Arboreal Galois groups with colliding critical points

*Rob Benedetto, Will DeGroot, Anna Dietrich, Xinyu Ni, Jesse Seid, Annie Wei, Min Winton

Amherst College

Friday, January 6, 2023
Notation

- $K$ is a field, usually a number field
- $\overline{K}$ is the algebraic closure of $K$
- $f \in K(z)$ is a rational function of degree $d \geq 2$
- $f^n = f \circ f \circ \cdots \circ f$ is the $n$-th iterate of $f$
- $f^{-n}(x_0)$ is the set of roots of $f^n(z) = x_0$.

Goal: Given $x_0 \in \mathbb{P}^1(K) = K \cup \{\infty\}$, to understand the action of Galois on the backward orbit

$$\text{Orb}_f^-(x_0) := \{x_0\} \cup f^{-1}(x_0) \cup f^{-2}(x_0) \cup \cdots$$
A Tower of Extension Fields

For each \( n \geq 0 \), let \( K_n = K(f^{-n}(x_0)) \) and \( G_n = \text{Gal}(K_n/K) \).

We call the groups \( G_n \) arboreal Galois groups.

Note \( G_n \subseteq \text{Aut}(T_{d,n}) \), where \( T_{d,n} \) is a \( d \)-ary rooted tree of \( n \) levels.
How big is $G_n$ in $\text{Aut}(T_{d,n})$?

**Recall:** $K_n = K(f^{-n}(x_0))$ and $G_n = \text{Gal}(K_n/K)$.

$T_{d,n}$ is a $d$-ary rooted tree with $n$ levels, so $G_n \subseteq \text{Aut}(T_{d,n})$.

**Expectation:**

$[\text{Aut}(T_{d,n}) : G_n]$ is bounded as $n \to \infty$, **unless** there is an obvious reason not.

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One such “obvious” reason is that $f$ is **PCF**:

**Definition**

$f(z)$ is **postcritically finite**, or **PCF**, if every critical point of $f$ has finite forward orbit.
Colliding Critical Points

Another “obvious” reason \([\text{Aut}(T_{d,n}) : G_n]\) could be unbounded:

**Definition**
Let \(f \in K(z)\) be a separable map, let \(\gamma_1, \gamma_2 \in \mathbb{P}^1(K)\) be critical points of \(f\), and let \(\ell \geq 1\) be an integer.
We say that \(\gamma_1\) and \(\gamma_2\) **collide** (at \(\ell\) iterations) if
\[
f^\ell(\gamma_1) = f^\ell(\gamma_2), \text{ but } f^i(\gamma_1) \neq f^i(\gamma_2) \text{ for } 0 \leq i < \ell.
\]

Pink (2013) observed that if \(f\) has only two critical points, and if these critical points collide at \(\ell \geq 1\) iterations, then \([\text{Aut}(T_{d,n}) : G_n]\) is unbounded.
Iterated Discriminants

Recall if \( P(z) = a_dz^d + \cdots + a_0 = a_d \prod_{i=1}^{d}(z - \alpha_i) \in K[z] \), the \textit{discriminant} of \( P \) is
\[
\Delta(P) = a_d^{2d-2} \prod_{i<j}(\alpha_i - \alpha_j)^2 \in K.
\]

Theorem (Jones, Manes, 2012; variant)

Let \( f(z) \in K(z) \) of degree \( d \geq 2 \).

Let \( \gamma_1, \ldots, \gamma_{2d-2} \in \mathbb{P}^1(K) \) be the critical points of \( f \).

Write \( f^n(z) = \frac{p_n(z)}{q_n(z)} \) with \( p_n, q_n \in K[z] \) relatively prime.

For any root point \( x_0 \), let \( \Delta_n := \Delta(p_n - x_0q_n) \).

Then for each \( n \geq 1 \), we have
\[
\Delta_n = c_n \Delta_{n-1}^{d-2} \prod_{i=1}^{2d-2} (f^n(\gamma_i) - x_0).
\]

Here, \( c_n \) is a fudge factor involving various lead coefficients, resultants, etc.
Quadratic rational maps with colliding critical points

[Assume char $K \neq 2$.]

Fix $\ell \geq 2$. Let $f \in K(z)$ with $\deg f = 2$ and such that the two critical points $\gamma_1, \gamma_2$ collide at the $\ell$-th iteration.

(The moduli space of such maps is one-dimensional.)

Then (by iterated discriminant formula), any $\sigma \in G_n = \text{Gal}(K_n/K)$ acts as an even permutation on the $2^\ell$ points of $f^{-\ell}(x_0)$.

But this parity restriction applies $\ell$ levels above every node of $T_{2,n}$.

Let $M_{\ell,n}$ be the subgroup of $\text{Aut}(T_{2,n})$ carved out by this (repeated) parity restriction.

So $G_n \subseteq M_{\ell,n}$. 
Special Case: $d = 2$ and $\ell = 2$

\[ G_n = \text{Gal}(K_n/K) \subseteq M_{\ell,n} \subseteq \text{Aut}(T_{2,n}) \]

$M_{\ell,n}$ is the subgroup of $\text{Aut}(T_{2,n})$ determined by:

For each node $y$ of the tree, each $\sigma \in M_{\ell,n}$ acts as an even permutation of the $2^\ell$ nodes lying $\ell$ levels above $y$. 
Odd Cousins (at level $\ell \geq 2$). [Joint with Anna Dietrich]

Fix a node $y$ in the tree. We say $\sigma$

- acts *positively* above $y$ if $\sigma$ is even above both $\alpha$ and $\beta$.
- acts *negatively* above $y$ if $\sigma$ is odd above both $\alpha$ and $\beta$. 
Odd Cousins Maps

Definition
For $\sigma \in M_{\ell,n}$, if $\sigma$ acts negatively above an odd number of nodes at level $n - \ell$, we say $\sigma$ is an odd-cousins map at level $n$. 
Colliding critical points for \( \deg(f) = 2 \)

\[
K_n = K(f^{-n}(x_0)), \quad G_n = \text{Gal}(K_n/K), \quad M_{\ell,n} \subseteq \text{Aut}(T_{2,n})
\]

Theorem (RB, Dietrich, 2023*)

Let \( K \) be a field with \( \text{char} \ K \neq 2 \), let \( x_0 \in \mathbb{P}^1(K) \), and let \( f \in K(z) \) with \( \deg(f) = 2 \) such that the two critical points of \( f \) collide at the \( \ell \)-th iterate, where \( \ell \geq 2 \).

Assume \( x_0 \) is not periodic and not postcritical.

There are \( \kappa_n \in K \), given by explicit expressions involving \( f \) and \( x_0 \), such that the following are equivalent:

1. \( G_n = M_{\ell,n} \) for all \( n \geq 1 \).
2. No product \( \kappa_{i_1} \cdots \kappa_{i_m} \) (for \( i_1 < \cdots < i_m \)) is a square in \( K \).

Idea: For \( n \geq \ell \), \( \kappa_n \) is a square in \( K \) if and only if there are no odd cousins maps in \( \text{Gal}(K_n/K) \).
Cubic polynomials with colliding critical points [REU 2022]

Let \( f \in K[z] \) with \( \deg f = 3 \) and such that the two critical points \( \gamma_1, \gamma_2 \) collide at the 2nd iteration. (We fix \( \ell = 2 \).)

The moduli space of such maps is one-dimensional: \( f(z) = Az^3 + Bz + 1, \) with \( A = \frac{1}{81}(4B^3 - 27B) \)

Fix \( x_0 \in K \) not periodic and not in the forward orbit of the critical points \( \gamma_1, \gamma_2 = \pm \sqrt{-B/(3A)} \)

This time, the discriminant formula for \( \Delta_n := \Delta(f^n - x_0) \) gives:

\[
\Delta_n = (\text{square in } K) \cdot (-3) \cdot \Delta_{n-1} \quad \text{for } n \geq \ell = 2.
\]

So any \( \sigma \in G_n = \text{Gal}(K_n/K) \) acts with same parity at every even level of the tree, and same parity at every odd level.

So \( G_n \subseteq Q_n \), the subgroup of \( \text{Aut}(T_{3,n}) \) carved out by this parity restriction, above every node of the tree.
Building towards $\text{Aut}(T_{3,2})$

Recall $f(z) = Az^3 + Bz + 1 \in K[z]$, with root point $x_0 \in K$.

It’s not hard to set conditions on $K, A, B, x_0$ to guarantee that

- $f^n(z) - x_0$ is irreducible over $K$ for all $n \geq 1$,
- $[K(\sqrt{\Delta_1}, \sqrt{-3}) : K] = 4$

which gives $|G_2| \geq \frac{1}{4} \cdot 6^4$, vs. $|Q_2| = |\text{Aut}(T_{3,2})| = 6^4$:

BUT $G_2$ is not necessarily all of $Q_2 = \text{Aut}(T_{3,2})$
A proper subgroup of $\text{Aut}(T_{3,2})$

$$H = \left\{ (\lambda, (\tau_1, \tau_2, \tau_3)) \mid \text{sgn}(\tau_1) = \text{sgn}(\tau_2) = \text{sgn}(\tau_3) \right\}$$

Four conjugate subgroups: $H_1 = H, H_2, H_3, H_4$ of $\text{Aut}(T_{3,2})$.

$H_1$ fixes $\delta_1 := \beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3$, where $\beta_i = \sqrt{\Delta(f - \alpha_i)}$

So we need $g(y) = (y - \delta_1)(y - \delta_2)(y - \delta_3)(y - \delta_4) \in K[y]$ to have no roots in $K$. 
Levels 3 and higher

One can specify further local conditions to guarantee that $g$ has no roots in $K$, (so $G_2 \not\subset H_i$), and hence that $G_2 = Q_2$.

But even if we also assume the same over each of the three subtrees $T_{3,2}$ based at $\alpha_1$, $\alpha_2$, $\alpha_3$, there’s another quartic obstacle to $G_3$ being all of $Q_3$.

That (more complicated) quartic obstacle is similarly surmountable.

And then — no more further conditions are needed!
The Theorem

Let $K$ be a number field, let $f(z) = Az^3 + Bz + 1 \in K[z]$ with $A = \frac{1}{81}(4B^3 - 27B)$, and let $x_0 \in K$. Let $G_n = \text{Gal}(K(f^{-n}(x_0)/K)$

Condition ($\dagger$) on $(K, x_0)$: There are primes $p, q, r$ of $K$ such that

1. $v_p(A) = v_p(B) = 0$ and $v_p(x_0) = -1$,
2. $6 | v_q(A) = v_p(B)$, with $-5v_q(A) \leq v_q(x_0) < 0$ and $v_q(x_0) \equiv \pm 1 \pmod{6}$,
3. $v_r(A) = v_r(B) > 0 > v_r(x_0)$, with $v_r(a) \not\equiv v_r(x_0) \pmod{3}$ and $4 \nmid v_r(x_0)$.
4. $[K(\sqrt{\Delta(f - x_0)}, \sqrt{-3}) : K] = 4$

Theorem (RB,DeGroot,Ni,Seid,Wei,Winton,2023)

With notation as above,

1. If ($\dagger$) holds for $(K, x_0)$, then it holds for $(K(\alpha), \alpha)$ for each $n \geq 1$ and each $\alpha \in f^{-n}(x_0)$.
2. If ($\dagger$) holds for $(K, x_0)$, then $G_n = Q_n$ for all $n \geq 1$. 
Summary: Maps with 2 critical points colliding at iterate $\ell$

**Quadratic Rational** (with Anna Dietrich):
- Description of group $M_{\ell,n} \subseteq \text{Aut}(T_{2,n})$ with $G_n \subseteq M_{\ell,n}$ for all $n \geq 1$.
- Sufficient condition to force $G_n = M_{\ell,n}$ for all $n \geq 1$.
  (Infinitely many “$\kappa_n$ is not a square in $K$” conditions.)

**Cubic Polynomial** (with Will DeGroot, Xinyu Ni, Jesse Seid, Annie Wei, and Samantha (Min) Winton): For $\ell = 2$ only:
- Description of group $Q_n \subseteq \text{Aut}(T_{3,n})$ with $G_n \subseteq Q_n$ for all $n \geq 1$.
- Sufficient condition to force $G_n = Q_n$ for all $n \geq 1$.
  (Finite condition: existence of primes with certain properties, inherited up the tree.)