Arboreal Galois groups with colliding critical points

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Notation

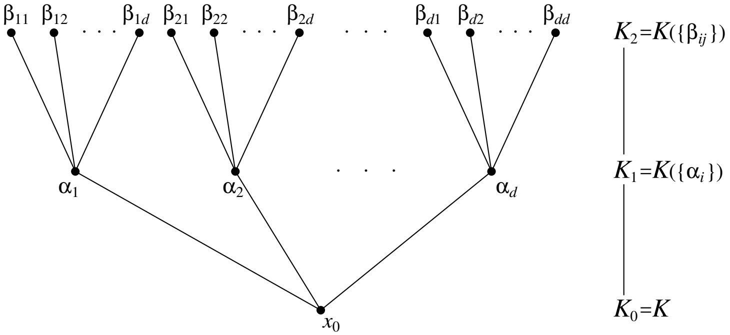
- K is a field, usually a number field
- $\overline{K}$ is the algebraic closure of K
- $f \in K(z)$ is a rational function of degree $d \geq 2$
- $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$ is the n-th iterate of f
- $f^{-n}(x_0)$ is the set of roots of $f^n(z) = x_0$.

Goal: Given $x_0 \in \mathbb{P}^1(K) = K \cup \{\infty\}$,
to understand the action of Galois on the backward orbit

$$\mathrm{Orb}_f^-(x_0) := \{x_0\} \cup f^{-1}(x_0) \cup f^{-2}(x_0) \cup \cdots$$

A Tower of Extension Fields

For each $n \geq 0$, let $K_n = K\big(f^{-n}(x_0)\big)$ and $G_n = \mathrm{Gal}(K_n/K)$.



We call the groups G_n *arboreal Galois groups*.

Note $G_n \subseteq \mathrm{Aut}(T_{d,n})$, where $T_{d,n}$ is a d-ary rooted tree of n levels.

How big is G_n in $\mathrm{Aut}(T_{d,n})$?

Recall: $K_n = K\big(f^{-n}(x_0)\big)$ and $G_n = \mathrm{Gal}(K_n/K)$.

$T_{d,n}$ is a d-ary rooted tree with n levels, so $G_n \subseteq \mathrm{Aut}(T_{d,n})$.

Expectation:
$[\mathrm{Aut}(T_{d,n}) : G_n]$ is bounded as $n \to \infty$,
unless there is an obvious reason not.

One such "obvious" reason is that f **is PCF**:

Definition

$f(z)$ is **postcritically finite**, or **PCF**, if every critical point of f has finite forward orbit.

Colliding Critical Points

Another "obvious" reason $[\mathrm{Aut}(T_{d,n}) : G_n]$ could be unbounded:

Definition

Let $f \in K(z)$ be a separable map, let $\gamma_1, \gamma_2 \in \mathbb{P}^1(\overline{K})$ be critical points of f, and let $\ell \geq 1$ be an integer.
We say that γ_1 and γ_2 *collide* (at ℓ iterations) if

$$f^\ell(\gamma_1) = f^\ell(\gamma_2), \text{ but } f^i(\gamma_1) \neq f^i(\gamma_2) \text{ for } 0 \leq i < \ell.$$

$$\begin{array}{ccccccccc}
\gamma_1 \longmapsto f(\gamma_1) \longmapsto \cdots \longmapsto f^{\ell-1}(\gamma_1) & \searrow & & & \\
& & f^\ell(\gamma_1) = f^\ell(\gamma_2) & \longmapsto & \cdots \\
\gamma_2 \longmapsto f(\gamma_2) \longmapsto \cdots \longmapsto f^{\ell-1}(\gamma_2) & \nearrow & & &
\end{array}$$

Pink (2013) observed that if f has only two critical points, and if these critical points collide at $\ell \geq 1$ iterations, then $[\mathrm{Aut}(T_{d,n}) : G_n]$ is unbounded.

Iterated Discriminants

Recall if $P(z) = a_d z^d + \cdots + a_0 = a_d \prod_{i=1}^{d}(z - \alpha_i) \in K[z]$, the *discriminant* of P is

$$\Delta(P) = a_d^{2d-2} \prod_{i<j} (\alpha_i - \alpha_j)^2 \in K.$$

Theorem (Jones, Manes, 2012; variant)

Let $f(z) \in K(z)$ of degree $d \geq 2$.
Let $\gamma_1, \ldots, \gamma_{2d-2} \in \mathbb{P}^1(\overline{K})$ be the critical points of f.
Write $f^n(z) = \dfrac{p_n(z)}{q_n(z)}$ with $p_n, q_n \in K[z]$ relatively prime.
For any root point x_0, let $\Delta_n := \Delta(p_n - x_0 q_n)$.
Then for each $n \geq 1$, we have

$$\Delta_n = c_n \, \Delta_{n-1}^d \prod_{i=1}^{2d-2} \big(f^n(\gamma_i) - x_0\big).$$

Here, c_n is a fudge factor involving various lead coefficients, resultants, etc.

Quadratic rational maps with colliding critical points

[Assume char $K \neq 2$.]

Fix $\ell \geq 2$. Let $f \in K(z)$ with $\deg f = 2$ and such that the two critical points γ_1, γ_2 collide at the ℓ-th iteration.

(The moduli space of such maps is one-dimensional.)

Then (by iterated discriminant formula), any $\sigma \in G_n = \mathrm{Gal}(K_n/K)$ acts as an **even permutation** on the 2^ℓ points of $f^{-\ell}(x_0)$.

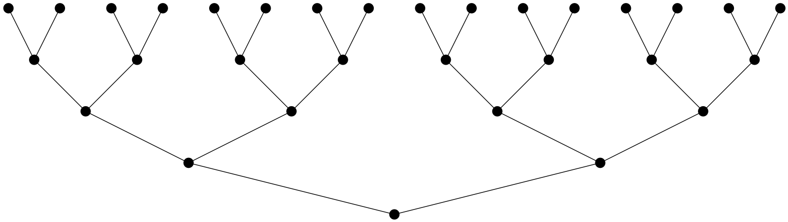
But this parity restriction applies ℓ levels above **every** node of $T_{2,n}$.

Let $M_{\ell,n}$ be the subgroup of $\mathrm{Aut}(T_{2,n})$ carved out by this (repeated) parity restriction.

So $G_n \subseteq M_{\ell,n}$.

Special Case: $d = 2$ and $\ell = 2$

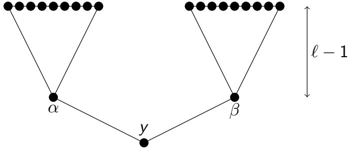
$$G_n = \mathrm{Gal}(K_n/K) \subseteq M_{\ell,n} \subseteq \mathrm{Aut}(T_{2,n})$$



$M_{\ell,n}$ is the subgroup of $\mathrm{Aut}(T_{2,n})$ determined by:

For each node y of the tree, each $\sigma \in M_{\ell,n}$ acts as an even permutation of the 2^ℓ nodes lying ℓ levels above y.

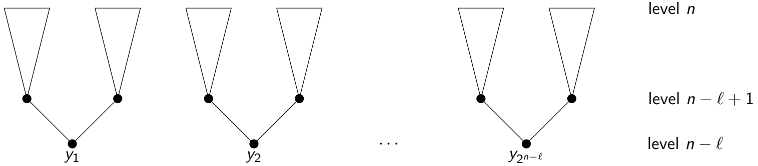
Odd Cousins (at level $\ell \geq 2$). [Joint with Anna Dietrich]



Fix a node y in the tree. We say σ

- acts *positively* above y if σ is even above both α and β.
- acts *negatively* above y if σ is odd above both α and β.

Odd Cousins Maps



Definition

For $\sigma \in M_{\ell,n}$, if σ acts negatively above an odd number of nodes at level $n - \ell$, we say σ is an **odd-cousins map** at level n.

Colliding critical points for $\deg(f) = 2$

$$K_n = K(f^{-n}(x_0)), \quad G_n = \mathrm{Gal}(K_n/K), \quad M_{\ell,n} \subseteq \mathrm{Aut}(T_{2,n})$$

Theorem (RB, Dietrich, 2023*)

Let K be a field with $\mathrm{char}\, K \neq 2$, let $x_0 \in \mathbb{P}^1(K)$, and let $f \in K(z)$ with $\deg(f) = 2$ such that the two critical points of f collide at the ℓ-th iterate, where $\ell \geq 2$.
Assume x_0 is not periodic and not postcritical.
There are $\kappa_n \in K$, given by explicit expressions involving f and x_0, such that the following are equivalent:

1. $G_n = M_{\ell,n}$ *for all* $n \geq 1$.
2. *No product* $\kappa_{i_1} \cdots \kappa_{i_m}$ *(for* $i_1 < \cdots < i_m$*) is a square in* K.

Idea: For $n \geq \ell$, κ_n is a square in K if and only if there are no odd cousins maps in $\mathrm{Gal}(K_n/K)$.

Cubic polynomials with colliding critical points [REU 2022]

Let $f \in K[z]$ with $\deg f = 3$ and such that the two critical points γ_1, γ_2 collide **at the 2nd iteration**. (We fix $\ell = 2$.)

The moduli space of such maps is one-dimensional:
$f(z) = Az^3 + Bz + 1$, with $A = \frac{1}{81}(4B^3 - 27B)$

Fix $x_0 \in K$ not periodic and not in the forward orbit of the critical points $\gamma_1, \gamma_2 = \pm\sqrt{-B/(3A)}$

This time, the discriminant formula for $\Delta_n := \Delta(f^n - x_0)$ gives:

$$\Delta_n = (\text{square in } K) \cdot (-3) \cdot \Delta_{n-1} \quad \text{for } n \geq \ell = 2.$$

So any $\sigma \in G_n = \mathrm{Gal}(K_n/K)$ acts with same parity at every even level of the tree, and same parity at every odd level.

So $G_n \subseteq Q_n$, the subgroup of $\mathrm{Aut}(T_{3,n})$ carved out by this parity restriction, above **every** node of the tree.

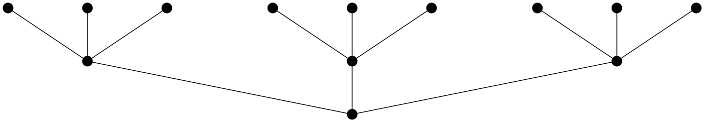
Building towards Aut($T_{3,2}$)

Recall $f(z) = Az^3 + Bz + 1 \in K[z]$, with root point $x_0 \in K$.

It's not hard to set conditions on K, A, B, x_0 to guarantee that

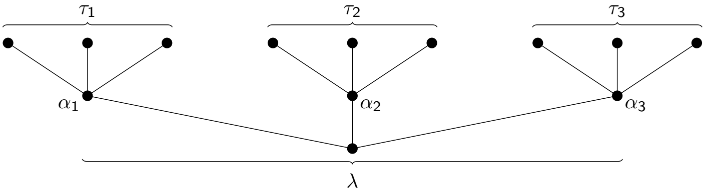
- $f^n(z) - x_0$ is irreducible over K for all $n \geq 1$,
- $[K(\sqrt{\Delta_1}, \sqrt{-3}) : K] = 4$

which gives $|G_2| \geq \frac{1}{4} \cdot 6^4$, vs. $|Q_2| = |\operatorname{Aut}(T_{3,2})| = 6^4$:



BUT G_2 is not necessarily all of $Q_2 = \operatorname{Aut}(T_{3,2})$

A proper subgroup of Aut($T_{3,2}$)



$$H = \{(\lambda, (\tau_1, \tau_2, \tau_3)) \mid \mathrm{sgn}(\tau_1) = \mathrm{sgn}(\tau_2) = \mathrm{sgn}(\tau_3)\}$$

Four conjugate subgroups: $H_1 = H, H_2, H_3, H_4$ of Aut($T_{3,2}$).

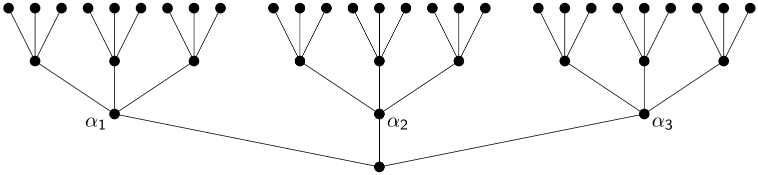
H_1 fixes $\delta_1 := \beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3$, where $\beta_i = \sqrt{\Delta(f - \alpha_i)}$

So we need $g(y) = (y - \delta_1)(y - \delta_2)(y - \delta_3)(y - \delta_4) \in K[y]$ to have no roots in K.

Levels 3 and higher

One can specify further local conditions to guarantee that g has no roots in K, (so $G_2 \not\subseteq H_i$), and hence that $G_2 = Q_2$.

But even if we also assume the same over each of the three subtrees $T_{3,2}$ based at α_1, α_2, α_3,
there's **another quartic obstacle** to G_3 being all of Q_3.



That (more complicated) quartic obstacle is similarly surmountable.

And then — **no more further conditions are needed**!

The Theorem

Let K be a number field, let $f(z) = Az^3 + Bz + 1 \in K[z]$ with $A = \frac{1}{81}(4B^3 - 27B)$, and let $x_0 \in K$. Let $G_n = \mathrm{Gal}(K(f^{-n}(x_0)/K)$

Condition ($\dagger$) on (K, x_0): There are primes $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ of K such that

1. $v_{\mathfrak{p}}(A) = v_{\mathfrak{p}}(B) = 0$ and $v_{\mathfrak{p}}(x_0) = -1$,
2. $6 | v_{\mathfrak{q}}(A) = v_{\mathfrak{p}}(B)$, with $-5v_{\mathfrak{q}}(A) \leq v_{\mathfrak{q}}(x_0) < 0$ and $v_{\mathfrak{q}}(x_0) \equiv \pm 1 \pmod 6$,
3. $v_{\mathfrak{r}}(A) = v_{\mathfrak{r}}(B) > 0 > v_{\mathfrak{r}}(x_0)$, with $v_{\mathfrak{r}}(a) \not\equiv v_{\mathfrak{r}}(x_0) \pmod 3$ and $4 \nmid v_{\mathfrak{r}}(x_0)$.
4. $[K(\sqrt{\Delta(f - x_0)}, \sqrt{-3}) : K] = 4$

Theorem (RB,DeGroot,Ni,Seid,Wei,Winton,2023)

With notation as above,

1. *If ($\dagger$) holds for (K, x_0), then it holds for $(K(\alpha), \alpha)$ for each $n \geq 1$ and each $\alpha \in f^{-n}(x_0)$.*
2. *If ($\dagger$) holds for (K, x_0), then $G_n = Q_n$ for all $n \geq 1$.*

Summary: Maps with 2 critical points colliding at iterate ℓ

Quadratic Rational (with Anna Dietrich):

- Description of group $M_{\ell,n} \subseteq \text{Aut}(T_{2,n})$ with $G_n \subseteq M_{\ell,n}$ for all $n \geq 1$.
- Sufficient condition to force $G_n = M_{\ell,n}$ for all $n \geq 1$. (Infinitely many "κ_n is not a square in K" conditions.)

Cubic Polynomial (with Will DeGroot, Xinyu Ni, Jesse Seid, Annie Wei, and Samantha (Min) Winton): For $\ell = 2$ only:

- Description of group $Q_n \subseteq \text{Aut}(T_{3,n})$ with $G_n \subseteq Q_n$ for all $n \geq 1$.
- Sufficient condition to force $G_n = Q_n$ for all $n \geq 1$. (Finite condition: existence of primes with certain properties, inherited up the tree.)