Hyperbolicity and J-stability in Non-archimedean Dynamics

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Hyperbolicity in Complex Dynamics

$f \in \mathbb{C}(z)$ has spherical derivative $f^{\#}(z) := |f'(z)| \cdot \dfrac{1+|z|^2}{1+|f(z)|^2}$

Theorem

Let $f \in \mathbb{C}(z)$ with Julia set $\mathcal{J}_f$. The following are equivalent:

1. *There exists $\sigma : \mathcal{J}_f \to (0,\infty)$ continuous such that $f^{\#}(z)\sigma(f(z)) > \sigma(z)$ for all $z \in \mathcal{J}_f$.*
2. *There exist $C > 0$ and $\lambda > 1$ such that $(f^n)^{\#}(z) \geq C\lambda^n$ for all $z \in \mathcal{J}_f$ and $n \geq 1$.*
3. *$\mathcal{J}_f$ is disjoint from the closure of the postcritical set of f.*
4. *All critical points of f are attracted to attracting cycles.*

Definition

A rational function $f \in \mathbb{C}(z)$ satisfying any (and hence all) of the above properties is said to be *hyperbolic*.

J-stability in Complex Dynamics

Let $\mathrm{Rat}_d(\mathbb{C})$ denote the space of rational functions of degree d.

Definition

Let $f \in \mathrm{Rat}_d(\mathbb{C})$ with Julia set $\mathcal{J}_f$.

1. $g \in \mathrm{Rat}_d(\mathbb{C})$ is J-**equivalent** to f if there is a homeomorphism $h : \mathcal{J}_f \to \mathcal{J}_g$ such that $h \circ f = g \circ h$.
2. f is J-**stable** if there is a neighborhood $W \subseteq \mathrm{Rat}_d(\mathbb{C})$ of f such that every $g \in W$ is J-equivalent to f.

(There's also a continuity condition for J-stability).

Theorem (Mañé, Sad, Sullivan 1983)

(At least for one-parameter families in $\mathrm{Rat}_d(\mathbb{C})$*),*
If $f \in \mathbb{C}(z)$ is hyperbolic, then f is J-stable.

(Recall: hyperbolic means expanding on the Julia set.)

Non-Archimedean Fields

A **non-archimedean absolute value** on a field K is a function $|\cdot|_v : K \to [0, \infty)$ such that for all $x, y \in K$,

- $|x|_v \geq 0, \quad \text{with } |x|_v = 0 \Leftrightarrow x = 0,$
- $|xy|_v = |x|_v \cdot |y|_v,$
- $|x + y|_v \leq \max\{|x|_v, |y|_v\}.$

Fact: If $|x|_v \neq |y|_v$, then $|x + y|_v = \max\{|x|_v, |y|_v\}$.

Just as one can go from $\mathbb{Q}$ to $\mathbb{C}$ by completing (to get $\mathbb{R}$) and taking algebraic closure (to get $\mathbb{C}$),

one can expand K to a complete and algebraically closed non-archimedean field $\mathbb{C}_v$.

Example: Puiseux series

Example. $K = \mathbb{C}((t))$, with

$$\left| \sum_{n \geq m} a_n t^n \right|_v := e^{-m}, \quad \text{assuming } a_m \neq 0.$$

Idea: formal power series with a pole at $t = 0$ are "big", and those with a zero at $t = 0$ are "small".

We have $\mathbb{C}_v$ = field of **Puiseux series**:

$g \in \mathbb{C}_v$ is of the form $g = \sum_{i=0}^{\infty} c_i t^{q_i}$,

where $c_i \in \mathbb{C}$ and $q_i \in \mathbb{Q}$, with $q_i \nearrow \infty$.

We have $|g|_v = e^{-q_0}$ where $c_0 \neq 0$.

Disks in $\mathbb{C}_v$

Let $\mathbb{C}_v$ be an algebraically closed field
that is complete w.r.t. a non-archimedean absolute value $|\cdot|_v$.

Given $a \in \mathbb{C}_v$ and $r > 0$,

$$D(a,r) := \{x \in \mathbb{C}_v : |x-a|_v < r\} \quad \text{and}$$
$$\overline{D}(a,r) := \{x \in \mathbb{C}_v : |x-a|_v \leq r\}$$

are the associated open disk and closed disk.

- if $r \notin |\mathbb{C}_v^\times|_v$, then $D(a,r) = \overline{D}(a,r)$ is an **irrational disk**
- if $r \in |\mathbb{C}_v^\times|_v$, then $D(a,r) \subsetneq \overline{D}(a,r)$
  - $D(a,r)$ is a **rational open disk**
  - $\overline{D}(a,r)$ is a **rational closed disk**

More About Disks in $\mathbb{C}_v$

A rational closed disk $\overline{D}(a,r)$ is an infinite disjoint union of rational open disks: $\overline{D}(a,r) = \coprod_{b \in S} D(b,r)$

Example: For $\mathbb{C}_v$ = Puiseux series,

$$\overline{D}(0,1) = \left\{ \sum c_i t^{q_i} \middle| q_i \geq 0 \text{ for all } i \right\} = \coprod_{a \in \mathbb{C}} D(a,1),$$

where

$$D(a,1) = \left\{ a + \sum c_i t^{q_i} \middle| q_i > 0 \text{ for all } i \right\}$$

- All disks are (topologically) **both** open and closed
- Any point of a disk is a center. i.e.,
  - If $b \in D(a,r)$, then $D(a,r) = D(b,r)$
  - If $b \in \overline{D}(a,r)$, then $\overline{D}(a,r) = \overline{D}(b,r)$

The Berkovich Projective Line: Idea

$\mathbb{C}_v$: a complete, algebraically closed non-archimedean field
with absolute value $|\cdot|_v$.

Any rational function $f \in \mathbb{C}_v(z)$ acts on $\mathbb{P}^1(\mathbb{C}_v)$, but even better, f acts on the $\boxed{\text{Berkovich line } \mathbb{P}^1_{\text{an}}}$, which:

- contains $\mathbb{P}^1(\mathbb{C}_v)$ as a subspace ("Type I points")
- contains one point $\zeta(a,r)$ for each closed disk $\overline{D}(a,r) \subseteq \mathbb{C}_v$ ("Type II and type III points")
- is compact and Hausdorff
- is path-connected.

Multiplicative Seminorms on $\mathbb{C}_v[z]$

Definition

A **multiplicative seminorm** on $\mathbb{C}_v[z]$ is a function $\zeta = \|\cdot\|_\zeta : \mathbb{C}_v[z] \to [0, \infty)$ such that

- $\|c\|_\zeta = |c|_v$ for all constants $c \in \mathbb{C}_v$,
- $\|fg\|_\zeta = \|f\|_\zeta \cdot \|g\|_\zeta$ for all $f, g \in \mathbb{C}_v[z]$, and
- $\|f + g\|_\zeta \leq \|f\|_\zeta + \|g\|_\zeta$ for all $f, g \in \mathbb{C}_v[z]$.

Note: We do **not** require that $\|f\|_\zeta = 0$ implies $f = 0$.

By the way: we get $\|f + g\|_\zeta \leq \max\{\|f\|_\zeta, \|g\|_\zeta\}$ for free.

Examples of Multiplicative Seminorms on $\mathbb{C}_v[z]$

1. For any $x \in \mathbb{C}_v$, define $\| \cdot \|_x$ by $\|f\|_x := |f(x)|_v$.
2. For any disk $D \subseteq \mathbb{C}_v$, define $\| \cdot \|_D$ by

$$\|f\|_D := \sup\{|f(x)|_v : x \in D\}.$$

FYI: for $D_1 = \overline{D}(a,r)$ and $D_2 = D(a,r)$, we have $\| \cdot \|_{D_1} = \| \cdot \|_{D_2}$.

So we denote both by $\| \cdot \|_{\zeta(a,r)}$.

The Berkovich Projective Line: Formal Definition

Definition

The **Berkovich affine line** $\mathbb{A}^1_{\mathrm{an}}$ is the set of all multiplicative seminorms on $\mathbb{C}_v[z]$.

The **Berkovich projective line** $\mathbb{P}^1_{\mathrm{an}}$ is $\mathbb{A}^1_{\mathrm{an}} \cup \{\infty\}$.

As topological spaces, we equip $\mathbb{A}^1_{\mathrm{an}}$ and $\mathbb{P}^1_{\mathrm{an}}$ with the **Gel'fand topology**.

This is the weakest topology such that for every $f \in \mathbb{C}_v[z]$, the map $\mathbb{A}^1_{\mathrm{an}} \to \mathbb{R}$ given by

$$\zeta \mapsto \|f\|_\zeta$$

is continuous.

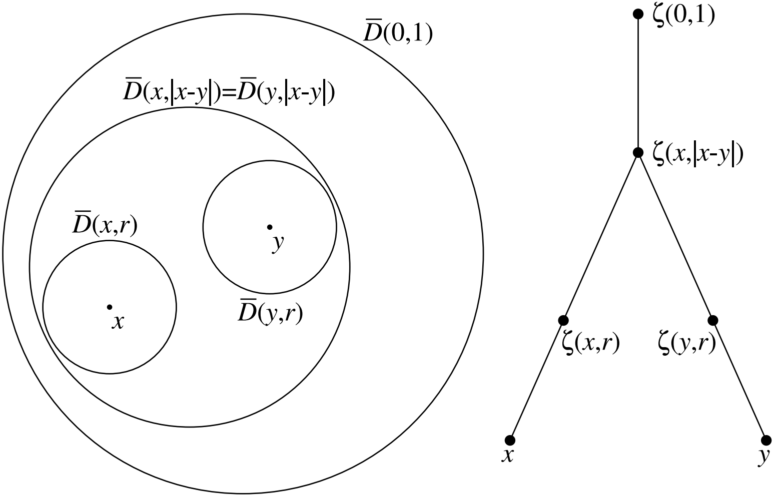
Some Properties of the Berkovich Projective Line (Recall)

The Berkovich projective line $\mathbb{P}^1_{\mathrm{an}}$:

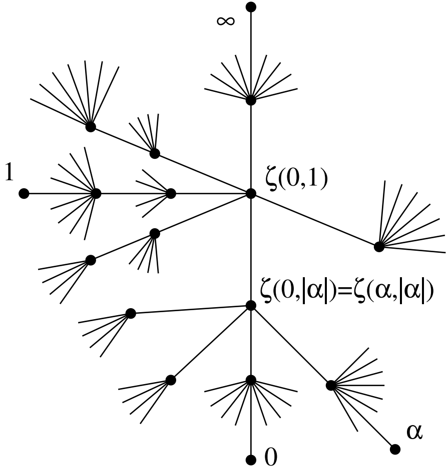
- contains $\mathbb{P}^1(\mathbb{C}_v)$ as a subspace ("Type I points")
- contains one point $\zeta(a, r)$ for each closed disk $\overline{D}(a, r) \subseteq \mathbb{C}_v$ ("Type II and type III points")
- is compact and Hausdorff
- is path-connected.

Each disk $D(a, r)$ or $\overline{D}(a, r)$ in $\mathbb{C}_v$ has a natural extension to $D_{\mathrm{an}}(a, r)$ or $\overline{D}_{\mathrm{an}}(a, r)$ in $\mathbb{P}^1_{\mathrm{an}}$.

Path-connectedness, intuitively



The Berkovich Projective Line $\mathbb{P}^1_{an}$



Rational Functions Acting on $\mathbb{P}^1_{\mathrm{an}}$

For $f \in \mathbb{C}_v(z)$ of degree $d \geq 2$,

$$f : \mathbb{P}^1_{\mathrm{an}} \to \mathbb{P}^1_{\mathrm{an}} \quad \text{by} \quad \|h\|_{f(\zeta)} := \|h \circ f\|_\zeta.$$

- f maps $\mathbb{P}^1_{\mathrm{an}}$ continuously onto itself.
- for $x \in \mathbb{P}^1(\mathbb{C}_v)$ of type I, $f(x)$ is the usual $f(x) \in \mathbb{P}^1(\mathbb{C}_v)$.
- If $f\big(\overline{D}(a,r)\big) = \overline{D}(b,s)$, then $f\big(\zeta(a,r)\big) = \zeta(b,s)$.

Non-archimedean Dynamics

$f \in \mathbb{C}_v(z)$ has an associated

- (Berkovich) *Fatou set* $\mathcal{F}_{\mathrm{an},f}$, and
- (Berkovich) *Julia set* $\mathcal{J}_{\mathrm{an},f} := \mathbb{P}^1_{\mathrm{an}} \setminus \mathcal{F}_{\mathrm{an},f}$

contained in $\mathbb{P}^1_{\mathrm{an}}$, such that:

- $\mathcal{F}_{\mathrm{an},f}$ is open in $\mathbb{P}^1_{\mathrm{an}}$, and $\mathcal{J}_{\mathrm{an},f}$ is closed (and hence compact).
- $f^{-1}(\mathcal{F}_{\mathrm{an},f}) = \mathcal{F}_{\mathrm{an},f}$ and $f^{-1}(\mathcal{J}_{\mathrm{an},f}) = \mathcal{J}_{\mathrm{an},f}$.
- Both $\mathcal{F}_{\mathrm{an},f}$ and $\mathcal{J}_{\mathrm{an},f}$ are nonempty.
- $\mathcal{J}_{\mathrm{an},f}$ is the smallest nonempty closed subset of $\mathbb{P}^1_{\mathrm{an}}$ that is invariant under f.

Fact: f has (explicit) good reduction iff $\mathcal{J}_{\mathrm{an},f} = \{\zeta(0,1)\}$.

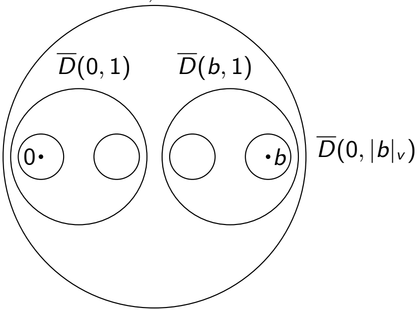
Example: Quadratic Polynomials

Let $f(z) = z^2 - bz \in \mathbb{C}_v[z]$.

Case 1: $|b|_v \leq 1$: then f has explicit good reduction, and $\mathcal{J}_{\mathrm{an},f} = \{\zeta(0,1)\}$.

- if $x, y \in \mathbb{C}_v$ with $|x|_v, |y|_v > 1$, then $f^n(x), f^n(y) \to \infty$.
- if $x, y \in \mathbb{C}_v$ with $|x|_v, |y|_v \leq 1$, then $|f^n(x) - f^n(y)|_v \leq |x - y|_v$ for all $n \geq 0$.

Case 2: $|b|_v > 1$: then $\mathcal{J}_{\mathrm{an},f}$ is a Cantor set in $\mathbb{C}_v$:

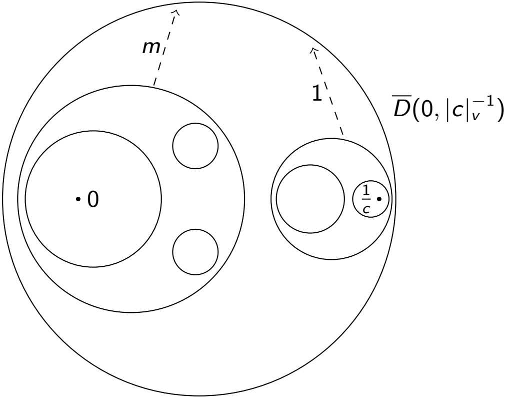


Example: A Higher-Degree Polynomial

Example Fix $m \geq 2$, and fix $c \in \mathbb{C}_v$ with $0 < |c|_v < 1$. Let

$$f(z) := cz^{m+1} - z^m + z \in \mathbb{C}_v[z].$$

For $x \in \mathbb{C}_v$ with $|x|_v > |c|_v^{-1}$, we have $f^n(x) \to \infty$.



Both $c^{-1} \in \mathbb{C}_v$ and $\zeta(0,1)$ are fixed points in $\mathcal{J}_{\mathrm{an},f}$.

Example: A Lattès Map

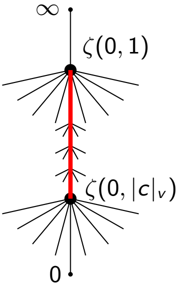
Example Fix $c \in \mathbb{C}_v$ with $0 < |c|_v \leq 1$. Let

$$f(z) := \frac{z^4 - 8c^2 z - c^2}{4z^3 + z^2 + 4c^2} \in \mathbb{C}_v(z),$$

the Lattès map for multiplication-by-2 on $E : y^2 + xy = x^3 + c^2$.

Case 1: $|c|_v = 1$: then f has explicit good reduction, and $\mathcal{J}_{\mathrm{an},f} = \{\zeta(0,1)\}$.

Case 2: $|c|_v < 1$: then $\mathcal{J}_{\mathrm{an},f}$ is the line segment in $\mathbb{P}^1_{\mathrm{an}}$ from $\zeta(0,|c|_v)$ to $\zeta(0,1)$.



It maps 2-to-1 onto itself, as a tent map.

Two Previous Non-Archimedean J-Stability Results

Theorem (T. Silverman, 2017)

Let $\{f_x\}_{x \in U}$ be a one-parameter analytic family for $U \subseteq \mathbb{A}^1_{\mathrm{an}}$ connected and open. Suppose

- *f_y has a type I repelling fixed point for some $y \in U$, and*
- *for all $x \in U$, f_x has no type I repelling periodic points of higher multiplicity, and no unstably indifferent periodic points.*

Then the family $\{f_x\}$ is J-stable on U.

Theorem (J. Lee, 2018)

Assume $f \in \mathrm{Rat}_d(\mathbb{C}_v)$ satisfies

- *$\mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v) \neq \varnothing$, and*
- *there exist $C > 0$ and $\lambda > 1$ such that*
$$(f^n)^{\#}(x) \geq C\lambda^n \text{ for all } x \in \mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v) \text{ and } n \geq 1.$$

Then f is J-stable, at least on $\mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v)$.

Thoughts on the Spherical Derivative

Recall the spherical derivative of $f \in \mathbb{C}(z)$

$$f^{\#}(z) := |f'(z)| \cdot \frac{1+|z|^2}{1+|f(z)|^2}$$

Idea: To non-archimedeanize something, replace $|A| + |B|$ by $\max\{|A|, |B|\}$.

So the non-archimedean spherical derivative of $f \in \mathbb{C}_v(z)$ on $\mathbb{P}^1(\mathbb{C}_v)$ should be $f^{\#}(z) := |f'(z)|_v \cdot \dfrac{\max\{1, |z|^2\}}{\max\{1, |f(z)|^2\}}$.

Idea: To Berkovichize something, replace $|f(x)|_v$ by $\|f\|_\zeta$.

So maybe the non-archimedean spherical derivative of $f \in \mathbb{C}_v(z)$ on $\mathbb{P}^1_{\mathrm{an}}$ should be $f^{\#}(\zeta) := \|f'\|_\zeta \cdot \dfrac{\max\{1, \|z\|_\zeta^2\}}{\max\{1, \|f\|_\zeta^2\}}$.

A (better) Berkovich Spherical Derivative

The **diameter** of $\zeta \in \mathbb{P}^1_{\text{an}}$ is:

- If $\zeta = x \in \mathbb{C}_v$ is of Type I, then $\text{diam}(\zeta) = 0$
- If $\zeta = \zeta(a, r)$ is of Type II or III, then $\text{diam}(\zeta) = r$.

Define the **spherical derivative** of f to be

$$f^{\natural}(x) := |f'(x)|_v \cdot \frac{\max\{1, |x|_v^2\}}{\max\{1, |f(x)|_v^2\}} \quad \text{if } x \in \mathbb{P}^1(\mathbb{C}_v),$$

and

$$f^{\natural}(\zeta) := \frac{\text{diam}(f(\zeta))}{\text{diam}(\zeta)} \cdot \frac{\max\{1, \|z\|_\zeta^2\}}{\max\{1, \|f\|_\zeta^2\}} \quad \text{if } \zeta \in \mathbb{P}^1_{\text{an}} \smallsetminus \mathbb{P}^1(\mathbb{C}_v).$$

A Fairly General Example

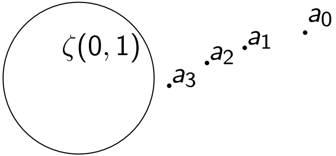
Fix $m \geq 2$ with $|m|_v = 1$, and fix $c \in \mathbb{C}_v$ with $0 < |c|_v < 1$. Let

$$f(z) := cz^{m+1} - z^m + z \in \mathbb{C}_v[z].$$

Then both $c^{-1} \in \mathbb{C}_v$ and $\zeta(0,1)$ are fixed points in $\mathcal{J}_{\mathrm{an},f}$.

Define a sequence $\{a_n\}_{n\geq 0}$ by $a_0 := c^{-1}$, and

$$f(a_{n+1}) = a_n \quad \text{and} \quad |a_n|_v = |c|_v^{-1/m^n} \quad \text{for every } n \geq 0.$$



A simple computation shows $(f^i)^\natural(a_n) < |c|_v^{-3}$ for every $0 \leq i \leq n$, even though $a_n \in \mathcal{J}_{\mathrm{an},f}$.

Also, $\zeta = \zeta(0,1)$ has $(f^n)^\natural(\zeta) = 1$ for all $n \geq 0$.

Another Example

Assume $\mathbb{C}_v$ has residue characteristic 0, and fix $0 < |c|_v < 1$. Let

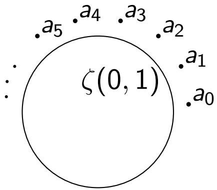
$$f(z) := \frac{(z+c)(z+1)}{z} = z + (c+1) + \frac{c}{z} \in \mathbb{C}_v(z).$$

There is a type I Julia point $a_0 \in D(0,1)$, and $\zeta(0,1)$ is a type II fixed point in $\mathcal{J}_{\mathrm{an},f}$.

Note $f\big(D_{\mathrm{an}}(x,1)\big) = D_{\mathrm{an}}(x+1,1)$ for every $x \in \mathbb{C}_v$ with $|x|_v = 1$.

Define a sequence $\{a_n\}_{n\geq 0}$ by

$$f(a_{n+1}) = a_n \quad \text{and} \quad a_n \in D(-n,1) \quad \text{for every } n \geq 0.$$



Then $(f^n)^\natural(a_n) = 1$ for every $n \geq 0$, even though $a_n \in \mathcal{J}_{\mathrm{an},f}$.

Again, $\zeta = \zeta(0,1)$ has $(f^n)^\natural(\zeta) = 1$ for all $n \geq 0$.

Moral

Even if we care only about the type I points of the Julia set, any *strictly* expansive condition like:

1. There exists $\sigma : \mathcal{J}_f \to (0, \infty)$ continuous such that $f^{\#}(z)\sigma(f(z)) > \sigma(z)$ for all $z \in \mathcal{J}_{\mathrm{an},f}$.
2. There exist $C > 0$ and $\lambda > 1$ such that $(f^n)^{\#}(z) \geq C\lambda^n$ for all $z \in \mathcal{J}_{\mathrm{an},f}$ and $n \geq 1$.
3. All critical points of f are attracted to attracting cycles.

is **TOO RESTRICTIVE** in non-archimedean dynamics.

A Stability Theorem

Theorem (B-Lee, 2021)

Let $\mathbb{C}_v$ be a complete, algebraically closed non-archimedean field of characteristic 0. Let $f \in \mathbb{C}_v(z)$ with $d := \deg f \geq 2$. Suppose there exists $\delta > 0$ such that

$$(f^n)^\natural(\zeta) \geq \delta \quad \text{for all } \zeta \in \mathcal{J}_{\mathrm{an},f} \text{ and } n \geq 0.$$

Then f is J-stable. More precisely, there exist:

- *a neighborhood $W \subseteq \mathrm{Rat}_d(\mathbb{C}_v)$ of f and*
- *an open set $U \subseteq \mathbb{P}^1_{\mathrm{an}}$ containing $\mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v)$*

so that for each $g \in W$, there is a homeomorphism $h : U \cup \mathcal{J}_{\mathrm{an},f} \to U \cup \mathcal{J}_{\mathrm{an},g}$ for which

1. *h is an isometry on the type I points $U \cap \mathbb{P}^1(\mathbb{C}_v)$ of U,*
2. *h is the identity map on $\mathcal{J}_{\mathrm{an},f} \smallsetminus U$,*
3. *$h \circ f = g \circ h$, and*
4. *$h(\mathcal{J}_{\mathrm{an},f}) = \mathcal{J}_{\mathrm{an},g}$.*

Sketch of Proof: Setup

Change coordinates so that $\mathcal{J}_{\mathrm{an},f} \subseteq \overline{D}_{\mathrm{an}}(0,1)$.
Pick $\varepsilon > 0$ so that f is injective on $D_{\mathrm{an}}(a,\varepsilon)$ for every $a \in \mathbb{C}_v$ for which $D_{\mathrm{an}}(a,\varepsilon) \cap \mathcal{J}_{\mathrm{an},f} \neq \varnothing$.

Without loss of generality, assume $\delta, \varepsilon < 1$.

For each $\zeta \in \mathcal{J}_{\mathrm{an},f}$, define

$$\sigma(\zeta) := \inf\left\{ (f^n)^\natural(\zeta) \,\middle|\, n \geq 0 \right\}.$$

Then for all $\zeta \in \mathcal{J}_{\mathrm{an},f}$,

- $\delta \leq \sigma(\zeta) \leq 1$
- $f^\natural(\zeta)\sigma\big(f(\zeta)\big) \geq \sigma(\zeta)$

Sketch of Proof: Domain of the Conjugacy

For $\zeta \in \mathcal{J}_{\mathrm{an},f}$, recall $\sigma(\zeta) := \inf\left\{ \left(f^n\right)^\natural(\zeta) \,\middle|\, n \geq 0 \right\}$. Define

$$\nu(\zeta) := \frac{\delta^2 \varepsilon}{\sigma(\zeta)} \quad \text{and} \quad \mathcal{J}^0_{\mathrm{an},f} := \{\zeta \in \mathcal{J}_{\mathrm{an},f} \mid \mathrm{diam}(\zeta) < \nu(\zeta)\}$$

and

$$\Omega := \bigcup_{\zeta \in \mathcal{J}^0_{\mathrm{an},f}} D_{\mathrm{an}}\left(\zeta, \nu(\zeta)\right)$$

Using $f^\natural(\zeta)\sigma(f(\zeta)) \geq \sigma(\zeta)$, we can show $f^{-1}(\Omega) \subseteq \Omega$.

Let $U := f^{-1}(\Omega)$. The conjugacy h will map $U \cup \mathcal{J}_{\mathrm{an},f}$ to itself, fixing all points of $\mathcal{J}_{\mathrm{an},f} \smallsetminus U$.

Sketch of Proof: Neighborhood in the Moduli Space

Lemma

There is an open neighborhood $W \subseteq \mathrm{Rat}_d(\mathbb{C}_v)$ of f such that for all $g \in W$,

- $\mathcal{J}_{\mathrm{an},g} \subseteq \overline{D}_{\mathrm{an}}(0,1)$
- $\left|g(x) - f(x)\right|_v < \dfrac{\delta^2 \varepsilon}{2}$ *for all* $x \in f^{-1}(\overline{D}(0,1))$

Moreover, for every $g \in W$ and open disk $D \subseteq U := f^{-1}(\Omega)$,

$$g \text{ maps } D \text{ bijectively onto } f(D).$$

In particular, g has a local inverse

$$G_D := \left(g\big|_D\right)^{-1} : f(D) \to D$$

Sketch of Proof: The Inductive Construction

$$\begin{array}{ccccccc}
\cdots f^{-3}(\Omega) & \xrightarrow{f} & f^{-2}(\Omega) & \xrightarrow{f} & f^{-1}(\Omega) & \xrightarrow{f} & \Omega \\
\downarrow h_3 & & \downarrow h_2 & & \downarrow h_1 & & \downarrow h_0 = \mathrm{id} \\
\cdots g^{-3}(\Omega) & \xrightarrow{g} & g^{-2}(\Omega) & \xrightarrow{g} & g^{-1}(\Omega) & \xrightarrow{g} & \Omega
\end{array}$$

On each open disk $D \subseteq U := f^{-1}(\Omega) = g^{-1}(\Omega)$:

$$\begin{array}{ccc}
f^{-n}(\Omega) \cap D & \xrightarrow{f} & f^{-(n-1)}(\Omega) \cap f(D) \\
 & & \downarrow h_{n-1} \\
g^{-n}(\Omega) \cap D & \xrightarrow{g} & g^{-(n-1)}(\Omega) \cap f(D)
\end{array}$$

Define h_n on $f^{-n}(\Omega) \cap D$ by $h_n := G_D \circ h_{n-1} \circ f$
where $G_D : f(D) \to D$ is the local inverse of $g : D \to f(D)$.

Sketch of Proof: The Inductive Construction

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\downarrow h_n & & \downarrow h_{n-1} \\
g^{-n}(\Omega) \cap D & \xleftarrow{G_D} & g^{-(n-1)}(\Omega) \cap f(D)
\end{array}
$$

Define h_n on $f^{-n}(\Omega) \cap D$ by $h_n := G_D \circ h_{n-1} \circ f$
where $G_D : f(D) \to D$ is the local inverse of $g : D \to f(D)$.

Finishing the Proof

$$\begin{array}{ccccccc}
\cdots f^{-3}(\Omega) & \xrightarrow{f} & f^{-2}(\Omega) & \xrightarrow{f} & f^{-1}(\Omega) & \xrightarrow{f} & \Omega \\
\downarrow h_3 & & \downarrow h_2 & & \downarrow h_1 & & \downarrow h_0 = \mathrm{id} \\
\cdots g^{-3}(\Omega) & \xrightarrow{g} & g^{-2}(\Omega) & \xrightarrow{g} & g^{-1}(\Omega) & \xrightarrow{g} & \Omega
\end{array}$$

There are many details to check, including:

- Each h_n is well-defined, a homeomorphism, and an isometry on $\mathbb{P}^1(\mathbb{C}_v)$.
- $\displaystyle\bigcap_{n\geq 0} f^{-n}(\Omega) = \mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v)$.
- The limit $h := \lim h_n$ converges and is a homeomorphism.
- $h \circ f = g \circ h$ on $U \cup \mathcal{J}_{\mathrm{an},f}$, and $h(\mathcal{J}_{\mathrm{an},f}) = \mathcal{J}_{\mathrm{an},g}$.

Reminder of the Theorem

Theorem (B-Lee)

Let $f \in \mathbb{C}_v(z)$ *with* $d := \deg f \geq 2$. *Suppose there exists* $\delta > 0$ *such that*

$$(f^n)^\natural(\zeta) \geq \delta \quad \textit{for all } \zeta \in \mathcal{J}_{\mathrm{an},f} \textit{ and } n \geq 0.$$

Then f *is J-stable. More precisely, there exist:*

- *a neighborhood* $W \subseteq \mathrm{Rat}_d(\mathbb{C}_v)$ *of* f *and*
- *an open set* $U \subseteq \mathbb{P}^1_{\mathrm{an}}$ *containing* $\mathcal{J}_{\mathrm{an},f} \cap \mathbb{P}^1(\mathbb{C}_v)$

so that for each $g \in W$*, there is a homeomorphism* $h : U \cup \mathcal{J}_{\mathrm{an},f} \to U \cup \mathcal{J}_{\mathrm{an},f}$ *for which*

1. h *is an isometry on the type I points* $U \cap \mathbb{P}^1(\mathbb{C}_v)$ *of* U,
2. h *is the identity map on* $\mathcal{J}_{\mathrm{an},f} \setminus U$,
3. $h \circ f = g \circ h$, *and*
4. $h(\mathcal{J}_{\mathrm{an},f}) = \mathcal{J}_{\mathrm{an},g}$.

And now we see $U := f^{-1}(\Omega)$.

What's good reduction?

$\mathbb{C}_v$ has a **ring of integers** $\mathcal{O}_v := \{x \in \mathbb{C}_v : |x|_v \le 1\}$, with maximal ideal $\mathcal{M}_v := \{x \in \mathbb{C}_v : |x|_v < 1\}$.

The **residue field** is $k := \mathcal{O}_v/\mathcal{M}_v$.

Any $f \in \mathbb{C}_v(z)$ of degree $d \ge 1$ can be written as $f = g/h$ with $g, h \in \mathcal{O}_v[z]$ relatively prime, and so that at least one coefficient c has $|c|_v = 1$, and $\max\{\deg g, \deg h\} = d$.

Let $\overline{g}, \overline{h} \in k[z]$ be the reduced polynomials, i.e., mod out the coefficients by $\mathcal{M}_v$.

If $\overline{f} := \overline{g}/\overline{h} \in k(z)$ has $\deg(\overline{f}) = d$, then f has **explicit good reduction**.

If f is conjugate to a map of explicit good reduction, we say f has **good reduction**.

So how are the Berkovich Fatou and Julia sets defined?

Let $f \in \mathbb{C}_v(z)$.

Definition

An open set $U \subseteq \mathbb{P}^1_{\text{an}}$ is *dynamically stable* under f if $\mathbb{P}^1_{\text{an}} \setminus \left(\bigcup_{n \geq 0} f^n(U) \right)$ is infinite.

Definition

The (*Berkovich*) *Fatou set* of f is

$$\mathcal{F}_{\text{an},f} := \{x \in \mathbb{P}^1_{\text{an}} \,|\, x \text{ has a dynamically stable neighborhood}\},$$

and the (*Berkovich*) *Julia set* of f is $\mathcal{J}_{\text{an},f} := \mathbb{P}^1_{\text{an}} \setminus \mathcal{F}_{\text{an},f}$.

So why is that Berkovich spherical derivative natural?

Recall: for $\zeta \in \mathbb{P}^1_{\mathrm{an}}$,

$$f^\natural(\zeta) = \begin{cases} |f'(\zeta)|_v \cdot \frac{\max\{1,\|z\|_\zeta^2\}}{\max\{1,\|f\|_\zeta^2\}} & \text{if } \zeta \in \mathbb{P}^1(\mathbb{C}_v), \\ \frac{\operatorname{diam}(f(\zeta))}{\operatorname{diam}(\zeta)} \cdot \frac{\max\{1,\|z\|_\zeta^2\}}{\max\{1,\|f\|_\zeta^2\}} & \text{otherwise.} \end{cases}$$

Why not $f^\natural(\zeta) \stackrel{?}{=} \|f'\|_\zeta \cdot \dfrac{\max\{1, \|z\|_\zeta^2\}}{\max\{1, \|f(z)\|_\zeta^2\}}$?

Example. $f(z) = z^p$, where $p \geq 2$ is the residue characteristic of $\mathbb{C}_v$.
f has good reduction, so $\mathcal{J}_{\mathrm{an},f} = \{\zeta(0,1)\}$.

Then $\|f'\|_\zeta \cdot \dfrac{\max\{1, \|z\|_\zeta^2\}}{\max\{1, \|f(z)\|_\zeta^2\}} = |p|_v < 1$ for $\zeta \in \mathcal{J}_{\mathrm{an},f}$.

So f would be *contracting* on the Julia set by that definition!

A natural way to define this Berkovich spherical derivative

The *spherical/chordal distance* on $\mathbb{P}^1(\mathbb{C}_v)$ is

$$\rho(x,y) = \frac{|x-y|}{\max\{1,|x|\}\max\{1,|y|\}},$$

which extends to the *spherical kernel*

$$\|\zeta,\xi\| := \limsup_{(x,y)\to(\zeta,\xi)} \rho(x,y)$$

where the limsup is over $(x,y) \in \mathbb{P}^1(\mathbb{C}_v) \times \mathbb{P}^1(\mathbb{C}_v)$.

The spherical kernel $\|\cdot,\cdot\| : \mathbb{P}^1_{\mathrm{an}} \times \mathbb{P}^1_{\mathrm{an}} \to [0,1]$ is:

- symmetric,
- upper semicontinuous,
- continuous in each variable separately,
- continuous at (ζ,ξ) *unless* $\zeta = \xi \in \mathbb{P}^1_{\mathrm{an}} \smallsetminus \mathbb{P}^1(\mathbb{C}_v)$.

We have $f^\natural(\zeta) = \lim_{\xi\to\zeta} \frac{\|f(\zeta),f(\xi)\|}{\|\zeta,\xi\|}$

How Can Non-archimedean J-Stability Fail?

Way # 1: Let $x \in \mathbb{P}^1(\mathbb{C}_v)$ be a type I critical point lying in the Julia set $\mathcal{J}_{\text{an},f}$.

- Clearly x fails the condition $(f^n)^\natural(\zeta) \geq \delta$, since $f^\natural(x) = 0$.
- Perturbing f in Rat_d (probably) makes the critical point leave $\mathcal{J}_{\text{an},f}$, so it's not J-stable.

Way # 2: Wild phenomena: locally m-to-1 mapping with $|m|_v < 1$.

- Let a periodic type II point of multiplier m interact with a repelling periodic type I point.
- In general, such dynamics appears to fail both J-stability and the $(f^n)^\natural(\zeta) \geq \delta$ condition.

A Wild Example

Let $\mathbb{C}_v = \mathbb{C}_p$, which is characteristic zero with $0 < |p|_v < 1$.

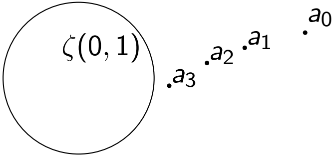
Fix $m \geq 2$ with $p|m$, so $0 < |m|_v < 1$. Fix $c \in \mathbb{C}_v$ with $0 < |c|_v < 1$. Let

$$f(z) := cz^{m+1} - z^m + z \in \mathbb{C}_v[z].$$

Then both $c^{-1} \in \mathbb{C}_v$ and $\zeta(0,1)$ are fixed points in $\mathcal{J}_{\text{an},f}$.

Define a sequence $\{a_n\}_{n\geq 0}$ by $a_0 := c^{-1}$, and

$$f(a_{n+1}) = a_n \quad \text{and} \quad |a_n|_v = |c|_v^{-1/m^n} \quad \text{for every } n \geq 0.$$



A simple computation shows $(f^i)^\natural(a_n) < |m|_v^i |c|_v^{-3}$ for every $0 \leq i \leq n$, even though $a_n \in \mathcal{J}_{\text{an},f}$.

But $\zeta = \zeta(0,1)$ has $(f^n)^\natural(\zeta) = 1$ for all $n \geq 0$.