Hyperbolicity and J-stability in Non-archimedean Dynamics

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Hyperbolicity in Complex Dynamics

$f \in \mathbb{C}(z)$ has spherical derivative $f^{\#}(z) := |f'(z)| \cdot \dfrac{1+|z|^2}{1+|f(z)|^2}$

Theorem

Let $f \in \mathbb{C}(z)$ with Julia set $\mathcal{J}_f$. The following are equivalent:

1. *There exists $\sigma : \mathcal{J}_f \to (0, \infty)$ continuous such that $f^{\#}(z)\sigma(f(z)) > \sigma(z)$ for all $z \in \mathcal{J}_f$.*
2. *There exist $C > 0$ and $\lambda > 1$ such that $(f^n)^{\#}(z) \geq C\lambda^n$ for all $z \in \mathcal{J}_f$ and $n \geq 1$.*
3. *$\mathcal{J}_f$ is disjoint from the closure of the postcritical set of f.*
4. *All critical points of f are attracted to attracting cycles.*

Definition

A rational function $f \in \mathbb{C}(z)$ satisfying any (and hence all) of the above properties is said to be *hyperbolic*.

J-stability in Complex Dynamics

Let $\mathrm{Rat}_d(\mathbb{C})$ denote the space of rational functions of degree d.

Definition

Let $f \in \mathrm{Rat}_d(\mathbb{C})$ with Julia set $\mathcal{J}_f$.

1. $g \in \mathrm{Rat}_d(\mathbb{C})$ is J-**equivalent** to f if there is a homeomorphism $h : \mathcal{J}_f \to \mathcal{J}_g$ such that $h \circ f = g \circ h$.
2. f is J-**stable** if there is a neighborhood $W \subseteq \mathrm{Rat}_d(\mathbb{C})$ of f such that every $g \in W$ is J-equivalent to f.

(There's also a continuity condition for J-stability, but never mind.)

Theorem (Mañé, Sad, Sullivan 1983)

(At least for one-parameter families in $\mathrm{Rat}_d(\mathbb{C})$*),*
If $f \in \mathbb{C}(z)$ is hyperbolic, then f is J-stable.

(Recall: hyperbolic means expanding on the Julia set.)

The Berkovich Projective Line

$\mathbb{C}_v$: a complete, algebraically closed non-archimedean field with absolute value $|\cdot|$, and with char $\mathbb{C}_v = 0$.

[E.g. p-adic field $\mathbb{C}_p$ or Puiseux series field $\widehat{\mathbb{C}((t))}$.]

$f \in \mathbb{C}_v(z)$ acts on $\mathbb{P}^1(\mathbb{C}_v)$, but even better, f acts on the Berkovich line $\mathbb{P}^1_{\text{an}}$, which:

- contains $\mathbb{P}^1(\mathbb{C}_v)$ as a subspace ("Type I points")
- contains one point $\zeta(a, r)$ for each closed disk $\overline{D}(a, r) \subseteq \mathbb{C}_v$ ("Type II and type III points")
- is compact and Hausdorff

Each disk $D(a, r)$ or $\overline{D}(a, r)$ in $\mathbb{C}_v$ has a natural extension to $D_{\text{an}}(a, r)$ or $\overline{D}_{\text{an}}(a, r)$ in $\mathbb{P}^1_{\text{an}}$.

Rational Functions Acting on $\mathbb{P}^1_{\text{an}}$

For $f \in \mathbb{C}_v(z)$ of degree $d \geq 2$,

- f maps $\mathbb{P}^1_{\text{an}}$ continuously onto itself.
- for $x \in \mathbb{P}^1(\mathbb{C}_v)$ of type I, $f(x)$ is the usual $f(x) \in \mathbb{P}^1(\mathbb{C}_v)$.
- If $f\big(\overline{D}(a,r)\big) = \overline{D}(b,s)$, then $f\big(\zeta(a,r)\big) = \zeta(b,s)$.

Non-archimedean Dynamics

$f \in \mathbb{C}_v(z)$ has an associated

- (Berkovich) *Fatou set* $\mathcal{F}_{\mathrm{an},f}$, and
- (Berkovich) *Julia set* $\mathcal{J}_{\mathrm{an},f} := \mathbb{P}^1_{\mathrm{an}} \setminus \mathcal{F}_{\mathrm{an},f}$

contained in $\mathbb{P}^1_{\mathrm{an}}$, such that:

- $\mathcal{F}_{\mathrm{an},f}$ is open in $\mathbb{P}^1_{\mathrm{an}}$, and $\mathcal{J}_{\mathrm{an},f}$ is closed (and hence compact).
- $f^{-1}(\mathcal{F}_{\mathrm{an},f}) = \mathcal{F}_{\mathrm{an},f}$ and $f^{-1}(\mathcal{J}_{\mathrm{an},f}) = \mathcal{J}_{\mathrm{an},f}$.
- Both $\mathcal{F}_{\mathrm{an},f}$ and $\mathcal{J}_{\mathrm{an},f}$ are nonempty.
- $\mathcal{J}_{\mathrm{an},f}$ is the smallest nonempty closed subset of $\mathbb{P}^1_{\mathrm{an}}$ that is invariant under f.

Fact: f has good reduction iff $\mathcal{J}_{\mathrm{an},f} = \{\zeta(0,1)\}$.

Two Previous Non-Archimedean J-Stability Results

Theorem (T. Silverman, 2017)

Let $\{f_x\}_{x \in U}$ *be a one-parameter analytic family for* $U \subseteq \mathbb{A}^1_{\mathrm{an}}$ *connected and open. Suppose*

- f_y *has a type I repelling fixed point for some* $y \in U$*, and*
- *for all* $x \in U$*,* f_x *has no type I repelling periodic points of higher multiplicity, and no unstably indifferent periodic points.*

Then the family $\{f_x\}$ *is J-stable on* U*.*

Theorem (J. Lee, 2018)

Assume $f \in \mathrm{Rat}_d(\mathbb{C}_v)$ *satisfies*

- $\mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v) \neq \varnothing$*, and*
- *there exist* $C > 0$ *and* $\lambda > 1$ *such that*
  $$(f^n)^{\#}(x) \geq C\lambda^n \text{ for all } x \in \mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v) \text{ and } n \geq 1.$$

Then f *is J-stable, at least on* $\mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v)$*.*

A Berkovich Spherical Derivative

The **diameter** of $\zeta \in \mathbb{P}^1_{\text{an}}$ is:

- If $\zeta = x \in \mathbb{C}_v$ is of Type I, then $\text{diam}(\zeta) = 0$
- If $\zeta = \zeta(a, r)$ is of Type II or III, then $\text{diam}(\zeta) = r$.

Define the **spherical derivative** of f to be

$$f^\natural(x) := |f'(x)| \cdot \frac{\max\{1, |x|^2\}}{\max\{1, |f(x)|^2\}} \qquad \text{if } x \in \mathbb{P}^1(\mathbb{C}_v),$$

and

$$f^\natural(\zeta) := \frac{\text{diam}(f(\zeta))}{\text{diam}(\zeta)} \cdot \frac{\max\{1, |\zeta|^2\}}{\max\{1, |f(\zeta)|^2\}} \qquad \text{if } \zeta \in \mathbb{P}^1_{\text{an}} \setminus \mathbb{P}^1(\mathbb{C}_v).$$

Wait, about that extension of the spherical derivative...

From previous slide: for $\zeta \in \mathbb{P}^1_{\mathrm{an}}$,

$$f^\natural(\zeta) = \begin{cases} |f'(\zeta)| \cdot \frac{\max\{1,|\zeta|^2\}}{\max\{1,|f(\zeta)|^2\}} & \text{if } \zeta \in \mathbb{P}^1(\mathbb{C}_v), \\ \frac{\mathrm{diam}(f(\zeta))}{\mathrm{diam}(\zeta)} \cdot \frac{\max\{1,|\zeta|^2\}}{\max\{1,|f(\zeta)|^2\}} & \text{otherwise.} \end{cases}$$

Why not $f^\natural(\zeta) \stackrel{?}{=} \|f'\|_\zeta \cdot \dfrac{\max\{1, \|z\|_\zeta^2\}}{\max\{1, \|f(z)\|_\zeta^2\}}$?

Example. $f(z) = z^p$, where $p \geq 2$ is the residue characteristic of $\mathbb{C}_v$.
f has good reduction, so $\mathcal{J}_{\mathrm{an},f} = \{\zeta(0,1)\}$.

Then $\|f'\|_\zeta \cdot \dfrac{\max\{1, \|z\|_\zeta^2\}}{\max\{1, \|f(z)\|_\zeta^2\}} = |p| < 1$ for $\zeta \in \mathcal{J}_{\mathrm{an},f}$.

So f would be *contracting* on the Julia set by that definition!

A Fairly General Example

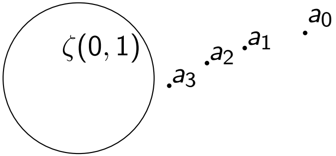
Fix $m \geq 2$ with $|m| = 1$, and fix $c \in \mathbb{C}_v$ with $0 < |c| < 1$. Let

$$f(z) := cz^{m+1} - z^m + z \in \mathbb{C}_v[z].$$

Then both $c^{-1} \in \mathbb{C}_v$ and $\zeta(0,1)$ are fixed points in $\mathcal{J}_{\text{an},f}$.

Define a sequence $\{a_n\}_{n\geq 0}$ by $a_0 := c^{-1}$, and

$$f(a_{n+1}) = a_n \quad \text{and} \quad |a_n| = |c|^{-1/m^n} \quad \text{for every } n \geq 0.$$



A simple computation shows $(f^i)^\natural(a_n) < |c|^{-3}$ for every $0 \leq i \leq n$, even though $a_n \in \mathcal{J}_{\text{an},f}$.

Also, $\zeta = \zeta(0,1)$ has $(f^n)^\natural(\zeta) = 1$ for all $n \geq 0$.

Moral

Even if we care only about the type I points of the Julia set, any *strictly* expansive condition like:

1. There exists $\sigma : \mathcal{J}_f \to (0, \infty)$ continuous such that $f^{\#}(z)\sigma(f(z)) > \sigma(z)$ for all $z \in \mathcal{J}_{\mathrm{an},f}$.
2. There exist $C > 0$ and $\lambda > 1$ such that $(f^n)^{\#}(z) \geq C\lambda^n$ for all $z \in \mathcal{J}_{\mathrm{an},f}$ and $n \geq 1$.
3. All critical points of f are attracted to attracting cycles.

is **TOO RESTRICTIVE** in non-archimedean dynamics.

A Stability Theorem

Theorem (B-Lee)

Let $f \in \mathbb{C}_v(z)$ *with* $d := \deg f \geq 2$. *Suppose there exists* $\delta > 0$ *such that*

$$(f^n)^\natural(\zeta) \geq \delta \quad \textit{for all } \zeta \in \mathcal{J}_{\mathrm{an},f} \textit{ and } n \geq 0.$$

Then f *is J-stable. More precisely, there exist:*

- *a neighborhood* $W \subseteq \mathrm{Rat}_d(\mathbb{C}_v)$ *of* f *and*
- *an open set* $U \subseteq \mathbb{P}^1_{\mathrm{an}}$ *containing* $\mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v)$

so that for each $g \in W$, *there is a homeomorphism* $h : U \cup \mathcal{J}_{\mathrm{an},f} \to U \cup \mathcal{J}_{\mathrm{an},f}$ *for which*

1. h *is an isometry on the type I points* $U \cap \mathbb{P}^1(\mathbb{C}_v)$ *of* U,
2. h *is the identity map on* $\mathcal{J}_{\mathrm{an},f} \setminus U$,
3. $h \circ f = g \circ h$, *and*
4. $h(\mathcal{J}_{\mathrm{an},f}) = \mathcal{J}_{\mathrm{an},g}$.

Sketch of Proof: Setup

Change coordinates so that $\mathcal{J}_{\text{an},f} \subseteq \overline{D}_{\text{an}}(0,1)$.
Pick $\varepsilon > 0$ so that f is injective on $D_{\text{an}}(a,\varepsilon)$ for every $a \in \mathbb{C}_v$ for which $D_{\text{an}}(a,\varepsilon) \cap \mathcal{J}_{\text{an},f} \neq \varnothing$.

Without loss of generality, assume $\delta, \varepsilon < 1$.

For each $\zeta \in \mathcal{J}_{\text{an},f}$, define

$$\sigma(\zeta) := \inf\left\{ \left(f^n\right)^\natural(\zeta) \,\middle|\, n \geq 0 \right\}.$$

Then for all $\zeta \in \mathcal{J}_{\text{an},f}$,

- $\delta \leq \sigma(\zeta) \leq 1$
- $f^\natural(\zeta)\sigma\big(f(\zeta)\big) \geq \sigma(\zeta)$

Sketch of Proof: Domain of the Conjugacy

For $\zeta \in \mathcal{J}_{\mathrm{an},f}$, recall $\sigma(\zeta) := \inf\left\{ \left(f^n\right)^\natural(\zeta) \,\middle|\, n \geq 0 \right\}$. Define

$$\nu(\zeta) := \frac{\delta^2 \varepsilon}{\sigma(\zeta)} \quad \text{and} \quad \mathcal{J}^0_{\mathrm{an},f} := \{\zeta \in \mathcal{J}_{\mathrm{an},f} \mid \mathrm{diam}(\zeta) < \nu(\zeta)\}$$

and

$$\Omega := \bigcup_{\zeta \in \mathcal{J}^0_{\mathrm{an},f}} D_{\mathrm{an}}\big(\zeta, \nu(\zeta)\big)$$

Using $f^\natural(\zeta)\sigma(f(\zeta)) \geq \sigma(\zeta)$, we can show $f^{-1}(\Omega) \subseteq \Omega$.

Let $U := f^{-1}(\Omega)$. The conjugacy h will map $U \cup \mathcal{J}_{\mathrm{an},f}$ to itself, fixing all points of $\mathcal{J}_{\mathrm{an},f} \smallsetminus U$.

Sketch of Proof: Neighborhood in the Moduli Space

Lemma

There is an open neighborhood $W \subseteq \mathrm{Rat}_d(\mathbb{C}_v)$ of f such that for all $g \in W$,

- $\mathcal{J}_{\mathrm{an},g} \subseteq \overline{D}_{\mathrm{an}}(0,1)$
- $\left|g(x) - f(x)\right| < \dfrac{\delta^2 \varepsilon}{2}$ *for all* $x \in f^{-1}(\overline{D}(0,1))$

Moreover, for every $g \in W$ and open disk $D \subseteq U := f^{-1}(\Omega)$,

$$g \text{ maps } D \text{ bijectively onto } f(D).$$

In particular, g has a local inverse

$$G_D := \left(g\big|_D\right)^{-1} : f(D) \to D$$

Sketch of Proof: The Inductive Construction

$$\begin{array}{ccccccc} \cdots f^{-3}(\Omega) & \xrightarrow{f} & f^{-2}(\Omega) & \xrightarrow{f} & f^{-1}(\Omega) & \xrightarrow{f} & \Omega \\ \downarrow h_3 & & \downarrow h_2 & & \downarrow h_1 & & \downarrow h_0 = \mathrm{id} \\ \cdots g^{-3}(\Omega) & \xrightarrow{g} & g^{-2}(\Omega) & \xrightarrow{g} & g^{-1}(\Omega) & \xrightarrow{g} & \Omega \end{array}$$

On each open disk $D \subseteq U := f^{-1}(\Omega) = g^{-1}(\Omega)$:

$$\begin{array}{ccc} f^{-n}(\Omega) \cap D & \xrightarrow{f} & f^{-(n-1)}(\Omega) \cap f(D) \\ & & \downarrow h_{n-1} \\ g^{-n}(\Omega) \cap D & \xrightarrow{g} & g^{-(n-1)}(\Omega) \cap f(D) \end{array}$$

Define h_n on $f^{-n}(\Omega) \cap D$ by $h_n := G_D \circ h_{n-1} \circ f$
where $G_D : f(D) \to D$ is the local inverse of $g : D \to f(D)$.

Sketch of Proof: The Inductive Construction

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Finishing the Proof

$$\begin{array}{ccccccc}
\cdots f^{-3}(\Omega) & \xrightarrow{f} & f^{-2}(\Omega) & \xrightarrow{f} & f^{-1}(\Omega) & \xrightarrow{f} & \Omega \\
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\cdots g^{-3}(\Omega) & \xrightarrow{g} & g^{-2}(\Omega) & \xrightarrow{g} & g^{-1}(\Omega) & \xrightarrow{g} & \Omega
\end{array}$$

There are many details to check, including:

- Each h_n is well-defined, a homeomorphism, and an isometry on $\mathbb{P}^1(\mathbb{C}_v)$.
- $\bigcap_{n \geq 0} f^{-n}(\Omega) = \mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v)$.
- The limit $h := \lim h_n$ converges and is a homeomorphism.
- $h \circ f = g \circ h$ on $U \cup \mathcal{J}_{\mathrm{an},f}$, and $h(\mathcal{J}_{\mathrm{an},f}) = \mathcal{J}_{\mathrm{an},g}$.

Reminder of the Theorem

Theorem (B-Lee)

Let $f \in \mathbb{C}_v(z)$ *with* $d := \deg f \geq 2$. *Suppose there exists* $\delta > 0$ *such that*

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Then f *is J-stable. More precisely, there exist:*

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so that for each $g \in W$, *there is a homeomorphism* $h : U \cup \mathcal{J}_{\mathrm{an},f} \to U \cup \mathcal{J}_{\mathrm{an},f}$ *for which*

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3. $h \circ f = g \circ h$, *and*
4. $h(\mathcal{J}_{\mathrm{an},f}) = \mathcal{J}_{\mathrm{an},g}$.

And now we see $U := f^{-1}(\Omega)$.