Misiurewicz polynomials and dynamical units

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Notation

- K is a characteristic zero field, usually a number field
- $\overline{K}$ is the algebraic closure of K
- $f(z) \in K[z]$ is a polynomial of degree $d \geq 2$
- $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$ is the n-th iterate of f

Definition

We say $x \in \mathbb{P}^1(\overline{K})$ is

- *periodic* if $f^n(x) = x$ for some $n \geq 1$,
- *preperiodic* if $f^n(x) = f^m(x)$ for some $n > m \geq 0$,
- *strictly preperiodic* if x is preperiodic but not periodic.

Postcritically Finite Maps

Definition

We say $f \in K[z]$ is *postcritically finite*, or *PCF*, if every critical point $c \in \mathbb{P}^1(\overline{K})$ of f is preperiodic under f.

Example. $f(z) = z^2 - 3$ is *not* PCF, since the critical point $x = 0$ is not preperiodic.

$$x = 0 \mapsto -3 \mapsto 6 \mapsto 33 \mapsto 1086 \mapsto \cdots$$

Examples of Postcritically Finite Maps

Example. $f(z) = z^d$: $\infty \mapsto \infty$ $0 \mapsto 0$

Example. $f(z) = z^2 - 1$: $\infty \mapsto \infty$ $0 \mapsto -1 \mapsto 0$

Example. $f(z) = z^2 - 2$: $\infty \mapsto \infty$ $0 \mapsto -2 \mapsto 2 \mapsto 2$

Example. $f(z) = z^2 + i$:
$\infty \mapsto \infty$ $0 \mapsto i \mapsto i - 1 \mapsto -i \mapsto i - 1$

Example. $f(z) = -2z^3 + 3z^2$: $\infty \mapsto \infty$ $0 \mapsto 0$ $1 \mapsto 1$

Why should we care about PCF maps?

One of many reasons:

Let K be a global field and $x_0 \in K$. Define

- $K_n := K(f^{-n}(x_0))$,
- $K_\infty := \bigcup_{n \geq 1} K_n$,
- $G_\infty := \mathrm{Gal}(K_\infty/K)$.

If f is PCF, then:

- The tower $\cdots K_3/K_2/K_1/K$ is ramified over only finitely many primes. (Aitken, Hajir, Maire 2005).
- G_∞ has infinite index in the otherwise expected automorphism group $\mathrm{Aut}(T_{d,\infty})$ of an infinite rooted d-ary tree.

Idea:

Elliptic Curve		Dynamical System
torsion point	$\longleftrightarrow$	preperiodic point
CM curve	$\longleftrightarrow$	PCF map

The Quadratic Polynomial Family

Define $f_c(z) = z^2 + c$. Critical points are ∞ (fixed) and 0.

$$0 \mapsto c \mapsto c^2 + c \mapsto (c^2 + c)^2 + c \mapsto \cdots$$

We say c is a **PCF parameter** if $f_c^n(0) = f_c^m(0)$ for some $n > m \geq 0$.

$\boxed{m = 0,\ n = 1}$: $\boxed{c = 0}$ $\qquad 0 \mapsto 0$

$\boxed{m = 0,\ n = 2}$: $c^2 + c = 0$, so $(c = 0)$, $\boxed{c = -1}$ $\qquad 0 \mapsto -1 \mapsto 0$

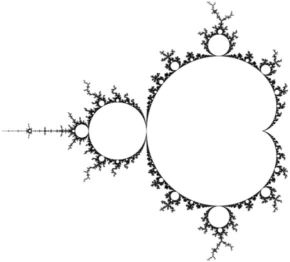
$\boxed{m = 2,\ n = 3}$: $(c^2 + c)^2 + c = c^2 + c$, so
$c^4 + 2c^3 = 0$, so $(c = 0)$, $\boxed{c = -2}$ $\qquad 0 \mapsto -2 \mapsto 2 \mapsto 2$

$\boxed{m = 2,\ n = 4}$: $\left((c^2 + c)^2 + c\right)^2 + c = c^2 + c$, so
$c^3(c+1)^2(c+2)(c^2+1) = 0$, so $(c = 0, -1, -2)$, $\boxed{c = \pm i}$
$0 \mapsto i \mapsto i - 1 \mapsto -i \mapsto i - 1$

PCF parameters and the Mandelbrot set

Let $K = \mathbb{C}$ and let $f_c(z) = z^2 + c$.

The **Mandelbrot set** is $\boxed{\mathcal{M} = \{c \in \mathbb{C} \,|\, \{f_c^n(0)\}_{n \geq 0} \text{ is bounded}\}}$



Facts:

• The PCF parameters c for which 0 is periodic lie in the interior of $\mathcal{M}$.

• The other (strictly preperiodic) PCF parameters c are dense in $\partial\mathcal{M}$.

The Unicritical Polynomial Family

Fix $d \geq 2$. Define $f_{d,c}(z) = z^d + c$. Critical points are ∞ (fixed) and 0.

$$0 \mapsto c \mapsto c^d + c \mapsto (c^d + c)^d + c \mapsto \cdots$$

We say $c \in \overline{K}$ is a **PCF parameter** if $f_{d,c}^{m+n}(0) = f_{d,c}^{m}(0)$ for some $m \geq 0$ and $n \geq 1$. In that case,

- c is a *Gleason parameter* if 0 is periodic under $f_{d,c}$, or
- c is a *Misiurewicz parameter* if 0 is strictly preperiodic under $f_{d,c}$.

Note: $f_{d,c}^{m+n}(0) - f_{d,c}^{m}(0) = \prod_{\zeta^d=1} \left(f_{d,c}^{m+n-1}(0) - \zeta f_{d,c}^{m-1}(0)\right)$

$$0 \mapsto a_1 \mapsto a_2 \cdots \mapsto a_{m-1} \mapsto \boxed{a_m} \mapsto \cdots \mapsto a_{m+n-1} \mapsto \boxed{a_{m+n}}$$

$$a_{m+n} = a_m \implies a_{m+n-1} = \zeta a_{m-1}$$

Gleason and Misiurewicz Polynomials

For the family $f_{d,c}(z) = z^d + c$, with $a_k = f_{d,c}^k(0)$.
(**Note**: $\deg a_k = d^{k-1}$)

The *n-th Gleason polynomial* is

$$G_{d,0,n}(c) := \prod_{k|n} \left(a_k(c)\right)^{\mu(n/k)}.$$

Fix a d-th root of unity $\zeta \neq 1$. The (m,n)-*Misiurewicz polynomial* is

$$G_{d,m,n}^{\zeta} := \prod_{k|n} \left(a_{m+k-1} - \zeta a_{m-1}\right)^{\mu(n/k)} \cdot \begin{cases} (G_{d,0,n})^{-1} & \text{if } n | m-1, \\ 1 & \text{if } n \nmid m-1. \end{cases}$$

Simple Roots

$$f_{d,c}(z) = z^d + c, \qquad a_k = f_{d,c}^k(0)$$

$$G_{d,0,n}(c) := \prod_{k|n} \left(a_k(c)\right)^{\mu(n/k)}$$

$$G_{d,m,n}^{\zeta} := \prod_{k|n} \left(a_{m+k-1} - \zeta a_{m-1}\right)^{\mu(n/k)} \cdot \begin{cases} (G_{d,0,n})^{-1} & \text{if } n|m-1, \\ 1 & \text{if } n \nmid m-1. \end{cases}$$

Theorem

Let $d \geq 2$, $m \geq 2$, and $n \geq 1$. Let $\zeta \neq 1$ be a d-th root of unity. Then

- $G_{d,0,n}$ *is a monic polynomial in $\mathbb{Z}[c]$ with only simple roots.*
- $G_{d,m,n}^{\zeta}$ *is a monic polynomial in $\mathbb{Z}[\zeta][c]$ with only simple roots.*

Irreducibility?

$$f_{d,c}(z) = z^d + c, \qquad a_k = f_{d,c}^k(0)$$

$$G_{d,0,n}(c) := \prod_{k|n} \left(a_k(c)\right)^{\mu(n/k)}$$

$$G_{d,m,n}^{\zeta} := \prod_{k|n} \left(a_{m+k-1} - \zeta a_{m-1}\right)^{\mu(n/k)} \cdot \begin{cases} \left(G_{d,0,n}\right)^{-1} & \text{if } n|m-1, \\ 1 & \text{if } n\nmid m-1. \end{cases}$$

Question/Conjecture: For every $d \geq 2$, $m \geq 2$, and $n \geq 1$,

- Is $G_{d,0,n}$ irreducible over $\mathbb{Q}$?
- Is $G_{d,m,n}^{\zeta}$ irreducible over $\mathbb{Q}(\zeta)$?

Fact: (Buff 2017): If $d \equiv 1 \pmod 6$, then $G_{d,0,3}$ is divisible by $c^{2(d-1)} + c^{d-1} + 1$

But in all other cases, computations suggest not only irreducibility, but that the Galois group of the polynomial is S_N.

Iterates of Misiurewicz parameters

$f_{d,c}(z) = z^d + c, \qquad a_k = f_{d,c}^k(0)$

Fix $d \geq 2$, $m \geq 2$, $n \geq 1$, and $\zeta \neq 1$ with $\zeta^d = 1$.

Roots of $G^{\zeta}_{d,m,n}$ satisfy $a_{n+m-1} = \zeta a_{m-1}$

Theorem

Let c_0 be a root of $G_{d,m,n}$, let $K = K(c_0)$, and let $v \in M_K$.

- *If $v(d) = 0$, then $v(a_i(c_0)) = 0$ for all $i \geq 1$*
- *If $d = p^e$ and $v(d) > 0$, then $v(a_i(c_0)) = 0$ for $n \nmid i$ but $v(a_i(c_0)) > 0$ for $n | i$.*
  (And there's an exact formula, $v(a_i(c_0)) \approx d^{-m} v(p)$.)

Proof considers $v(a_i - a_j)$ and $v(\zeta - 1)$ in:

$$\cdots \mapsto a_{m-1} \mapsto \boxed{a_m} \mapsto \cdots \mapsto a_{m+n-1} = \zeta a_{m-1} \mapsto \boxed{a_m}$$

Evaluating a special polynomial at special parameter

Example: Let Φ_m be the m-th cyclotomic polynomial, and let ζ_n be a primitive n-th root of unity, with $m > n \geq 1$. Then

$$\Phi_m(\zeta_n) \text{ is } \textit{not} \text{ a unit} \quad \Leftrightarrow \quad m = p^k n \ \ (\text{some prime power } p^k)$$

Essentially: when is $\zeta_m - \zeta_n$ an algebraic unit?

Example (Morton, Silverman, 1995): Let $f \in \mathbb{Z}[z]$ be a monic polynomial of degree $d \geq 2$. Let $a, b \in \overline{\mathbb{Q}}$ be periodic points of exact periods $m, n \geq 1$ (respectively), with $m \nmid n$ and $n \nmid m$. Then $a - b$ is an algebraic unit.

I.e., plugging a into the polynomial $\prod_{d|n} \left(f^d(z) - z\right)^{\mu(n/d)}$ defining b gives an algebraic unit.

So what about $G^{\zeta}_{d,j,\ell}(c_0)$ if c_0 is a root of a *different* Misiurewicz polynomial?

When is $G^{\zeta}_{d,j,\ell}(c_0)$ a unit?

$f_{d,c}(z) = z^d + c$

Let $d, m \geq 2$, $n \geq 1$, and $\zeta \neq 1$ a d-th root of unity.

Let c_0 be a root of $G^{\zeta}_{d,m,n}$.

So $f^{m+n}_{d,c_0}(0) = f^{m}_{d,c_0}(0)$

Question: For which integers $j \geq 2$ and $\ell \geq 1$ is $G^{\zeta}_{d,j,\ell}(c_0)$ an algebraic unit?

Theorem (B., Goksel)

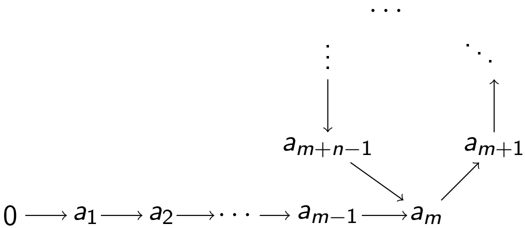
With notation as above, if $j \neq m$, then

$$G_{d,j,\ell}(c_0) \text{ is a unit} \quad \Longleftrightarrow \quad \ell \neq n.$$

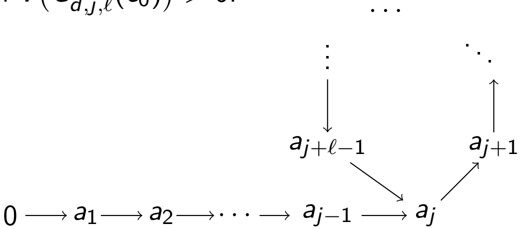
And when $\ell = n$, we have a precise description of $\langle G_{d,j,\ell}(c_0) \rangle$.

Proof ideas for $G^{\zeta}_{d,j,\ell}(c_0)$ with $j \neq m$

Actual:



Modulo v, if $v\big(G^{\zeta}_{d,j,\ell}(c_0)\big) > 0$:



When is $G^{\zeta}_{d,j,\ell}(c_0)$ a unit if $j = m$?

$f_{d,c}(z) = z^d + c$

Let $d, m \geq 2$, $n \geq 1$, and $\zeta \neq 1$ a d-th root of unity.

Let c_0 be a root of $G^{\zeta}_{d,m,n}$.

So $f^{m+n}_{d,c_0}(0) = f^{m}_{d,c_0}(0)$

Theorem (B., Goksel)

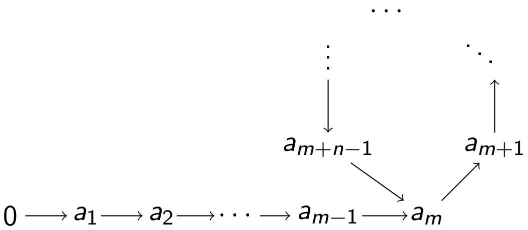
With notation as above, if $1 \leq \ell \leq n$ and $\ell \nmid n$, then $G^{\zeta}_{d,m,\ell}(c_0)$ is an algebraic unit.

Conjecture

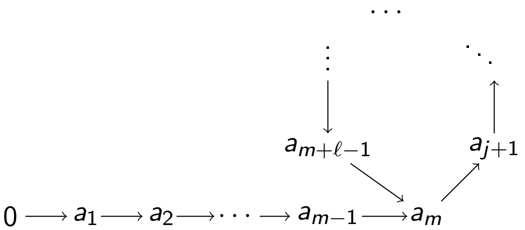
With notation as above, if $1 \leq \ell \leq n$ and $\ell | n$, then $G^{\zeta}_{d,m,\ell}(c_0)$ is **not** *an algebraic unit.*

Proof ideas for $G^{\zeta}_{d,m,\ell}(c_0)$, i.e., $j = m$

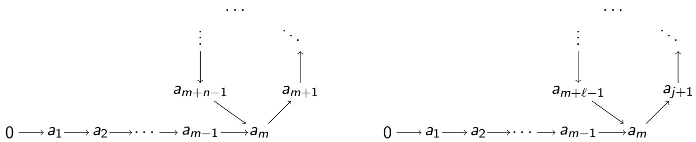
Actual:



Modulo v:



The hard case: $j = m$ and $\ell | n$



Let c_0 be a root of $G^{\zeta}_{d,m,n}$, and let α_0 be a root of $G^{\zeta}_{d,m,\ell}$

Goal: $\exists$ finite place v s.t. $v(G^{\zeta}_{d,m,\ell}(c_0)) > 0$

Assume: $G^{\zeta}_{d,m,n}$ irreducible over $\mathbb{Q}(\zeta)$. Then:

(Goal) $\Longleftarrow$ $\exists$ finite place v s.t. $v(G^{\zeta}_{d,m,n}(\alpha_0)) > 0$

$\Longleftarrow$ the multiplier $\lambda(\alpha_0)$ of the periodic cycle in the critical orbit of f_{d,α_0} is v-adically close to $\zeta_{n/\ell}$

$\Longleftarrow$ $v(\Phi_{n/\ell}(\lambda(\alpha_0))) > 0$

The multiplier polynomial

Definition

Let $d, m \geq 2$ and $n \geq 1$ be integers, and let $\zeta \neq 1$ be a d-th root of unity. The *multiplier polynomial* associated with $G^{\zeta}_{d,m,n}$ is

$$P^{\zeta}_{d,m,n}(x) = \prod_{j} \left(x - \lambda(c_j)\right) \in \mathbb{Z}[\zeta][x],$$

where $\lambda(c_j) = \displaystyle\prod_{i=m}^{m+n-1} d(a_i(c_j))^{d-1}$ is the multiplier of the periodic cycle of f_{d,c_j}.

Conjecture

For d, m, n, ζ as above, and for every $i \geq 1$, the resultant $\mathrm{Res}(P^{\zeta}_{d,m,n}, \Phi_i)$ is **not** *a unit in $\mathbb{Z}[\zeta]$.*

The case $d = 2$: the quadratic family $f_c(z) = z^2 + c$

$\zeta^2 = 1$ with $\zeta \neq 1$ means $\zeta = -1$, so write $G_{m,n}$ instead of $G^{\zeta}_{d,m,n}$

Prior results (Goksel et al.):
$G_{m,n}$ is irreducible for all $m \geq 1$ and $1 \leq n \leq 3$.

Theorem (B., Goksel)

If $f = x^k + A_{k-1}x^{k-1} + \cdots + A_0 \in \mathbb{Z}[x]$ is 2-special, i.e.,

- $v_2(A_{k-1}) > v_2(2)$, *and*
- $v_2(A_j) > v_2(A_{k-1})$ *for* $j = 0, 1, \ldots, k-2$,

then for all $\ell \geq 1$, we have $|\mathrm{Res}(f, \Phi_\ell)| > 1$.

Theorem (B., Goksel)

For every $m \geq 2$, both $P_{m,1}$ and $P_{m,2}$ are 2-special.

Combining various implications

Corollary

Let $d = 2$*, and let* $m \geq 2$*. For* $n = 1, 2, 3$*:*
For any root c_0 *of* $G_{m,n}$*, and any integer* $1 \leq \ell \leq n$*,*

$$G_{m,\ell}(c_0) \text{ is a unit} \quad \Longleftrightarrow \quad \ell \nmid n$$