An Introduction to the Berkovich Projective Line

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p-adic numbers: $\mathbb{Q}_p$ and $\mathbb{C}_p$

Fix $p \geq 2$ prime. The p-adic absolute value on $\mathbb{Q}$ is given by

$$\left| \frac{r}{s} p^n \right|_p = p^{-n} \qquad \text{for } r, s \in \mathbb{Z} \text{ not divisible by } p.$$

Idea: numbers divisible by large powers of p are "small".
Note: for all $x, y \in \mathbb{Q}$,

- $|x|_p \geq 0$, with $|x|_p = 0 \Leftrightarrow x = 0$,
- $|xy|_p = |x|_p \cdot |y|_p$,
- $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.
  "**ultrametric**" or "**non-archimedean**" triangle inequality

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$$\mathbb{Q}_p := \left\{ \sum_{n \geq n_0} a_n p^n : n_0 \in \mathbb{Z}, a_n \in \{0, 1, \ldots, p-1\} \right\}$$

is the completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$.
$\mathbb{C}_p :=$ the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

The absolute value $|\cdot|_p$ extends uniquely to $\mathbb{C}_p$.

Also, $\mathbb{C}_p$ is both complete and algebraically closed.

Disks in $\mathbb{C}_v$

Let $\mathbb{C}_v$ be an algebraically closed field
that is complete w.r.t. a non-archimedean absolute value $|\cdot|_v$.

Given $a \in \mathbb{C}_v$ and $r > 0$,

$$D(a,r) := \{x \in \mathbb{C}_v : |x-a|_v < r\} \quad \text{and}$$
$$\overline{D}(a,r) := \{x \in \mathbb{C}_v : |x-a|_v \le r\}$$

are the associated open disk and closed disk.

- if $r \notin |\mathbb{C}_v^\times|_v$, then $D(a,r) = \overline{D}(a,r)$ is an **irrational disk**
- if $r \in |\mathbb{C}_v^\times|_v$, then $D(a,r) \subsetneq \overline{D}(a,r)$
- $D(a,r)$ is a **rational open disk**
- $\overline{D}(a,r)$ is a **rational closed disk**

- All disks are (topologically) **both** open and closed
- Any point of a disk is a center. i.e.,
  - If $b \in D(a,r)$, then $D(a,r) = D(b,r)$
  - If $b \in \overline{D}(a,r)$, then $\overline{D}(a,r) = \overline{D}(b,r)$

Residue classes

Let $\mathcal{O}_v := \{x \in \mathbb{C}_v : |x|_v \leq 1\} = \overline{D}(0,1)$
and $\mathcal{M}_v := \{x \in \mathbb{C}_v : |x|_v < 1\} = D(0,1)$.

Then $\mathcal{O}_v$ is a ring with unique maximal ideal $\mathcal{M}_v$.

The quotient field $k := \mathcal{O}_v/\mathcal{M}_v$ is an algebraically closed field.

[For $\mathbb{C}_v = \mathbb{C}_p$, we have $k \cong \overline{\mathbb{F}}_p$.]

k is called the **residue field** of $\mathbb{C}_v$, and
the elements of k are called **residue classes**.

Decomposing $\mathcal{O}_v$ into cosets modulo $\mathcal{M}_v$,
we see that $\overline{D}(0,1)$ is an infinite disjoint union

$$\overline{D}(0,1) = \coprod_{\overline{c} \in k} D(c,1)$$

of one open disk for each residue class $\overline{c} \in k$.

Similarly, each rational closed disk $\overline{D}(a,r)$ is a disjoint union of infinitely many rational open disks $D(a,r)$,
but an irrational disk $\overline{D}(a,r)$ is simply $\overline{D}(a,r) = D(a,r)$.

$\mathbb{P}$-Disks and Connected Affinoids

Recall $\mathbb{P}^1(\mathbb{C}_v) := \mathbb{C}_v \cup \{\infty\}$.

Definition

- A $\mathbb{P}$-**disk** is either D or $\mathbb{P}^1(\mathbb{C}_v) \smallsetminus D$, where D is a disk.
- A **connected affinoid** is a nonempty intersection of finitely many $\mathbb{P}$-disks.
  Equivalently, a connected affinoid is $\mathbb{P}^1$ with finitely many $\mathbb{P}$-disks removed.

Theorem

Let $g(z) \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 1$, and let $D, U \subseteq \mathbb{P}^1(\mathbb{C}_v)$ be a $\mathbb{P}$-disk and a connected affinoid. Then

- *$g(D)$ is either $\mathbb{P}^1(\mathbb{C}_v)$ or a $\mathbb{P}$-disk.*
- *$g(U)$ is either $\mathbb{P}^1(\mathbb{C}_v)$ or a connected affinoid.*
- *$g^{-1}(D)$ (respectively $g^{-1}(U)$) is a union of connected affinoids $V_1, \ldots, V_\ell$, each mapping onto D (respectively, U).*

From $\mathbb{P}^1(\mathbb{C}_v)$ to $\mathbb{P}^1_{\mathrm{an}}$

- $\mathbb{P}^1(\mathbb{C}_v)$ is not compact, or even locally compact.
- $\mathbb{P}^1(\mathbb{C}_v)$ is totally disconnected.

There is a nicer space $\mathbb{P}^1_{\mathrm{an}}$ that:

- contains $\mathbb{P}^1(\mathbb{C}_v)$ as a subspace,
- is compact,
- is (still) Hausdorff, and
- is path-connected.

Multiplicative Seminorms on $\mathbb{C}_v[z]$

Definition

A **multiplicative seminorm** on $\mathbb{C}_v[z]$ is a function $\zeta = \|\cdot\|_\zeta : \mathbb{C}_v[z] \to [0, \infty)$ such that

- $\|c\|_\zeta = |c|_v$ for all constants $c \in \mathbb{C}_v$,
- $\|fg\|_\zeta = \|f\|_\zeta \cdot \|g\|_\zeta$ for all $f, g \in \mathbb{C}_v[z]$, and
- $\|f + g\|_\zeta \leq \|f\|_\zeta + \|g\|_\zeta$ for all $f, g \in \mathbb{C}_v[z]$.

Note: We do **not** require that $\|f\|_\zeta = 0$ implies $f = 0$.

By the way: we get $\|f + g\|_\zeta \leq \max\{\|f\|_\zeta, \|g\|_\zeta\}$ for free.

Examples of Multiplicative Seminorms on $\mathbb{C}_v[z]$

1. For any $x \in \mathbb{C}_v$, define $\|\cdot\|_x$ by $\|f\|_x := |f(x)|_v$.
2. For any disk $D \subseteq \mathbb{C}_v$, define $\|\cdot\|_D$ by

$$\|f\|_D := \sup\{|f(x)|_v : x \in D\}.$$

Notes on $\|\cdot\|_D$:

- If $D = \overline{D}(a,r)$ or $D = D(a,r)$, and $f(z) = \sum c_n(z-a)^n$, then

$$\|f\|_D = \max\{|c_n|_v r^n : n \geq 0\}.$$

- Since $\|\cdot\|_{\overline{D}(a,r)} = \|\cdot\|_{D(a,r)}$, we can denote both by $\|\cdot\|_{\zeta(a,r)}$.

The Berkovich Line

Definition

The **Berkovich affine line** $\mathbb{A}^1_{\mathrm{an}}$ is the set of all multiplicative seminorms on $\mathbb{C}_v[z]$.

The **Berkovich projective line** $\mathbb{P}^1_{\mathrm{an}}$ is $\mathbb{A}^1_{\mathrm{an}} \cup \{\infty\}$.

As topological spaces, we equip $\mathbb{A}^1_{\mathrm{an}}$ and $\mathbb{P}^1_{\mathrm{an}}$ with the **Gel'fand topology**.

This is the weakest topology such that for every $f \in \mathbb{C}_v[z]$, the map $\mathbb{A}^1_{\mathrm{an}} \to \mathbb{R}$ given by

$$\zeta \mapsto \|f\|_\zeta$$

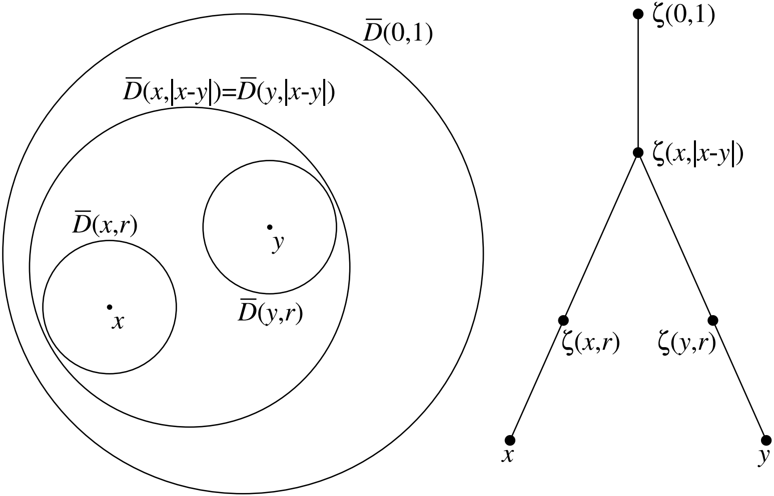
is continuous.

Berkovich's Classification of Points

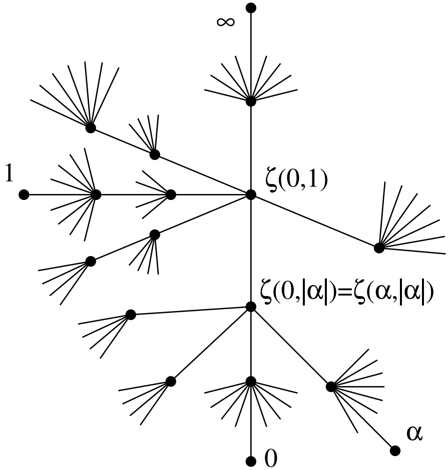
There are four kinds of points in $\mathbb{P}^1_{\mathrm{an}}$.

1. Type I: seminorms $\|\cdot\|_x$ corresponding to (classical) points $x \in \mathbb{C}_v$, as well as the point at ∞.
2. Type II: norms $\|\cdot\|_{\zeta(a,r)}$ corresponding to **rational** closed disks $\overline{D}(a,r) \subseteq \mathbb{C}_v$.
3. Type III: norms $\|\cdot\|_{\zeta(a,r)}$ corresponding to **irrational** disks $\overline{D}(a,r) \subseteq \mathbb{C}_v$.
4. Type IV: norms $\|\cdot\|_{\zeta}$ corresponding to (equivalence classes of) decreasing chains $D_1 \supseteq D_2 \supseteq \cdots$ of disks with **empty intersection**.

Path-connectedness, intuitively



The Berkovich Projective Line $\mathbb{P}^1_{\mathrm{an}}$



Berkovich Disks

Let $a \in \mathbb{C}_v$ and $r > 0$.

- The **closed Berkovich disk** $\overline{D}_{\mathrm{an}}(a, r)$ is

$$\overline{D}_{\mathrm{an}}(a, r) := \{\zeta \in \mathbb{A}^1_{\mathrm{an}} \,:\, \|z - a\|_\zeta \leq r\},$$

  which is the set of all $\zeta \in \mathbb{P}^1_{\mathrm{an}}$ corresponding to a point/disk/chain of disks contained in $\overline{D}(a, r)$.

- The **open Berkovich disk** $D_{\mathrm{an}}(a, r)$ is

$$D_{\mathrm{an}}(a, r) := \{\zeta \in \mathbb{A}^1_{\mathrm{an}} \,:\, \|z - a\|_\zeta < r\},$$

  which is the set of all $\zeta \in \mathbb{P}^1_{\mathrm{an}}$ corresponding to a point/disk/chain of disks contained in $D(a, r)$, **except** $\zeta(a, r)$ itself.

Berkovich Disks and Connected Affinoids

Recall:

$$\overline{D}_{\mathrm{an}}(a,r) := \{\zeta \in \mathbb{A}^1_{\mathrm{an}} \ : \ \|z-a\|_\zeta \le r\}$$

and

$$D_{\mathrm{an}}(a,r) := \{\zeta \in \mathbb{A}^1_{\mathrm{an}} \ : \ \|z-a\|_\zeta < r\}$$

- A Berkovich $\mathbb{P}$-disk is either a Berkovich disk or the complement in $\mathbb{P}^1_{\mathrm{an}}$ of a Berkovich disk.
- A **Berkovich connected affinoid** is a nonempty intersection of finitely many Berkovich $\mathbb{P}$-disks.

Facts:

- $D_{\mathrm{an}}(a,r)$ is open but not closed.
- $\overline{D}_{\mathrm{an}}(a,r)$ is closed but not open.
- The open Berkovich connected affinoids form a basis for the Gel'fand topology.

Rational Functions Acting on $\mathbb{P}^1_{\mathrm{an}}$

Let $\phi(z) \in \mathbb{C}_v(z)$. Then for each point $\zeta \in \mathbb{P}^1_{\mathrm{an}}$, there is a unique point $\phi(\zeta) \in \mathbb{P}^1_{\mathrm{an}}$ such that

$$\|h\|_{\phi(\zeta)} = \|h \circ \phi\|_\zeta \quad \text{for all } h \in \mathbb{C}_v(z).$$

- If ζ is type I, then $\phi(\zeta)$ is what you think.
- If ζ is a point of type II or III, corresponding to a disk $D \subseteq \mathbb{C}_v$, and if ϕ has no poles in D, then $\phi(\zeta)$ is the type II or III point corresponding to the disk $\phi(D)$.

Dynamics on $\mathbb{P}^1_{\mathrm{an}}$: Periodic Points

Let $\phi(z) \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$.

Note: $\deg(F/G) := \max\{\deg F, \deg G\}$.

For $n \geq 1$, write $\phi^n := \underbrace{\phi \circ \cdots \circ \phi}_{n}$.

(And ϕ^0 =identity map on $\mathbb{P}^1_{\mathrm{an}}$.)

Consider a point $\zeta \in \mathbb{P}^1_{\mathrm{an}}$.

- If $\phi(\zeta) = \zeta$, we say that ζ is a **fixed** point of ϕ.
- If $\phi^n(\zeta) = \zeta$ for some $n \geq 1$, we say ζ is a **periodic** point of ϕ, of period n.
  The smallest such $n \geq 1$ is the **minimal period** of ζ.
- If $\phi^n(\zeta) = \phi^m(\zeta)$ for some $n > m \geq 0$, we say ζ is a **preperiodic** point of ϕ.

Dynamics on $\mathbb{P}^1_{\text{an}}$: Multipliers of Periodic Points

Let $\phi(z) \in \mathbb{C}_v(z)$ of degree $d \geq 2$, and
let $\zeta \in \mathbb{P}^1_{\text{an}}$ be a periodic point of ϕ of minimal period n.

- If $\zeta \in \mathbb{P}^1(\mathbb{C}_v)$ is of type I,
  then $\lambda := (\phi^n)'(\zeta)$ is the **multiplier** of ζ.
- If ζ is of type II, III, or IV, then ϕ^n maps ζ to itself with some **local degree** (or **multiplicity**) $\ell \in \{1, 2, \ldots, d^n\}$.
  then $\lambda := \ell$ is the **multiplier** of ζ.

Note:

- The multiplier is the the same for all points in the periodic cycle $\{\zeta, \phi(\zeta), \ldots, \phi^{n-1}(\zeta)\}$ of ζ.
- The multiplier is coordinate-independent.
- If ζ is a periodic point of type III or IV, then its multiplier is necessarily $\lambda = 1$.

Classifying periodic points

Let $\phi(z) \in \mathbb{C}_v(z)$ of degree $d \geq 2$, and
let $\zeta \in \mathbb{P}^1_{\text{an}}$ be a periodic point of ϕ with multiplier λ.

If $\zeta \in \mathbb{P}^1(\mathbb{C}_v)$ is of type I (so $\lambda \in \mathbb{C}_v$), we say ζ is

- **attracting** if $|\lambda|_v < 1$.
- **repelling** if $|\lambda|_v > 1$.
- **indifferent** (or **neutral**) if $|\lambda|_v = 1$.

If ζ is of type II, III, or IV, (so λ is a positive integer), we say ζ is

- **repelling** if $\lambda \geq 2$.
- **indifferent** (or **neutral**) if $\lambda = 1$.

Berkovich Fatou and Julia Sets

Definition

An open set $U \subseteq \mathbb{P}^1_{\text{an}}$ is **dynamically stable** under $\phi \in \mathbb{C}_v(z)$ if $\bigcup_{n \geq 0} \phi^n(U)$ omits infinitely many points of $\mathbb{P}^1_{\text{an}}$.

The **(Berkovich) Fatou set of** ϕ is the set $\mathcal{F}_\phi = \mathcal{F}_{\phi,\text{an}}$ given by

$$\mathcal{F}_\phi := \{x \in \mathbb{P}^1_{\text{an}} : x \text{ has a dynamically stable neighborhood}\}.$$

The **(Berkovich) Julia set of** ϕ is the set

$$\mathcal{J}_\phi = \mathcal{J}_{\phi,\text{an}} := \mathbb{P}^1_{\text{an}} \smallsetminus \mathcal{F}_{\phi,\text{an}}.$$

Basic Properties of Berkovich Fatou and Julia Sets

- $\mathcal{F}_{\phi,\mathrm{an}}$ is open, and $\mathcal{J}_{\phi,\mathrm{an}}$ is closed.
- $\mathcal{F}_{\phi^n,\mathrm{an}} = \mathcal{F}_{\phi,\mathrm{an}}$, and $\mathcal{J}_{\phi^n,\mathrm{an}} = \mathcal{J}_{\phi,\mathrm{an}}$
- $\phi(\mathcal{F}_\phi) = \mathcal{F}_\phi = \phi^{-1}(\mathcal{F}_\phi)$, and $\phi(\mathcal{J}_\phi) = \mathcal{J}_\phi = \phi^{-1}(\mathcal{J}_\phi)$.
- All attracting periodic points are Fatou.
- All repelling periodic points are Julia.
- All indifferent periodic points **of type I, III, and IV** are Fatou.
- Indifferent periodic type II points are Fatou if the residue field is algebraic over a finite field (e.g., for $\mathbb{C}_v = \mathbb{C}_p$), but otherwise, they can be either Julia or Fatou.

Good Reduction

Recall $\mathcal{O}_v := \{x \in \mathbb{C}_v : |x|_v \leq 1\}$ is the ring of integers of $\mathbb{C}_v$, with unique maximal ideal $\mathcal{M}_v := \{x \in \mathbb{C}_v : |x|_v < 1\}$ and residue field $k = \mathcal{O}_v/\mathcal{M}_v$.

Given $\phi \in \mathbb{C}_v(z)$, we may write $\phi(z) = \dfrac{f(z)}{g(z)}$ with $f, g \in \mathcal{O}_v[z]$ relatively prime, and with at least one coefficient c of f or g having $|c|_v = 1$.

Define $\overline{\phi} := \overline{f}/\overline{g}$, where $\overline{f}, \overline{g} \in k[z]$ are the reductions of f and g.

Note that $\deg(\overline{\phi}) \leq \deg(\phi)$.

If $\deg(\overline{\phi}) = \deg(\phi)$, we say ϕ has **(explicit) good reduction**.

Theorem

Let $\phi \in \mathbb{C}_v(z)$ with $\deg(\phi) \geq 2$. The following are equivalent:

1. *ϕ has explicit good reduction.*
2. $\mathcal{J}_{\phi,\mathrm{an}} = \{\zeta(0,1)\}$.