A Beginner's Introduction to the Mandelbrot Set

Robert L. Benedetto

Amherst College

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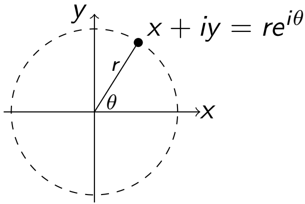
The Complex Plane

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\} = \{re^{i\theta} : r, \theta \in \mathbb{R}\}$$

where $re^{i\theta} = (r\cos\theta) + i(r\sin\theta)$.



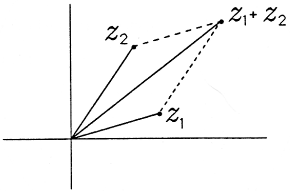
If $z = x + iy = re^{i\theta} \in \mathbb{C}$, we say:

- $x = \operatorname{Re} z$ is the **real part** of z,
- $y = \operatorname{Im} z$ is the **imaginary part** of z,
- $r = |z| = \sqrt{x^2 + y^2}$ is the **modulus** of z,
- $\theta = \arg z$ is the **argument** of z.

Arithmetic in $\mathbb{C}$

Addition:

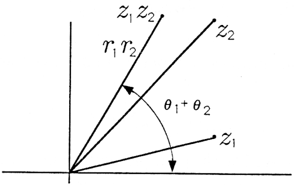
Vector-style: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$



Multiplication:

Multiply moduli (lengths); add arguments (angles):

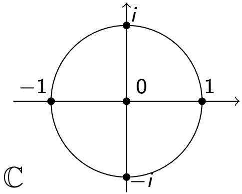
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

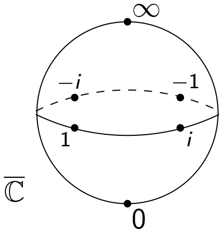


The Riemann Sphere

The "Riemann sphere" is the set

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$





Think of $\overline{\mathbb{C}}$ as the surface of a sphere, so points with large absolute value are "close to ∞."

Example: 1000, -1000, $1000i$, $-1000i$ are all very close to each other in $\overline{\mathbb{C}}$, even though they are very far apart in $\mathbb{C}$.

Dynamics of Polynomials

Let $f(z)$ be a polynomial of degree $d \geq 2$. So

$$f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

Write

$$f^1(z) = f(z), \qquad f^2(z) = f \circ f(z),$$
$$f^3(z) = f \circ f \circ f(z), \qquad \text{etc.}$$

Example. $f(z) = z^2$. Then $f^2(z) = z^4$, $f^3(z) = z^8$, $f^4(z) = z^{16}$, and in general, $f^n(z) = z^{(2^n)}$.

Example. $f(z) = z^2 + 1$. Then

- $f^2(z) = (z^2 + 1)^2 + 1 = z^4 + 2z^2 + 2$,
- $f^3(z) = (z^4 + 2z^2 + 2)^2 + 1 = z^8 + 4z^6 + 8z^4 + 8z^2 + 5$,
- $f^n(z) = z^{(2^n)} +$ big mess.

Periodic Points

Definition

A **fixed point** of f is a point $z_0 \in \overline{\mathbb{C}}$ such that $f(z_0) = z_0$.

Example. If $f(z) = z^2$, then 0, 1, and ∞ are fixed points of f.

(And that's it, since any fixed point besides ∞ must satisfy $f(z) = z$, which means $z^2 - z = 0$.)

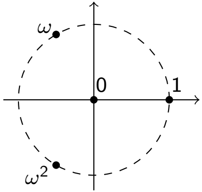
Definition

More generally, a **periodic point** of f of period $n \geq 1$ (a.k.a an n-**periodic point**) is a point $z_0 \in \overline{\mathbb{C}}$ such that $f^n(z_0) = z_0$.

The smallest positive integer n such that $f^n(z_0) = z_0$ is the **(exact) period** of z_0.

2-Periodic Points of z^2

Example. $\omega = e^{2\pi i/3} = \dfrac{-1 + i\sqrt{3}}{2}$ satisfies $\omega^3 = 1$.



As a result, ω is a 2-periodic point of $f(z) = z^2$:

We see $f(\omega) = \omega^2 = e^{4\pi i/3} = \dfrac{-1 - i\sqrt{3}}{2}$, $f^2(\omega) = \omega^4 = \omega$.

We say $\{\omega, \omega^2\}$ is a **2-cycle**.

To find them: Solving $f^2(z) = z$ gives $z^4 = z$,
i.e., [$z = \infty$ or] $\quad z(z-1)(z^2 + z + 1) = 0$,
i.e., $z = \infty, 0, 1, \omega, \omega^2$.

Some Periodic Points of $z^2 - 1$

Example. If $f(z) = z^2 - 1$, then the fixed points are ∞ and the roots of $z^2 - z - 1 = 0$, which means $\infty, \dfrac{1 \pm \sqrt{5}}{2}$.

To find the 2-periodic points, we solve $f^2(z) = z$:

$$(z^2 - 1)^2 - 1 = z,$$

that is, $z^4 - 2z^2 - z = 0$, which factors as $(z^2 - z - 1)(z^2 + z) = 0$.

Discard $z^2 - z - 1$ (those were fixed points, not 2-periodic points), and the only 2-periodic points are 0 and -1.

Sure enough, $f(0) = -1$ and $f(-1) = 0$. So $\{0, -1\}$ is a 2-cycle.

Classifying Periodic Points

Consider $f(z) = z^2$ near the fixed points at 0 and 1.

For z near 0 (say, $|z| < 1$), then $f(z)$ is **even closer** to 0.
(I.e., $|f(z)| < |z|$.)

For z near 1 (say, $|z - 1| < 1/2$), then $f(z)$ is **farther away** from 1.
(I.e., $|f(z) - 1| > |z - 1|$.)

What's going on?

More generally, if $f(a) = a$, let $\lambda = f'(a)$. The Taylor series is:

$$f(z) = a + \lambda(z - a) + c_2(z - a)^2 + c_3(z - a)^3 + \cdots .$$

So for z close to a (i.e., $|z - a|$ small):

$$f(z) - a \approx \lambda(z - a).$$

Multipliers of Periodic Points

Definition

Let f be a polynomial, and let $a \in \overline{\mathbb{C}}$ be a periodic point of exact period $n \geq 1$. The **multiplier** of a is

$$\begin{aligned} \lambda &= (f^n)'(a) \\ &= (f \circ f \circ \cdots \circ f)'(a) \\ &= [f'(a)] \cdot [f'(f(a))] \cdot [f'(f^2(a))] \cdot \cdots \cdot [f'(f^{n-1}(a))]. \end{aligned}$$

If $|\lambda| < 1$, we say a is **attracting**.

If $|\lambda| > 1$, we say a is **repelling**.

If $|\lambda| = 1$, we say a is **indifferent**.

Recall: For z close to a, $\left|f^n(z) - a\right| \approx |\lambda| \, |z - a|$.

Examples

Example. For $f(z) = z^2$,
0 is an attracting fixed point (since $f'(0) = 0$, and $|0| < 1$),
and
1 is a repelling fixed point (since $f'(1) = 2$, and $|2| > 1$).

(Note: ∞ is also attracting, for any polynomial of degree ≥ 2.)

Example. For $f(z) = z^2 - 1$,
$\{0, -1\}$ is an attracting 2-cycle,

because $f'(0) = 0$ and $f'(-1) = -2$,
so that $(f^2)'(0) = (f^2)'(-1) = 0 \cdot (-2) = 0$.

Fatou and Julia Sets

Definition

Let f be a polyomial. The **Fatou set** $\mathcal{F}$ of f is

$$\{z \in \overline{\mathbb{C}} : \text{there is a disk } D \ni z \text{ s.t. if } w_1, w_2 \in D, \text{ then } \forall n \geq 1,\ f^n(w_1) \text{ is close to } f^n(w_2)\}$$

The complement is the **Julia set** $\mathcal{J} = \overline{\mathbb{C}} \setminus \mathcal{F}$.

Fact: All attracting periodic points are in the Fatou set, and all repelling periodic points are in the Julia set.

Example: The Fatou and Julia Sets of z^2

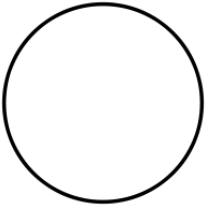
Example. $f(z) = z^2$:

If $|z| < 1$, then $f^n(w) \to 0$ for every nearby w. So $\{z \in \mathbb{C} : |z| < 1\} \subseteq \mathcal{F}$.

If $|z| > 1$, then $f^n(w) \to \infty$ for every nearby w. So $\{z \in \mathbb{C} : |z| > 1\} \subseteq \mathcal{F}$.

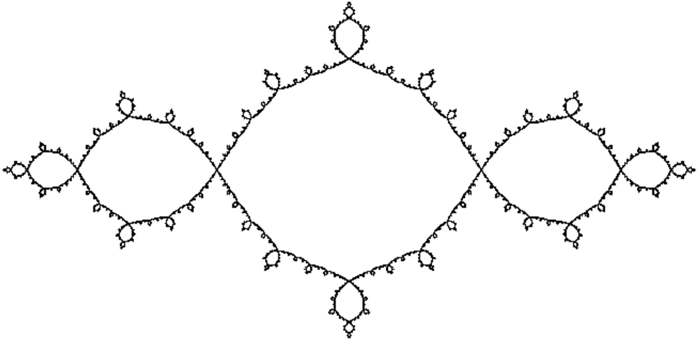
If $|z| = 1$, then: uh-oh.

So $\mathcal{J}$ is the unit circle:



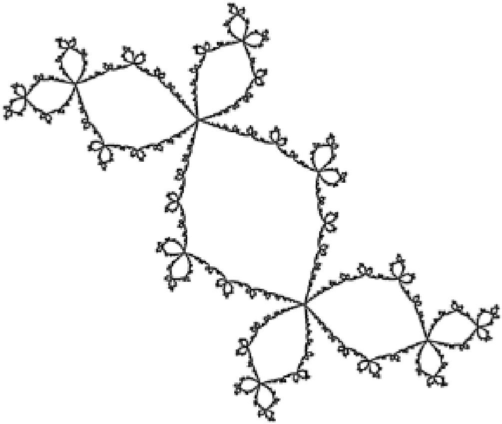
The Julia Set of $f(z) = z^2 + 1$

The Julia Set of $f(z) = z^2 - 1$



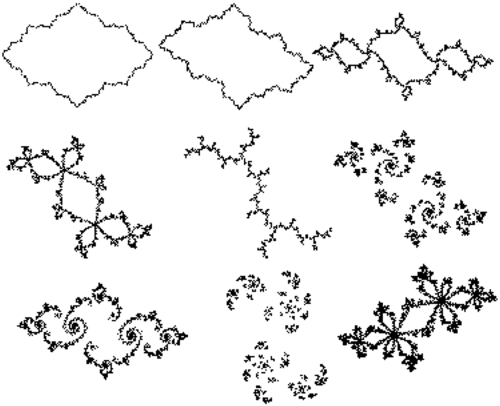
(“The Basilica”)

The Julia Set of $f(z) = z^2 + (.123 + .745i)$



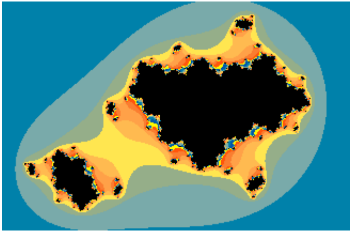
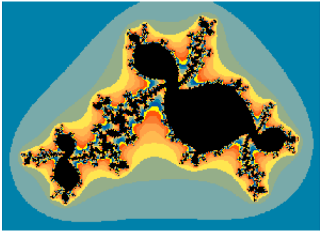
(Douady's "Rabbit")

The Julia Sets of $f(z) = z^2 + c$ for Various c



$c = -.5$	$c = -.5 + .3i$	$c = -1 + .16i$
$c = -.12 + .765i$	$c = i$	$c = -.3 + .71i$
$c = -.775 + .177i$	$c = .44 + .29i$	$c = -.513 - .579i$

Julia Sets of Some Cubic Polynomials


More Facts about Complex Dynamics

1. Points in $\mathcal{F}$ map to $\mathcal{F}$,
   and points in $\mathcal{J}$ map to $\mathcal{J}$.

2. If $f(z)$ has an attracting periodic point a, then **there must be a critical point b whose iterates $f^n(b)$ are attracted to a.**

3. (Special case of (2) for quadratic polynomials):
   Suppose $f_c(z) = z^2 + c$.
   Then besides the attracting fixed point at ∞,
   f_c has **at most one attracting periodic cycle in $\mathbb{C}$.**
   (f_c has only one critical point, at $z = 0$.)

From now on, let's only consider $f_c(z) = z^2 + c$.

The orbit of the critical point 0

Note: If $f_c(z) = z^2 + c$ has an attracting cycle (besides ∞), then it attracts 0, so

$$\{f_c^n(0) : n \geq 1\}$$

is a **bounded** set.

Note: Lots of other f_c have this property, too.

Example: $f_{-2}(z) = z^2 - 2$ has $0 \mapsto -2 \mapsto 2 \mapsto 2$

Example: $f_i(z) = z^2 + i$ has $0 \mapsto i \mapsto i - 1 \mapsto -i \mapsto i - 1$

BUT: $f_1(z) = z^2 + 1$ does **not**, since:
$0 \mapsto 1 \mapsto 2 \mapsto 5 \mapsto 26 \mapsto 677 \mapsto \cdots$

The Mandelbrot Set

Recall $f_c(z) = z^2 + c$.

Definition

The **Mandelbrot set** is

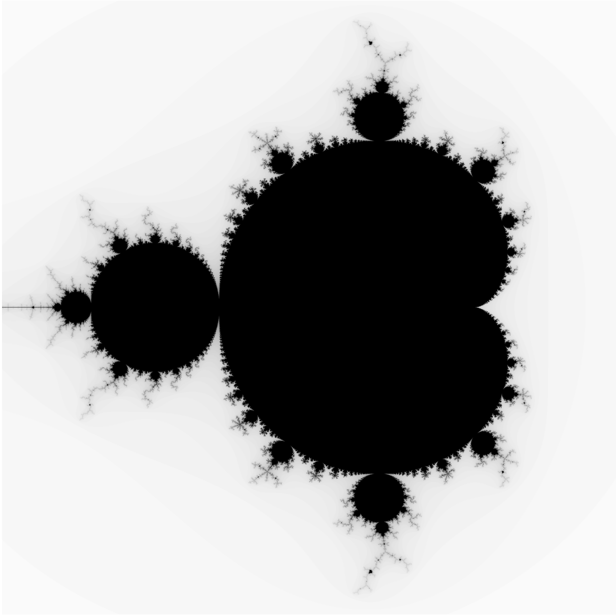
$$\mathcal{M} = \{c \in \mathbb{C} : \{f_c^n(0) : n \geq 1\} \text{ is bounded}\}.$$

(Benoit Mandelbrot, 1980)

Facts:

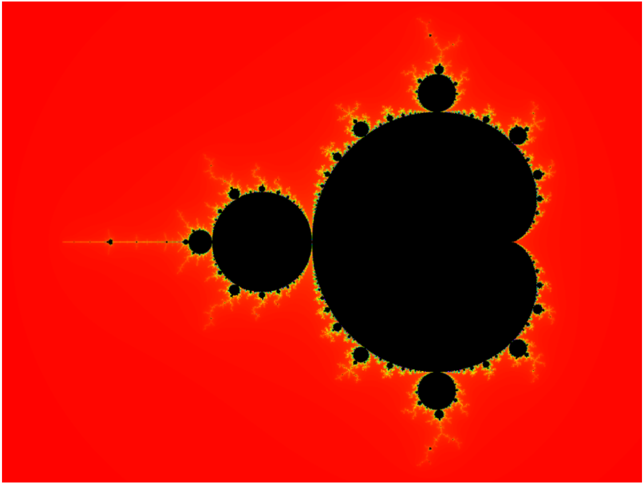
1. The Julia set $\mathcal{J}$ of f_c is connected **if and only if** $c \in \mathcal{M}$.
2. For every $c \in \mathcal{M}$, $|c| \leq 2$.
3. $\mathcal{M}$ is connected.
   (**Hard** Theorem: Douady and Hubbard, 1984.)

The Mandelbrot Set



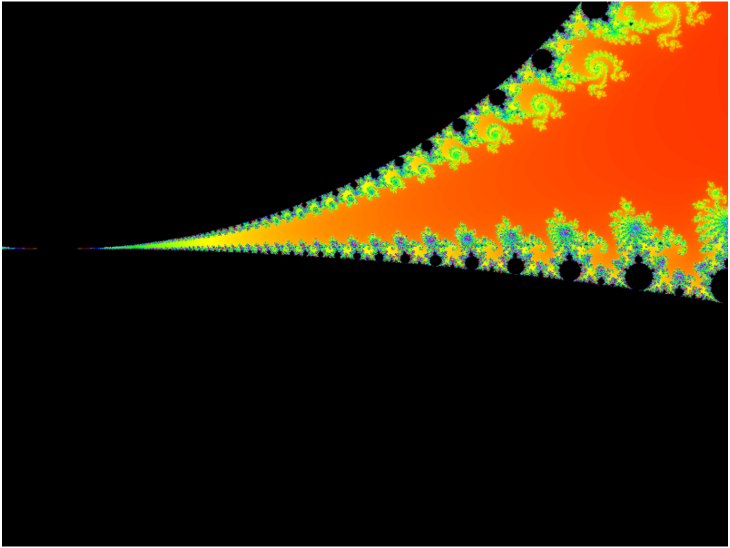
Zooming in: $-2.2 < x < 0.8$ and $-1.2 < y < 1.2$

Using Mandebrot viewer/explorer at:
`http://math.hws.edu/eck/js/mandelbrot/MB.html`

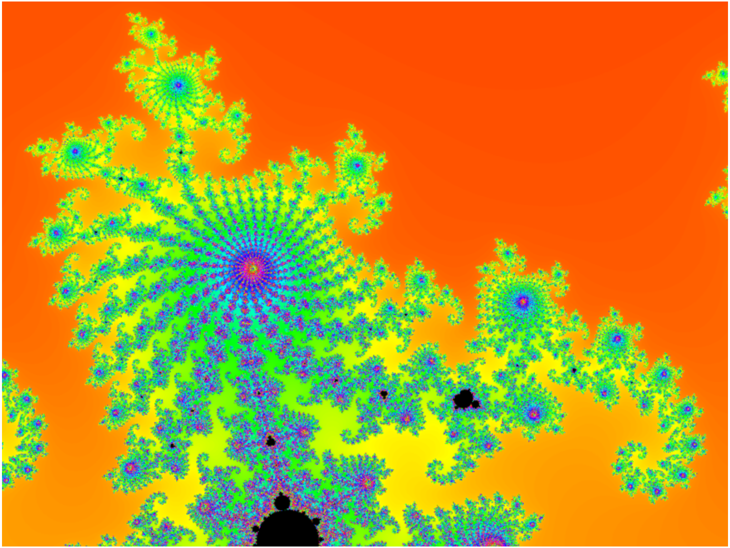


(credits: David Eck at Hobart and William Smith)

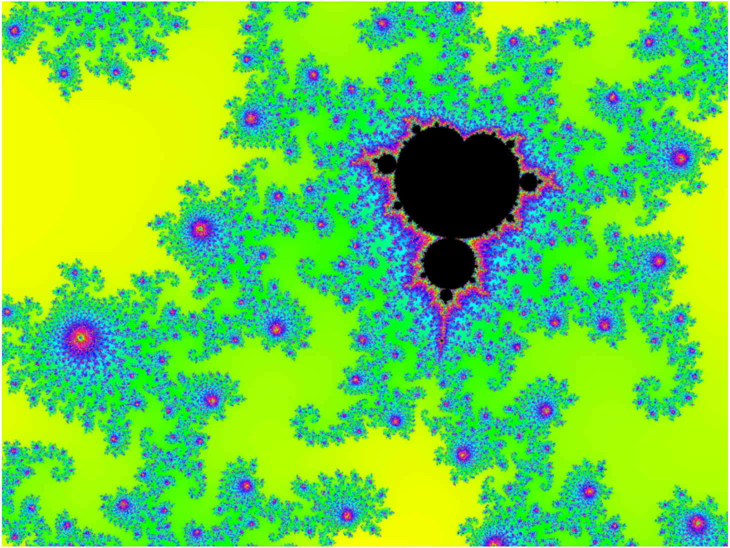
$-.132 < x < -.032$ and $.608 < y < .684$ (30X)



$-.0572 < x < -.0477$ and $.6483 < y < .6554$ (300X)



$-.04985 < x < -.04958$ and $.65044 < y < .65064$ (11000X)



Two Big Open Questions

1. Let

$$\mathcal{H} = \{c \in \mathbb{C} : f_c \text{ has an attracting cycle in } \mathbb{C}\}.$$

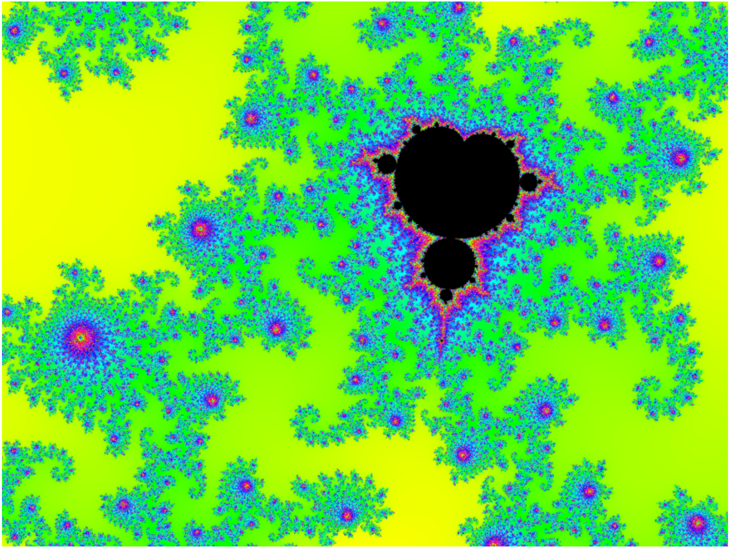
Big Conjecture: $\mathcal{H}$ is dense in $\mathcal{M}$.

The density of $\mathcal{H}$ would be implied by another
Big Conjecture: $\mathcal{M}$ is locally connected.

2. What is the area of the boundary $\partial\mathcal{M}$?

[Shishikura (1994) showed $\partial\mathcal{M}$ has Hausdorff dimension 2.]

$-.04985 < x < -.04958$ and $.65044 < y < .65064$



Where are the c's with Attracting Fixed Points?

Let's compute, for $f_c(z) = z^2 + c$, the set:

$$\mathcal{H}_1 = \{c \in \mathbb{C} : f_c \text{ has an attracting fixed point}\}.$$

---

1. The fixed points are roots of $z^2 - z + c = 0$. That means

$$z = \frac{1 \pm \sqrt{1-4c}}{2}.$$

Write $1 - 4c = re^{i\theta}$, so that $c = \frac{1}{4}(1 - re^{i\theta})$. $(r \geq 0.)$ So

$$z = \frac{1}{2}(1 \pm \sqrt{r}e^{i\theta/2}).$$

2. To be attracting, one must have $|f_c'(z)| < 1$. That is, $|2z| < 1$, which is to say

$$\left|1 \pm \sqrt{r}e^{i\theta/2}\right| < 1.$$

(for at least one choice of $+$ or $-$.)

3. Writing $e^{i\theta} = \cos\theta + i\sin\theta$, $\quad |1 \pm \sqrt{r}e^{i\theta/2}| < 1$ means

$$1 > \left| \left(1 \pm \sqrt{r}\cos\frac{\theta}{2}\right) \pm i\sqrt{r}\sin\frac{\theta}{2} \right|.$$

Squaring, that is

$$1 > \left(1 \pm \sqrt{r}\cos\frac{\theta}{2}\right)^2 + r\sin^2\frac{\theta}{2} = 1 \pm 2\sqrt{r}\cos\frac{\theta}{2} + r,$$

so that

$$r < \mp 2\sqrt{r}\cos\frac{\theta}{2},$$

for at least one of $-$ or $+$.

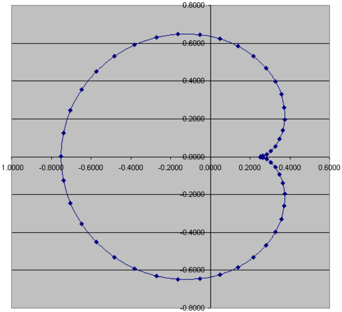
4. Squaring $r < \mp 2\sqrt{r}\cos(\theta/2)$ gives

$$r^2 < 4r\cos^2\frac{\theta}{2} = 2r(1+\cos\theta),$$

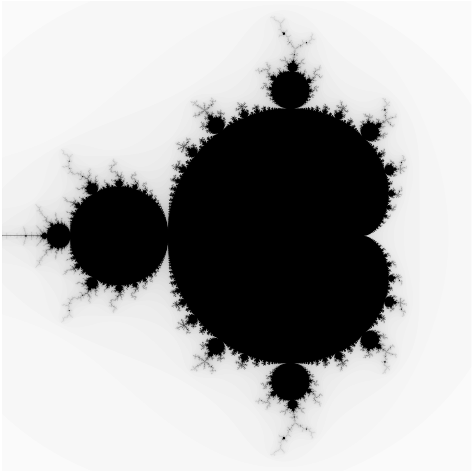
or in other words, $r < 2(1+\cos\theta)$.

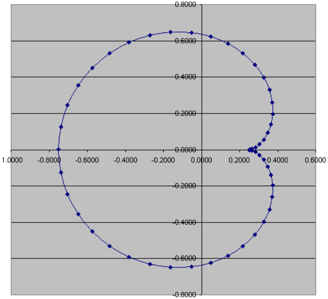
That means $re^{i\theta}$ is inside the cardioid $r = 2(1+\cos\theta)$.

And **that** means $c = \frac{1}{4}(1 - re^{i\theta})$ is inside this cardioid:



Reminder of The Mandelbrot Set





Where are the c's with Attracting 2-cycles?

Let's compute

$$\mathcal{H}_2 = \{c \in \mathbb{C} : f_c \text{ has an attracting 2-cycle}\}.$$

---

1. $f_c^2(z) = z^4 + 2cz^2 + (c^2 + c)$, so

$$f_c^2(z) - z = z^4 + 2cz^2 - z + (c^2 + c) = (z^2 - z + c)(z^2 + z + (c + 1)).$$

The first factor is the fixed points, so we discard it.
Thus, the 2-periodic points are the two roots of

$$z^2 + z + (c + 1) = 0.$$

2. We compute

$$(f_c^2)'(z) = 4z^3 + 4cz = 4zf(z).$$

If z is a 2-periodic point, so that $z^2 + z + c + 1 = 0$, we get $f(z) = z^2 + c = -z - 1$, so

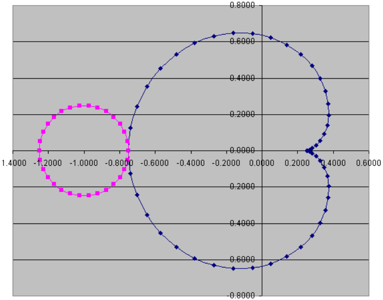
$$(f_c^2)'(z) = 4z(-z-1) = -4(z^2+z) = 4(c+1).$$

3. So $f_c(z) = z^2 + c$ has an attracting 2-cycle if $|4(c+1)| < 1$.

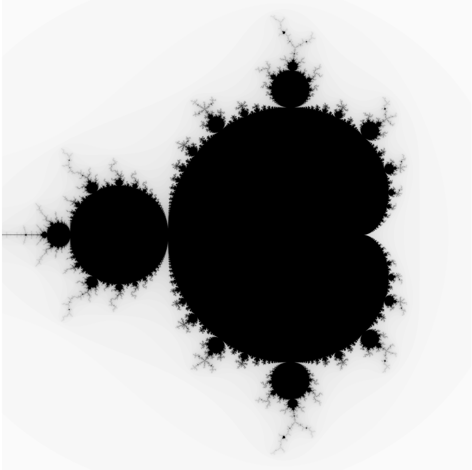
If we write $c = a + bi$ and square, this means

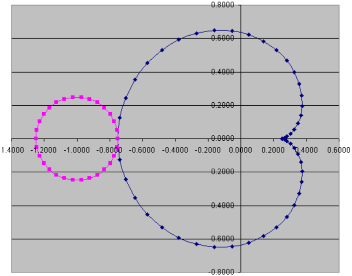
$$(a+1)^2 + b^2 < \left(\frac{1}{4}\right)^2,$$

which means c is inside the circle of radius $1/4$ centered at -1:

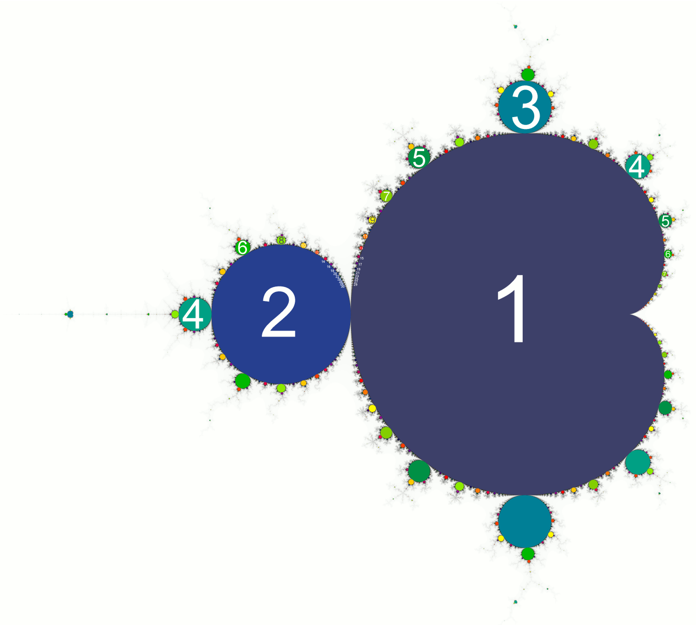


Another Reminder of The Mandelbrot Set

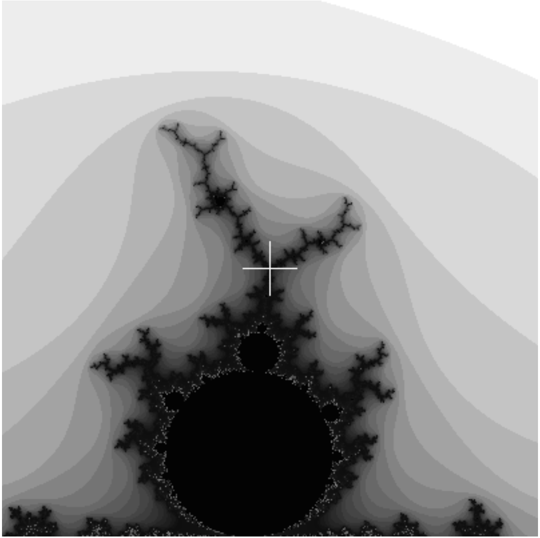




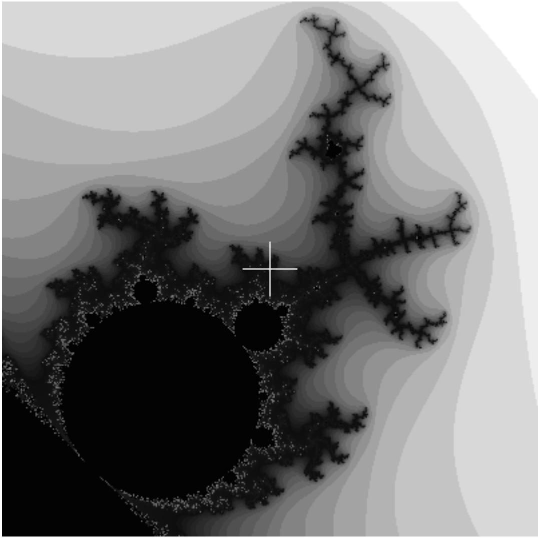
Periods of Some Other Bulbs



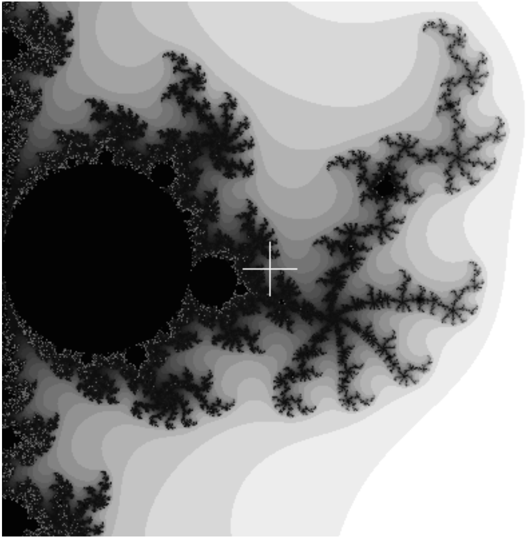
The 3-bulb

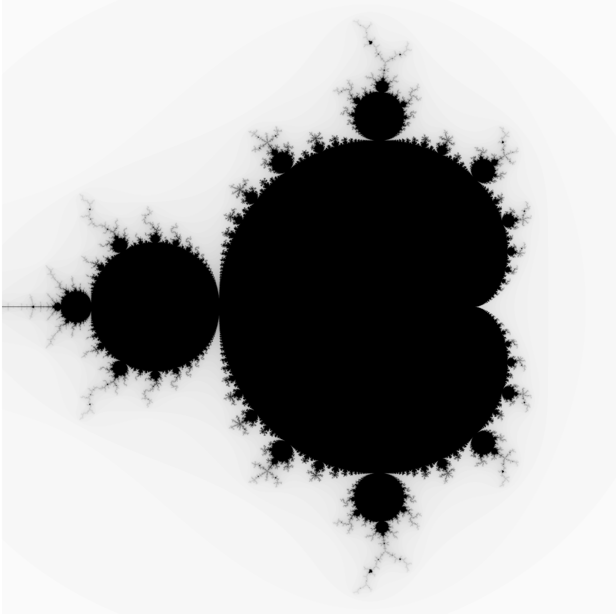


A 4-bulb

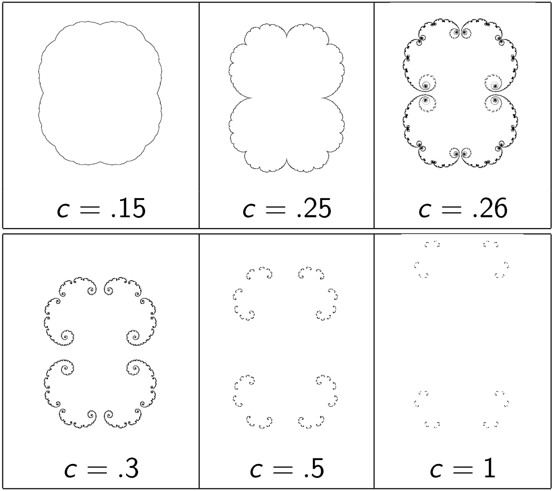


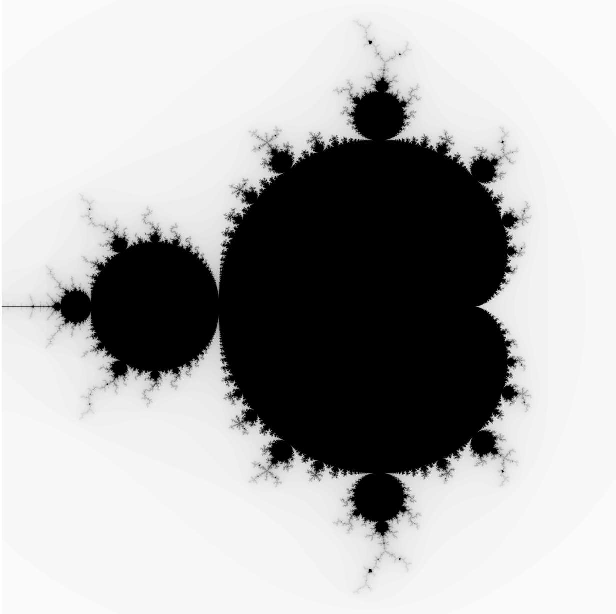
A 6-bulb



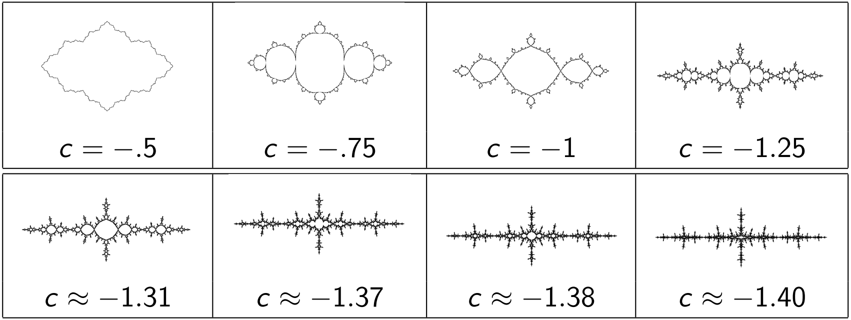


Moving Right out of the Cardioid: $f_c(z) = z^2 + c$

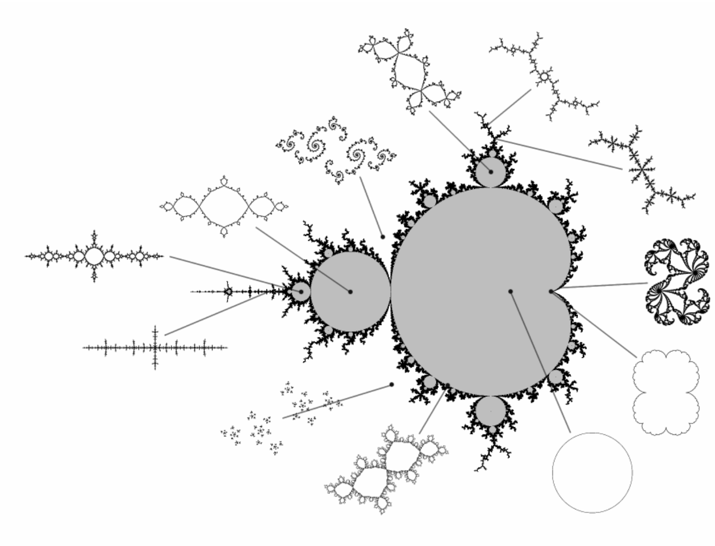




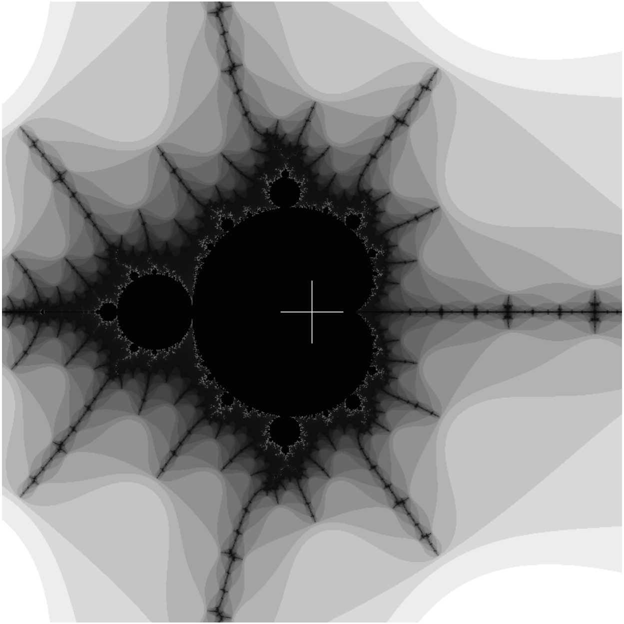
Moving Left out of the Cardioid: $f_c(z) = z^2 + c$



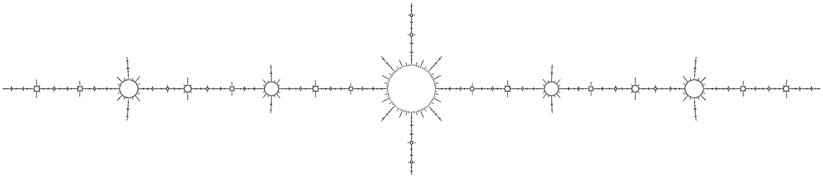
Julia Sets for Some Specific Parameters



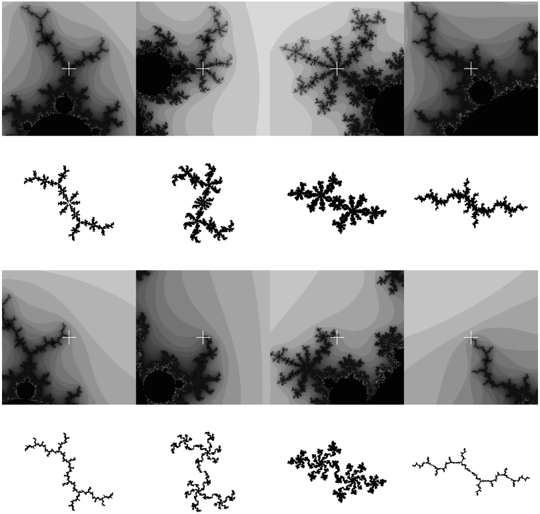
A Closeup of $\mathcal{M}$ near $c = -1.755$



The Airplane Julia Set: $c \approx -1.755$



Comparing Julia Sets to the Mandelbrot Set



(Tan's Theorem, 1990)

Mandelbrot's Picture of the Mandelbrot Set

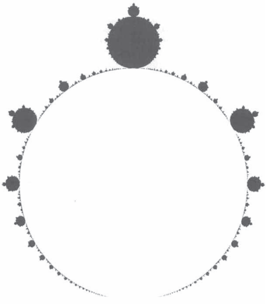


FIGURE 1. Complex plane map of the λ-domain Q. The real axis of the λ-plane points up from $\lambda = 1$. The center of the circle is $\lambda = 2$ and the tip of the whole is $\lambda = 4$.

1, the remainder of Q being symmetric to this figure with respect to the line $\mathrm{Re}(\lambda) = 1$.

A striking fact, which I think is new, becomes apparent here: FIGURE 1 is made of several disconnected portions, as follows.

The Domain of Confluence $\mathcal{L}$, and Its Fractal Boundary

The most visible feature of FIGURE 1 is the large connected domain $\mathcal{L}$ surrounding $\lambda = 2$. This $\mathcal{L}$ splits into a sequence of subdomains one can introduce in successive stages.

$$g_\lambda(z) = \lambda z(1 - z)$$

"A striking fact, which I think is new, becomes apparent here: FIGURE 1 is made of several disconnected portions, as follows."