Hyperbolicity and $J$-stability in Non-archimedean Dynamics

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Hyperbolicity in Complex Dynamics

\( f \in \mathbb{C}(z) \) has spherical derivative \( f^\#(z) := |f'(z)| \cdot \frac{1 + |z|^2}{1 + |f(z)|^2} \)

**Theorem**

Let \( f \in \mathbb{C}(z) \) with Julia set \( J_f \). The following are equivalent:

1. There exists \( \sigma : J_f \to (0, \infty) \) continuous such that
   \[ f^\#(z)\sigma(f(z)) > \sigma(z) \text{ for all } z \in J_f. \]
2. There exist \( C > 0 \) and \( \lambda > 1 \) such that
   \[ (f^n)^\#(z) \geq C\lambda^n \text{ for all } z \in J_f \text{ and } n \geq 1. \]
3. \( J_f \) is disjoint from the closure of the postcritical set of \( f \).
4. All critical points of \( f \) are attracted to attracting cycles.

**Definition**

A rational function \( f \in \mathbb{C}(z) \) satisfying any (and hence all) of the above properties is said to be hyperbolic.
$J$-stability in Complex Dynamics

Let $\text{Rat}_d(\mathbb{C})$ denote the space of rational functions of degree $d$.

**Definition**

Let $f \in \text{Rat}_d(\mathbb{C})$ with Julia set $\mathcal{J}_f$.

1. $g \in \text{Rat}_d(\mathbb{C})$ is **$J$-equivalent** to $f$ if there is a homeomorphism $h : \mathcal{J}_f \rightarrow \mathcal{J}_g$ such that $h \circ f = g \circ h$.

2. $f$ is **$J$-stable** if there is a neighborhood $W \subseteq \text{Rat}_d(\mathbb{C})$ of $f$ such that every $g \in W$ is $J$-equivalent to $f$.

(There’s also a continuity condition for $J$-stability, but never mind.)

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**Theorem (Mañé, Sad, Sullivan 1983)**

(At least for one-parameter families in $\text{Rat}_d(\mathbb{C})$),

If $f \in \mathbb{C}(z)$ is hyperbolic, then $f$ is $J$-stable.

(Recall: hyperbolic means expanding on the Julia set.)
The Berkovich Projective Line

\( \mathbb{C}_v \): a complete, algebraically closed non-archimedean field with absolute value \( |\cdot| \), and with char \( \mathbb{C}_v = 0 \).

[E.g. \( p \)-adic field \( \mathbb{C}_p \) or Puiseux series field \( \overline{\mathbb{C}}((t)) \).]

\( f \in \mathbb{C}_v(z) \) acts on \( \mathbb{P}^1(\mathbb{C}_v) \), but even better, \( f \) acts on the Berkovich line \( \mathbb{P}^1_{\text{an}} \), which:

- contains \( \mathbb{P}^1(\mathbb{C}_v) \) as a subspace (“Type I points”)
- contains one point \( \zeta(a, r) \) for each closed disk \( \overline{D}(a, r) \subseteq \mathbb{C}_v \) (“Type II and type III points”)
- is compact and Hausdorff

Each disk \( D(a, r) \) or \( \overline{D}(a, r) \) in \( \mathbb{C}_v \) has a natural extension to \( D_{\text{an}}(a, r) \) or \( \overline{D}_{\text{an}}(a, r) \) in \( \mathbb{P}^1_{\text{an}} \).
Rational Functions Acting on $\mathbb{P}^1_{\text{an}}$

For $f \in \mathbb{C}_v(z)$ of degree $d \geq 2$,

- $f$ maps $\mathbb{P}^1_{\text{an}}$ continuously onto itself.

- for $x \in \mathbb{P}^1(\mathbb{C}_v)$ of type I, $f(x)$ is the usual $f(x) \in \mathbb{P}^1(\mathbb{C}_v)$.

- If $f(D(a, r)) = D(b, s)$, then $f(\zeta(a, r)) = \zeta(b, s)$.
Non-archimedean Dynamics

$f \in \mathbb{C}_v(z)$ has an associated

- (Berkovich) Fatou set $\mathcal{F}_{an,f}$, and
- (Berkovich) Julia set $\mathcal{J}_{an,f} := \mathbb{P}^1_{an} \setminus \mathcal{F}_{an,f}$

contained in $\mathbb{P}^1_{an}$, such that:

- $\mathcal{F}_{an,f}$ is open in $\mathbb{P}^1_{an}$, and $\mathcal{J}_{an,f}$ is closed (and hence compact).
- $f^{-1}(\mathcal{F}_{an,f}) = \mathcal{F}_{an,f}$ and $f^{-1}(\mathcal{J}_{an,f}) = \mathcal{J}_{an,f}$.
- Both $\mathcal{F}_{an,f}$ and $\mathcal{J}_{an,f}$ are nonempty.
- $\mathcal{J}_{an,f}$ is the smallest nonempty closed subset of $\mathbb{P}^1_{an}$ that is invariant under $f$.

**Fact:** $f$ has good reduction iff $\mathcal{J}_{an,f} = \{\zeta(0,1)\}$. 
Two Previous Non-Archimedean J-Stability Results

Theorem (T. Silverman, 2017)

Let \( \{f_x\}_{x \in U} \) be a one-parameter analytic family for \( U \subseteq \mathbb{A}_{an}^1 \) connected and open. Suppose

- \( f_y \) has a type I repelling fixed point for some \( y \in U \), and
- for all \( x \in U \), \( f_x \) has no type I repelling periodic points of higher multiplicity, and no unstably indifferent periodic points.

Then the family \( \{f_x\} \) is J-stable on \( U \).

Theorem (J. Lee, 2018)

Assume \( f \in \text{Rat}_d(\mathbb{C}_v) \) satisfies

- \( \mathcal{J}_{an,f} \cap \mathbb{P}^1(\mathbb{C}_v) \neq \emptyset \), and
- there exist \( C > 0 \) and \( \lambda > 1 \) such that
  \[
  (f^n)\#(x) \geq C\lambda^n \text{ for all } x \in \mathcal{J}_{an,f} \cap \mathbb{P}^1(\mathbb{C}_v) \text{ and } n \geq 1.
  \]

Then \( f \) is J-stable, at least on \( \mathcal{J}_{an,f} \cap \mathbb{P}^1(\mathbb{C}_v) \).
A Berkovich Spherical Derivative

The **diameter** of $\zeta \in \mathbb{P}^1_{\text{an}}$ is:

- If $\zeta = x \in \mathbb{C}_v$ is of Type I, then $\text{diam}(\zeta) = 0$
- If $\zeta = \zeta(a, r)$ is of Type II or III, then $\text{diam}(\zeta) = r$.

Define the **spherical derivative** of $f$ to be

$$f^\sharp(x) := \left|f'(x)\right| \cdot \frac{\max\{1, |x|^2\}}{\max\{1, |f(x)|^2\}} \quad \text{if } x \in \mathbb{P}^1(\mathbb{C}_v),$$

and

$$f^\sharp(\zeta) := \frac{\text{diam}(f(\zeta))}{\text{diam}(\zeta)} \cdot \frac{\max\{1, |\zeta|^2\}}{\max\{1, |f(\zeta)|^2\}} \quad \text{if } \zeta \in \mathbb{P}^1_{\text{an}} \setminus \mathbb{P}^1(\mathbb{C}_v).$$
Wait, about that extension of the spherical derivative... 

From previous slide: for \( \zeta \in \mathbb{P}^1_{an} \),

\[
f^{\sharp}(\zeta) = \begin{cases} 
|f'(\zeta)| \cdot \frac{\max\{1,|\zeta|^2\}}{\max\{1,|f(\zeta)|^2\}} & \text{if } \zeta \in \mathbb{P}^1(\mathbb{C}_v), \\
\frac{\text{diam}(f(\zeta))}{\text{diam}(\zeta)} \cdot \frac{\max\{1,|\zeta|^2\}}{\max\{1,|f(\zeta)|^2\}} & \text{otherwise.}
\end{cases}
\]

Why not \( f^{\sharp}(\zeta) \overset{?}{=} \|f'\|_{\zeta} \cdot \frac{\max\{1,\|z\|^2_{\zeta}\}}{\max\{1,\|f(z)\|^2_{\zeta}\}} \)?

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**Example.** \( f(z) = z^p \), where \( p \geq 2 \) is the residue characteristic of \( \mathbb{C}_v \).

\( f \) has good reduction, so \( \mathcal{J}_{an,f} = \{\zeta(0,1)\} \).

Then \( \|f'\|_{\zeta} \cdot \frac{\max\{1,\|z\|^2_{\zeta}\}}{\max\{1,\|f(z)\|^2_{\zeta}\}} = |p| < 1 \) for \( \zeta \in \mathcal{J}_{an,f} \).

So \( f \) would be *contracting* on the Julia set by that definition!
A Fairly General Example

Fix $m \geq 2$ with $|m| = 1$, and fix $c \in \mathbb{C}_v$ with $0 < |c| < 1$. Let

$$f(z) := cz^{m+1} - z^m + z \in \mathbb{C}_v[z].$$

Then both $c^{-1} \in \mathbb{C}_v$ and $\zeta(0, 1)$ are fixed points in $\mathcal{J}_{an,f}$.

Define a sequence $\{a_n\}_{n \geq 0}$ by $a_0 := c^{-1}$, and

$$f(a_{n+1}) = a_n \quad \text{and} \quad |a_n| = |c|^{-1/m^n} \quad \text{for every } n \geq 0.$$

A simple computation shows $(f^i)^\sharp(a_n) < |c|^{-3}$ for every $0 \leq i \leq n$, even though $a_n \in \mathcal{J}_{an,f}$.

Also, $\zeta = \zeta(0, 1)$ has $(f^n)^\sharp(\zeta) = 1$ for all $n \geq 0$. 

Moral

Even if we care only about the type I points of the Julia set, any strictly expansive condition like:

1. There exists $\sigma : \mathcal{J}_f \to (0, \infty)$ continuous such that $f^\#(z)\sigma(f(z)) > \sigma(z)$ for all $z \in \mathcal{J}_{\text{an},f}$.

2. There exist $C > 0$ and $\lambda > 1$ such that $(f^n)^\#(z) \geq C\lambda^n$ for all $z \in \mathcal{J}_{\text{an},f}$ and $n \geq 1$.

3. All critical points of $f$ are attracted to attracting cycles.

is TOO RESTRICTIVE in non-archimedean dynamics.
A Stability Theorem

Theorem (B-Lee)

Let \( f \in \mathbb{C}_v(z) \) with \( d := \deg f \geq 2 \). Suppose there exists \( \delta > 0 \) such that

\[
(f^n)_{\sharp}(\zeta) \geq \delta \quad \text{for all } \zeta \in J_{\text{an},f} \text{ and } n \geq 0.
\]

Then \( f \) is \( J \)-stable. More precisely, there exist:

- a neighborhood \( W \subseteq \text{Rat}_d(\mathbb{C}_v) \) of \( f \) and
- an open set \( U \subseteq \mathbb{P}^1_{\text{an}} \) containing \( J_{\text{an},f} \cap \mathbb{P}^1(\mathbb{C}_v) \)

so that for each \( g \in W \), there is a homeomorphism \( h : U \cup J_{\text{an},f} \to U \cup J_{\text{an},f} \) for which

1. \( h \) is an isometry on the type I points \( U \cap \mathbb{P}^1(\mathbb{C}_v) \) of \( U \),
2. \( h \) is the identity map on \( J_{\text{an},f} \setminus U \),
3. \( h \circ f = g \circ h \), and
4. \( h(J_{\text{an},f}) = J_{\text{an},g} \).
Sketch of Proof: Setup

Change coordinates so that $\mathcal{J}_{an,f} \subseteq \overline{D}_{an}(0, 1)$.
Pick $\varepsilon > 0$ so that $f$ is injective on $D_{an}(a, \varepsilon)$ for every $a \in \mathbb{C}_v$ for which $D_{an}(a, \varepsilon) \cap \mathcal{J}_{an,f} \neq \emptyset$.

Without loss of generality, assume $\delta, \varepsilon < 1$.

For each $\zeta \in \mathcal{J}_{an,f}$, define

$$\sigma(\zeta) := \inf \{(f^n)^{\sharp}(\zeta) \mid n \geq 0\}.$$  

Then for all $\zeta \in \mathcal{J}_{an,f}$,

- $\delta \leq \sigma(\zeta) \leq 1$
- $f^{\sharp}(\zeta)\sigma(f(\zeta)) \geq \sigma(\zeta)$
Sketch of Proof: Domain of the Conjugacy

For $\zeta \in J_{an,f}$, recall $\sigma(\zeta) := \inf \{(f^n)^{\#}(\zeta) \mid n \geq 0\}$. Define

$$\nu(\zeta) := \frac{\delta^2 \varepsilon}{\sigma(\zeta)}$$

and

$$J_{an,f}^0 := \{\zeta \in J_{an,f} \mid \text{diam} (\zeta) < \nu(\zeta)\}$$

and

$$\Omega := \bigcup_{\zeta \in J_{an,f}^0} D_{an}(\zeta, \nu(\zeta))$$

Using $f^{\#}(\zeta)\sigma(f(\zeta)) \geq \sigma(\zeta)$, we can show $f^{-1}(\Omega) \subseteq \Omega$.

Let $U := f^{-1}(\Omega)$. The conjugacy $h$ will map $U \cup J_{an,f}$ to itself, fixing all points of $J_{an,f} \setminus U$. 
Lemma

There is an open neighborhood $W \subseteq \text{Rat}_d(\mathbb{C}_v)$ of $f$ such that for all $g \in W$,

- $\mathcal{I}_{\text{an},g} \subseteq \overline{D}_\text{an}(0,1)$

- $|g(x) - f(x)| < \frac{\delta^2 \varepsilon}{2}$ for all $x \in f^{-1}(\overline{D}(0,1))$

Moreover, for every $g \in W$ and open disk $D \subseteq U := f^{-1}(\Omega)$,

$g$ maps $D$ bijectively onto $f(D)$.

In particular, $g$ has a local inverse

$$G_D := (g|_D)^{-1} : f(D) \to D$$
Sketch of Proof: The Inductive Construction

\[
\cdots f^{-3}(\Omega) \xrightarrow{f} f^{-2}(\Omega) \xrightarrow{f} f^{-1}(\Omega) \xrightarrow{f} \Omega \\
\downarrow h_3 \quad \downarrow h_2 \quad \downarrow h_1 \quad \downarrow h_0 = \text{id} \\
\cdots g^{-3}(\Omega) \xrightarrow{g} g^{-2}(\Omega) \xrightarrow{g} g^{-1}(\Omega) \xrightarrow{g} \Omega
\]

On each open disk \( D \subseteq U := f^{-1}(\Omega) = g^{-1}(\Omega) \):

\[
f^{-n}(\Omega) \cap D \xrightarrow{f} f^{-(n-1)}(\Omega) \cap f(D) \\
\downarrow h_{n-1} \\
g^{-n}(\Omega) \cap D \xrightarrow{g} g^{-(n-1)}(\Omega) \cap f(D)
\]

Define \( h_n \) on \( f^{-n}(\Omega) \cap D \) by \( h_n := G_D \circ h_{n-1} \circ f \)
where \( G_D : f(D) \to D \) is the local inverse of \( g : D \to f(D) \).
Sketch of Proof: The Inductive Construction

\[ \cdots f^{-3}(\Omega) \xrightarrow{f} f^{-2}(\Omega) \xrightarrow{f} f^{-1}(\Omega) \xrightarrow{f} \Omega \]
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On each open disk \( D \subseteq U := f^{-1}(\Omega) = g^{-1}(\Omega) \):

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\[ \downarrow h_n \quad \downarrow h_{n-1} \]
\[ g^{-n}(\Omega) \cap D \xleftarrow{G_D} g^{-(n-1)}(\Omega) \cap f(D) \]

Define \( h_n \) on \( f^{-n}(\Omega) \cap D \) by \( h_n := G_D \circ h_{n-1} \circ f \)
where \( G_D : f(D) \to D \) is the local inverse of \( g : D \to f(D) \).
Finishing the Proof

\[ \ldots f^{-3}(\Omega) \xrightarrow{f} f^{-2}(\Omega) \xrightarrow{f} f^{-1}(\Omega) \xrightarrow{f} \Omega \]
\[ \downarrow h_3 \quad \downarrow h_2 \quad \downarrow h_1 \quad \downarrow h_0 = \text{id} \]
\[ \ldots g^{-3}(\Omega) \xrightarrow{g} g^{-2}(\Omega) \xrightarrow{g} g^{-1}(\Omega) \xrightarrow{g} \Omega \]

There are many details to check, including:

- Each \( h_n \) is well-defined, a homeomorphism, and an isometry on \( \mathbb{P}^1(\mathbb{C}_v) \).

- \( \bigcap_{n \geq 0} f^{-n}(\Omega) = \mathcal{J}_{an,f} \cap \mathbb{P}^1(\mathbb{C}_v) \).

- The limit \( h := \lim h_n \) converges and is a homeomorphism.

- \( h \circ f = g \circ h \) on \( U \cup \mathcal{J}_{an,f} \), and \( h(\mathcal{J}_{an,f}) = \mathcal{J}_{an,g} \).
Theorem (B-Lee)

Let $f \in C_v(z)$ with $d := \deg f \geq 2$. Suppose there exists $\delta > 0$ such that

$$(f^n)^\sharp(\zeta) \geq \delta \text{ for all } \zeta \in J_{an,f} \text{ and } n \geq 0.$$

Then $f$ is $J$-stable. More precisely, there exist:

1. a neighborhood $W \subseteq \text{Rat}_d(C_v)$ of $f$ and
2. an open set $U \subseteq \mathbb{P}^1_{an}$ containing $J_{an,f} \cap \mathbb{P}^1(C_v)$

so that for each $g \in W$, there is a homeomorphism $h : U \cup J_{an,f} \to U \cup J_{an,f}$ for which

1. $h$ is an isometry on the type I points $U \cap \mathbb{P}^1(C_v)$ of $U$,
2. $h$ is the identity map on $J_{an,f} \setminus U$,
3. $h \circ f = g \circ h$, and
4. $h(J_{an,f}) = J_{an,g}$.

And now we see $U := f^{-1}(\Omega)$. 