Some Open Problems in Non-archimedean Dynamics

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Dynamics on $\mathbb{P}^1(K)$

Let K be an algebraically closed field that is complete with respect to an absolute value $|\cdot|$.

[Examples: $K = \mathbb{C}$, $K = \mathbb{C}_p$, or K is the completion of an algebraic closure of $\mathbb{F}((t))$.]

Consider $\phi(z) \in K(z)$ as a function on $\mathbb{P}^1(K) = K \cup \{\infty\}$, i.e.,

$$\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K).$$

Writing $\phi = f/g$, define $\deg \phi = \max\{\deg f, \deg g\}$.

For $n \geq 0$, write

$$\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}$$

Periodic Points and Multipliers

Definition

Let $x \in \mathbb{P}^1(K)$ be a periodic point of exact period $n \geq 1$.
The **multiplier** of x is $\lambda := (\phi^n)'(x)$.

(Fact: the multiplier is independent of coordinate.)
We say x is

- **attracting** if $|\lambda| < 1$,
- **repelling** if $|\lambda| > 1$,
- **indifferent** (or **neutral**) if $|\lambda| = 1$.

Fatou and Julia sets

Definition

An open set $U \subseteq \mathbb{P}^1(K)$ is **dynamically stable** under ϕ if

$$\mathbb{P}^1(K) \smallsetminus \left[\bigcup_{n \geq 0} \phi^n(U) \right] \text{ is infinite.}$$

The **Fatou set** $\mathcal{F} = \mathcal{F}_\phi$ of ϕ is

$$\mathcal{F} = \{x \in \mathbb{P}^1(K) : x \text{ has a dynamically stable neighborhood}\}.$$

The **Julia set** $\mathcal{J} = \mathcal{J}_\phi$ of ϕ is $\mathcal{J} = \mathbb{P}^1(K) \smallsetminus \mathcal{F}$.

Note:

- $\mathcal{F}$ is open, and $\mathcal{J}$ is closed.
- $\phi(\mathcal{F}) = \phi^{-1}(\mathcal{F}) = \mathcal{F}$, and $\phi(\mathcal{J}) = \phi^{-1}(\mathcal{J}) = \mathcal{J}$.

Open Question #1: Repelling Density in $\mathbb{P}^1(K)$

- Attracting periodic points are Fatou.
- Repelling periodic points are Julia.
- If K is non-archimedean, then all indifferent periodic points are Fatou.

Let $\mathcal{R}$ denote the closure of the set of repelling periodic points. Clearly, $\mathcal{J} \supseteq \mathcal{R}$.

Question: Do we necessarily have $\mathcal{J} = \mathcal{R}$?

(If $K = \mathbb{C}$ is archimedean, Fatou and Julia proved the answer is **yes**.)

Hsia's Periodic Closure Theorem

Theorem (Hsia, 2000)

Let K be non-archimedean, and let $\phi \in K(z)$. Then $\mathcal{J}_\phi$ is contained in the closure of the set of **all** *periodic points.*

Maybe we can prove repelling density by showing nonrepelling periodic points stay away from $\mathcal{J}$?

But:

Example. If K has residue characteristic $p > 0$, then for any $a \in K$ with $0 < |a| < 1$, the polynomial

$$\phi(z) = (a^p - a)z^{p+1} + z^p$$

has $1/a \in \mathcal{J}$, and there is an infinite sequence of attracting periodic points accumulating at $1/a$.

Bézivin's Repelling Density Theorem

Theorem (Bézivin, 2001)

Let K be non-archimedean, and let $\phi \in K(z)$. Suppose that ϕ has **at least one** *repelling periodic point. Then $\mathcal{J}_\phi$ is the closure of the set of* **repelling** *periodic points.*

Maybe we can prove repelling density by showing that if $\mathcal{J} \neq \varnothing$, there is always a repelling periodic point of some bounded period?

But:

Example. If char $K = 0$ but K has residue characteristic $p > 0$ (e.g., $K = \mathbb{C}_p$), then for any $n \geq 0$, there is some $b \in K$ with $|b| > 1$ such that

$$\phi(z) = z^{p+1}(z-b)^{2p}$$

has a repelling periodic point of period $n+1$ but **no** repelling periodic points of period n or smaller.

Components of the Fatou Set

Definition

1. A **closed $\mathbb{P}^1(K)$-disk** is either a closed disk in K or the complement in $\mathbb{P}^1(K)$ of an open disk in K.

2. An **open connected affinoid** is a set of the form

$$\mathbb{P}^1(K) \smallsetminus (D_1 \cup \cdots \cup D_m),$$

where each D_i is a closed $\mathbb{P}^1(K)$-disk.

3. Given $\phi \in K(z)$ and $x \in \mathcal{F}_\phi$, the **Fatou component** of x is the union of all dynamically stable open connected affinoids containing x.

Theorem

If U is a Fatou component, then so is $\phi(U)$.

Classifying Periodic Fatou Components

Definition

Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

Let $U \subseteq \mathcal{F}$ be a Fatou component, and suppose that $\phi^m(U) = U$ for some (minimal) integer $m \geq 1$.

- We say U is an *indifferent component* if the mapping $\phi^m : U \twoheadrightarrow U$ is one-to-one.
- We say U is an *attracting component* if there is an attracting periodic point $x \in U$ of period m, and if $\lim_{n\to\infty} \phi^{mn}(z) = x$ for all $z \in U$.

A Fatou component that is not preperiodic is called a *wandering domain*.

Rivera-Letelier's Classification Theorem

Theorem (Rivera-Letelier, 2000)

Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$, and let U be a Fatou component.

Then exactly one of the following three possibilities occurs.

1. *Some iterate $\phi^n(U)$ is an indifferent periodic component.*

   In that case, $\phi^n(U)$ is a (rational open) connected affinoid.

2. *Some iterate $\phi^n(U)$ is an attracting periodic component.*

   In that case, $\phi^n(U)$ is either a (rational open) disk or a "component of Cantor type."

3. *U is a wandering domain.*

   In that case, there is some $N \geq 0$ such that for all $n \geq N$, $\phi^n(U)$ is a disk.

An Example: Nonconstant Reduction

Let $n \geq 2$ and let $0 < |a| < 1$. Let $\phi(z) = z^n + \dfrac{a}{z-1}$.

- Residue classes (except at 1) map to residue classes, and (most) are Fatou components.
- The residue class at 0 and the residue class at ∞ are attracting fixed components.
- The residue classes of n-power roots of unity eventually map onto the "bad" residue class of 1.
- The residue classes of other roots of unity are preperiodic. They are all attracting if $p|n$, or indifferent if $p \nmid n$.
- Any other residue classes (which exist if and only if the residue field is not algebraic over a finite field) are wandering domains.

What drives this example is the fact that the reduction $\overline{\phi}(z) = z^n$ is nonconstant.

(Equivalently, the Gauss point in Berkovich space is fixed.)

Examples: Cantor-Type Attracting Components

If $\phi(z) \in K[z]$ with $\deg \phi \geq 2$ is a polynomial, then the Fatou component U containing ∞ is fixed and attracting. If ϕ is not of "potentially good reduction," then U is of Cantor type.

Example. $\phi(z) = z^2 + az$, where $|a| > 1$.

The Julia set is a Cantor set, and the Fatou set is a single Cantor-type attracting component.

Example. $\phi(z) = bz^3 + z^2$, where $0 < |b| < 1$.

The Julia set is **not** a Cantor set, and the Fatou set has **more** than one component, but the attracting component U at ∞ is still of Cantor type.

Example: A Non-disk Indifferent Component

$$\phi(z) = \frac{1}{1-z} + a^2 z = \frac{a^2z^2 - a^2 z - 1}{z-1} \in K(z), \text{ with } 0 < |a| < 1.$$

Then ϕ maps the open connected affinoid

$$U := D(0, |a|^{-1}) \setminus \left(\overline{D}(0, |a|) \cup \overline{D}(1, |a|)\right)$$

bijectively onto itself.

In fact, U is a fixed indifferent Fatou component.

Open Question #2: Bounding Non-disk Component Cycles

Theorem

(Rivera-Letelier, 2000) *Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$. Then ϕ has at most $d-1$ cycles of attracting periodic Fatou components of Cantor type.*

Question: Is there a bound (or even a finiteness result) for the number of cycles of **indifferent** periodic Fatou components that are not disks?

Wandering Domains, and Open Question #3

If the residue field of K is not algebraic over a finite field, recall ϕ can have wandering domains, via nonconstant reduction.

There are subtler examples, at least in residue characteristic $p > 0$:

Theorem (RB, 2002)

Let K have residue characteristic $p > 0$. Then there is a parameter $a \in K$ such that

$$\phi_a(z) := (1-a)z^{p+1} + az^p$$

has a wandering domain not arising from nonconstant reduction.

Question: The subtler wandering domains constructed in residue characteristic p are disks. What are their radii?

In particular, can the radii lie in the value group $|K^\times|$? Can they lie outside $|K^\times|$?

Open Question #4

On the other hand:

Theorem (RB, 1998)

Let L be a locally compact non-archimedean subfield of K, and let $\phi \in L(z)$ be a rational function such that either

- char $K = p$ *and $\mathcal{J}_\phi$ contains no wild critical points, or*
- char $K = 0$ *and $\mathcal{J}_\phi$ contains no wild recurrent critical points.*

Then ϕ has no wandering domains.

However:

Theorem (Rivera-Letelier, 2005)

Let K be a complete non-archimedean field of residue characteristic p. Then there are polynomials $\phi \in K[z]$ with wild recurrent Julia critical points.

Question: Let L be a locally compact subfield of K. Can there be rational functions $\phi \in L(z)$ with wandering domains?

Open Question #5

Theorem (RB, 2005)

Suppose K has residue characteristic zero. Let L be a **discretely valued subfield** *of K, and let $\phi \in L(z)$ be a rational function.*

Then ϕ has no wandering domains besides those attached to nonconstant reduction.

Theorem (Trucco, 2009)

Suppose K has residue characteristic zero, and let $\phi \in K[z]$ be a **polynomial**.

Then ϕ has no wandering domains besides those attached to nonconstant reduction.

Question: If K has residue characteristic zero, can $\phi \in K(z)$ have wandering domains besides those attached to nonconstant reduction?