi(\phi(x)) = d \cdot \hat{h}_\phi(x)$
- $\hat{h}_\phi(x) = 0 \iff x$ is preperiodic under ϕ

$\hat{h}_\phi$ can be decomposed as a sum of **local canonical heights**:

$$\hat{h}_\phi(x) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K} [K_v : \mathbb{Q}_v] \hat{\lambda}_{\phi,v}(x) \quad \text{for } x \in \overline{\mathbb{Q}}$$

for $x \in K$ with $[K:\mathbb{Q}] < \infty$.

Points of small canonical height

Recall $\hat{h}_\phi(x) = 0 \iff x$ is preperiodic. What if $\hat{h}_\phi(x) > 0$ is small?

Example. $\phi(z) = -\frac{25}{24}z^3 + \frac{97}{24}z + 1$ with $x = -\frac{7}{5}$:

$$-\frac{7}{5} \mapsto -\frac{9}{5} \mapsto -\frac{1}{5} \mapsto \frac{1}{5} \mapsto \frac{9}{5} \mapsto \frac{11}{5} \mapsto -\frac{6}{5} \mapsto -\frac{41}{20} \mapsto \frac{4323}{2560} \mapsto \cdots$$

has $\hat{h}_\phi(-7) = 0.0011\ldots$

Example. $\phi(z) = \frac{7z^2 - 15z + 2}{49z^2 - 39z + 2}$ with $x = \infty$:

$$\infty \mapsto \frac{1}{7} \mapsto 0 \mapsto 1 \mapsto -\frac{1}{2} \mapsto \frac{1}{3} \mapsto \frac{2}{5} \mapsto \frac{1}{2} \mapsto \frac{5}{7} \mapsto 6$$
$$\mapsto \frac{41}{383} \mapsto -\frac{2320}{7889} \mapsto \frac{3465767}{8746087} \mapsto \cdots$$

has $\hat{h}_\phi(\infty) = 0.003606\ldots$

A brief review of some measure theory

Let X be a topological space (think $\mathbb{P}^1_{\mathrm{an}}$ or $\mathbb{P}^1(\mathbb{C})$).

- The **Borel σ-algebra** $\mathcal{B}$ on X is the set of all subsets of X generated from open sets by allowing complements and **countable** unions.
- A **signed (Borel) measure** μ on X is a function $\mu : \mathcal{B} \to \mathbb{R}$ such that
  - $\mu(\varnothing) = 0$, and
  - $\mu(E_1 \cup E_2 \cup \cdots) = \mu(E_1) + \mu(E_2) + \cdots$ for any countable set $\{E_n\}_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{B}$.

  Note: we require μ to take on only **finite** values.
- If $\mu \geq 0$ with $\mu(X) = 1$, we call μ a (Borel) **probability measure**.
- For any $a \in X$, the **delta-measure** at a is the probability measure

$$\delta_a(U) := \begin{cases} 1 & \text{if } a \in U, \\ 0 & \text{if } a \notin U. \end{cases}$$

Integration

Given a (signed Borel) measure μ on X, we can define integrals

$$\int_X f\,d\mu = \int_X f(x)\,d\mu(x)$$

for measurable functions $f : X \to \mathbb{R}$.

Theorem (Riesz Representation)

Let X be a locally compact Hausdorff space. Then

$$M_{\mathrm{Rad}}(X) \xrightarrow{\sim} C_c(X)^\vee \quad \text{by} \quad \mu \mapsto \left[f \mapsto \int_X f\,d\mu\right]$$

Recall:

A signed Borel measure μ is **Radon** if it satisfies certain technical regularity properties.

$C_c(X) = \{f : X \to \mathbb{R} \text{ continuous, with compact support}\}$.

So $C_c(X)^\vee = \{L : C_c(X) \to \mathbb{R} \text{ linear}\}$.

Pushforward Measures

Let X be a topological space (think $\mathbb{P}^1_{\text{an}}$ or $\mathbb{P}^1(\mathbb{C})$),
let μ be a signed Borel measure on X, and
let $\phi : X \to X$ be a measurable function
(think $\phi \in \mathbb{C}_v(z)$ or $\phi \in \mathbb{C}(z)$).

The **pushforward measure** $\phi_*\mu$ is the signed Borel measure

$$\phi_*\mu(U) := \mu\big(\phi^{-1}(U)\big).$$

- If $f : X \to \mathbb{R}$ is measurable, then
$$\int_X f\, d(\phi_*\mu) = \int_X (f \circ \phi) d\mu.$$
- If μ is a probability measure (i.e., if $\mu \geq 0$ and $\mu(X) = 1$), then so is $\phi_*\mu$.
- $\big(\phi \circ \psi\big)_*\mu = \phi_*\big(\psi_*\mu\big).$

Pullback Measures

Now specifically set $X = \mathbb{P}^1(\mathbb{C})$ or $X = \mathbb{P}^1_{\mathrm{an}}$, and $\phi \in \mathbb{C}(z)$ or $\phi \in \mathbb{C}_v(z)$ nonconstant.

Given a signed Radon Borel measure μ, the **pullback measure** $\phi^*\mu$ is the signed Radon Borel measure such that

$$\int_X f \, d\left(\phi^* \mu\right) = \int_X \sum_{y \in \phi^{-1}(x)} (\deg_y \phi) \cdot f(y) \, d\mu(x).$$

Example. $\phi^*\delta_a = \sum_{b \in \phi^{-1}(a)} (\deg_b \phi)\delta_b.$

- If μ is a probability measure, then so is $\dfrac{1}{\deg \phi}\phi^*\mu$.
- $(\phi \circ \psi)^*\mu = \psi^*(\phi^*\mu)$.

Some Properties

- $\mathrm{supp}\,(\phi_*\mu) = \phi(\,\mathrm{supp}(\mu))$
- $\mathrm{supp}\,(\phi^*\mu) = \phi^{-1}(\,\mathrm{supp}(\mu))$
- More specifically, for any $a \in \mathbb{P}^1_{\mathrm{an}}$,

$$\phi_*\delta_a = \delta_{\phi(a)} \qquad \text{and} \qquad \phi^*\delta_a = \sum_{b\in\phi^{-1}(a)} (\deg_b \phi)\delta_b.$$

- $\phi_*(\phi^*\mu) = (\deg\phi)\mu$.
- If $\phi^*\mu = (\deg\phi)\mu$, then we get $\phi_*\mu = \mu$ for free, from the previous line.

  (A pair (T, μ) where $T : X \to X$ satisfies $T_*\mu = \mu$, i.e., $\mu(T^{-1}(U)) = \mu(U)$, is a central object of study in ergodic theory. We say T is *measure-preserving* w.r.t. μ, and μ is *invariant* w.r.t. T.)

The Invariant/Canonical/Equilibrium Measure

Theorem (Favre & Rivera-Letelier; Baker & Rumely; Autissier, Chambert-Loir, & Thuillier, mid 2000s)

Let $\mathbb{C}_v$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$. Then there is a unique probability measure μ_ϕ on $\mathbb{P}^1_{\text{an}}$ with the following properties:

- $\phi^* \mu_\phi = d \cdot \mu_\phi$, *and*
- $\mu_\phi(E_\phi) = 0$,

where $E_\phi \subseteq \mathbb{P}^1(\mathbb{C}_v)$ is the (type I) exceptional set of ϕ.
Moreover, we have $\operatorname{supp}(\mu_\phi) = \mathcal{J}_{\phi,\text{an}}$.

Note: $E_\phi = \{\infty\}$ for a polynomial $\phi \in \mathbb{C}_v[z]$,
and $E_\phi = \{0, \infty\}$ for $\phi(z) = z^m$, $m \in \mathbb{Z}$.
For ϕ not conjugate to such a map, $E_\phi = \varnothing$.

Weak Convergence of Measures

Definition

Let X be a topological space, and let $\{\mu_n\}_{n\geq 1}$ be a sequence of signed Borel measures on X. We say that $\{\mu_n\}$ **converges weakly** to a signed Borel measure μ if for every continuous function $f : X \to \mathbb{R}$ with compact support,

$$\lim_{n\to\infty} \int_X f \, d\mu_n = \int_X f \, d\mu.$$

Example

$X = \mathbb{R}$. Then $\{\delta_{1/n}\}_{n\geq 1}$ converges weakly to δ_0, because for any $f \in C_c(\mathbb{R})$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f \, d\delta_{1/n} = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0) = \int_{\mathbb{R}} f \, d\delta_0.$$

Note: The assumption that f is continuous is **crucial** here.

Definition of the Equilibrium Measure

Given $\phi \in \mathbb{C}_v(z)$ of degree $d \geq 2$, choose any point $\xi \in \mathbb{P}^1_{\mathrm{an}}$ that is **not** an exceptional point in $\mathbb{P}^1(\mathbb{C}_v)$.

Set $\mu_0 := \delta_\xi$, and for each $n \geq 1$, set

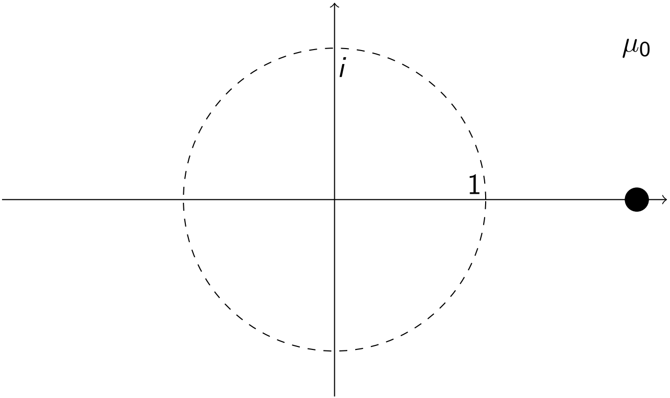
$$\mu_n := \frac{1}{d}\phi^* \mu_{n-1} = \frac{1}{d^n}(\phi^n)^* \delta_\xi.$$

Thus, μ_n is a probability measure with support $\mathrm{supp}(\mu_n) = \phi^{-n}(\xi)$.

Then the sequence $\{\mu_n\}_{n \geq 0}$ converges weakly to a measure, and this measure is the equilibrium measure $\mu = \mu_\phi$.

An Example over $\mathbb{C}$

Let $\phi(z) = z^2$, and $\xi = 2 \in \mathbb{C}$.



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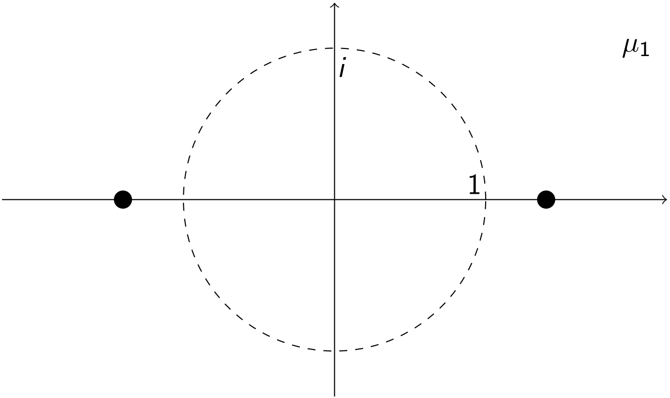

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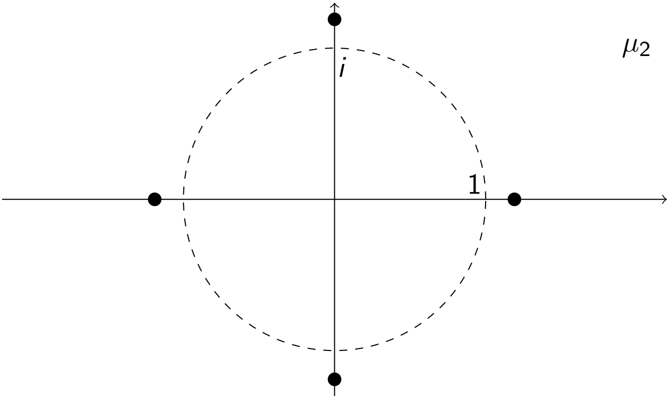

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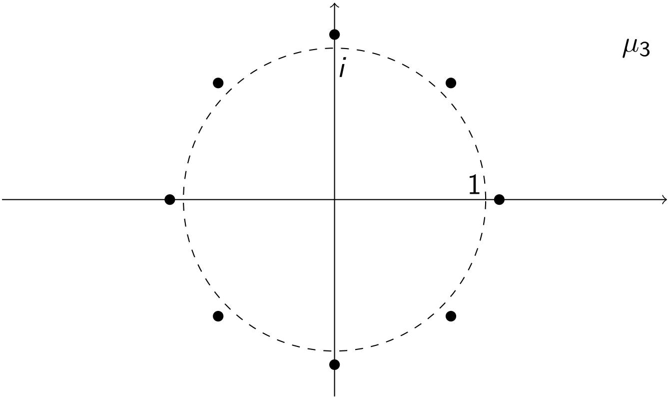

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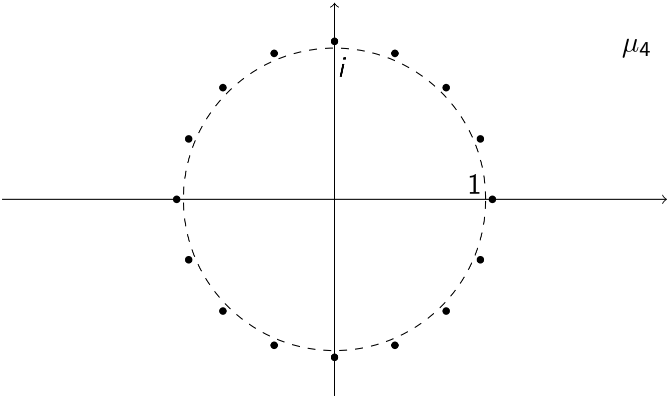

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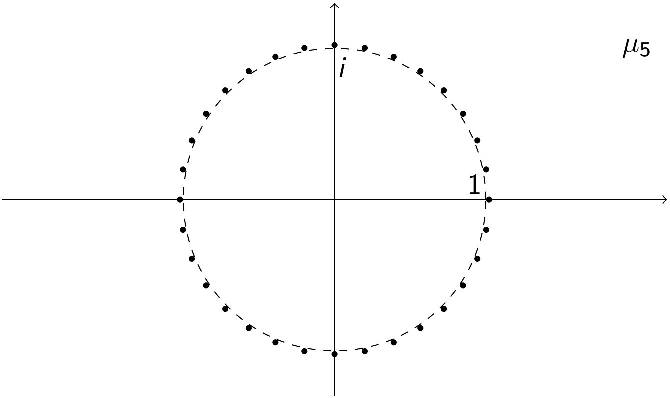

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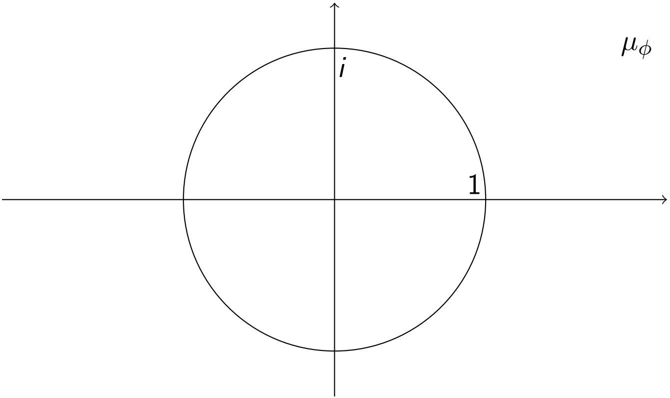


Complex Potential Theory in One Slide

Given a real-valued function $f \in C^2(\mathbb{C})$, recall the Laplacian of f is

$$\Delta f = \partial_x^2 f + \partial_y^2 f = 4\partial_z \partial_{\bar{z}} f.$$

Fact: $$\int_{\mathbb{C}} g\, \Delta f\, dA = \int_{\mathbb{C}} f\, \Delta g\, dA \quad \text{for all} \quad g \in C_c^\infty(\mathbb{C}).$$

(By Green's Theorem. Intuition: $\int gf'' = -\int f'g' = \int fg''$.)

For more general (nice enough) f, there is a unique (Radon) **measure** Δf such that

$$\int_{\mathbb{C}} g\, \Delta f = \int_{\mathbb{C}} f\, \Delta g\, dA \quad \text{for all} \quad g \in C_c^\infty(\mathbb{C}).$$

Examples:

$\Delta(\log|z|) = 2\pi\delta_0$.

$\Delta(\log\max\{|z|, 1\})$ is Lebesgue measure on the unit circle.

The Laplacian on $\mathbb{P}^1_{\text{an}}$

There is a metric on $\mathbb{P}^1_{\text{an}} \setminus \mathbb{P}^1(\mathbb{C}_v)$ as suggested by the tree: the distance from $\zeta(a, r)$ to $\zeta(a, s)$ is $|\log s - \log r|$.

Given a (nice enough) function $f : \mathbb{P}^1_{\text{an}} \to [-\infty, \infty]$, here's the **idea** of how to define Δf.

At a point ζ, for each direction $\vec{u}$ emanating from ζ, we can define the directional derivative

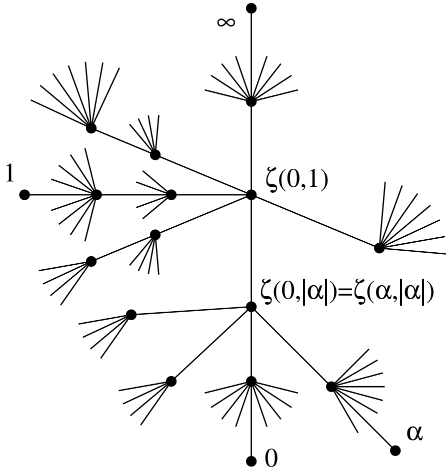
$$D_{\vec{u}} f(\zeta) \text{ ``}=\text{''} \lim_{h \to 0^+} \frac{f(\zeta + h\vec{u}) - f(\zeta)}{h}.$$

If S is the set of directions at ζ, then we define Δf to have a δ mass at ζ of mass $\sum_{\vec{u} \in S} D_{\vec{u}} f(\zeta)$.

Equip a segment $[\zeta(a, r), \zeta(a, s)]$, which is isometric to $[\log r, \log s] \subseteq \mathbb{R}$, with Lebesgue measure λ.

Define Δf along $[\log r, \log s]$ to be just $f''(x)\lambda$.

$\mathbb{P}^1_{an}$



Note: $\Delta f(\mathbb{P}^1_{an}) = 0$.

Example

Let $f(\zeta) =$ "$\log\max\{|\zeta|_v, 1\}$" $= \log\max\{\|z\|_\zeta, 1\}$.

- f has constant value 0 on $\overline{D}_{\mathrm{an}}(0,1)$.
- $f\big(\zeta(0,t)\big) = \log t$ for $t \geq 1$,
  so by isometrically identifying the segment from $\zeta(0,1)$ to ∞ with the real interval $[0,\infty]$, we have $f(x) = x$.
- f is constant (with value $\log t$) on any side branch emanating from $\zeta(0,t)$.

So $\Delta f = \delta_{\zeta(0,1)} - \delta_\infty$.

Some Properties of the Laplacian

There is a space $\mathcal{P}$ of "nice enough" functions $f : \mathbb{P}^1_{\text{an}} \to [-\infty, \infty]$ with

$$\Delta : \mathcal{P} \to \{\text{Radon measures on } \mathbb{P}^1_{\text{an}}\}.$$

- Δ is linear.
- $\Delta f = 0$ iff f is constant.
- $\Delta f = \Delta g$ iff $f - g$ is constant.
- $\displaystyle\int_{\mathbb{P}^1_{\text{an}}} f \Delta g = \int_{\mathbb{P}^1_{\text{an}}} g \Delta f \quad$ for $f, g \in \mathcal{P}$ continuous.
- $\phi^*(\Delta f) = \Delta(f \circ \phi)$.
- If $f_n \to f$ uniformly on $\mathbb{P}^1_{\text{an}}$ (and $|\Delta f_n|$ is uniformly bounded on $\mathbb{P}^1_{\text{an}}$), then $\Delta f_n \to \Delta f$ weakly.

  (Note: weak convergence can also be proven from weaker assumptions.)

Idea of Proving Convergence

Write $\phi(z) = F(z)/G(z)$, and assume $\infty \notin E_\phi$.
Let $u(\zeta) := d^{-1} \log \|G\|_\zeta$. (I.e., $u(x) = d^{-1} \log |G(x)|_v$.)
Then $\Delta u = d^{-1}\phi^*(\delta_\infty) - \delta_\infty$.

For each $n \geq 1$, define $f_n := \sum_{i=0}^{n-1} d^{-i} \cdot u \circ \phi^i$.

Then

$$\Delta f_n = \sum_{i=0}^{n-1} d^{-i} \Delta\big(u \circ \phi^i\big) = \sum_{i=0}^{n-1} d^{-i} (\phi^i)^*(\Delta u)$$

$$= \sum_{i=0}^{n-1} \left[d^{-i-1} (\phi^{i+1})^*(\delta_\infty) - d^{-i}(\phi^i)^*(\delta_\infty) \right]$$

$$= d^{-n}(\phi^n)^*(\delta_\infty) - \delta_\infty = \mu_n - \delta_\infty$$

Proving Convergence, Continued: The Invariant Potential

$$\mu_n = d^{-n}(\phi^n)^*(\delta_\infty),$$

$$u(x) = d^{-1}\log|G(x)|_v, \qquad \Delta u = d^{-1}\phi^*(\delta_\infty) - \delta_\infty,$$

$$f_n = \sum_{i=0}^{n-1} d^{-i}\cdot u\circ\phi^i, \quad \text{with} \quad \Delta f_n = \mu_n - \delta_\infty.$$

Let $\hat{\lambda}_\phi := \lim_{n\to\infty} f_n = \sum_{n=0}^{\infty} d^{-n}u\circ\phi^n$, and $\mu_\phi := \Delta\hat{\lambda}_\phi + \delta_\infty$.

Then $f_n \to \hat{\lambda}_\phi$, so $\Delta f_n \to \Delta\hat{\lambda}_\phi$ weakly,

So $\mu_n := d^{-n}(\phi^n)^*\delta_\infty \to \mu_\phi$ weakly, and $\phi^*\mu_\phi = d\cdot\mu_\phi$.

Moreover,

$$\hat{\lambda}_\phi\circ\phi = \sum_{n\geq 0} d^{-n}u\circ\phi^{n+1} = d\cdot\sum_{n\geq 1} d^{-n}u\circ\phi^n = d\cdot(\hat{\lambda}_\phi - u).$$

So $\hat{\lambda}_\phi(\phi(x)) = d\cdot\hat{\lambda}_\phi(x) - \log|G(x)|_v$.

The Punchline

Let K be a **number** field, with set of places M_K.

Let $\phi(z) = F(z)/G(z) \in K(z)$ with $\deg \phi = d \geq 2$.

For each $v \in M_K$, let $\hat{\lambda}_{\phi,v}$ be the v-adic invariant potential on $\mathbb{P}^1_{\mathrm{an}}$.

That is, $\Delta \hat{\lambda}_{\phi,v}$ is the equilibrium measure $\mu_{\phi,v}$ on $\mathbb{P}^1_{\mathrm{an},v}$, and

$$\hat{\lambda}_{\phi,v}(\phi(x)) = d \cdot \hat{\lambda}_{\phi,v}(x) - \log |G(x)|_v.$$

Then $\hat{h}_\phi(x) := [K : \mathbb{Q}]^{-1} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \hat{\lambda}_{\phi,v}(x)$ satisfies

$$\hat{h}_\phi\big(\phi(x)\big) = d \cdot \hat{h}_\phi(x),$$

since $\sum_v [K_v : \mathbb{Q}_v] \log |G(x)|_v = 0$.

Fact: the individual functions $\hat{\lambda}_{\phi,v}$ are the local canonical height functions for ϕ, and $\hat{h}_\phi$ is the canonical height function for ϕ.

Equidistribution

Theorem (Baker-Rumely, Favre-Rivera-Letelier, mid-2000s)

Let K be a global field, let $\phi \in K(z)$, let $v \in M_K$ be a nontrivial absolute value on K, and let $\{x_n\}_{n\geq 1}$ be a sequence of **distinct** *points in $\mathbb{P}^1(K^{\text{sep}})$ such that*

$$\lim_{n\to\infty} \hat{h}_\phi(x_n) = 0.$$

For each $n \geq 1$, let $X_n \subseteq \mathbb{P}^1(K^{\text{sep}})$ be the $\text{Gal}(K^{\text{sep}}/K)$-orbit of x_n, and define

$$\nu_n := \frac{1}{|X_n|} \sum_{y \in X_n} \delta_y.$$

which is a Borel probability measure on $\mathbb{P}^1_{\text{an},v}$. Then the sequence $\{\nu_n\}_{n\geq 1}$ converges weakly to $\mu_{\phi,v}$, where $\mu_{\phi,v}$ is the equilibrium measure for ϕ at v.