Computing the entropy of certain p-adic dynamical systems

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AMS Special Session on Number Theory
in Ergodic Theory and Dynamical Systems

Saturday, March 7, 2015

p-adic dynamics

Fix a prime number p. (In this talk, usually $p = 3$.)

$\mathbb{C}_p$ is a complete, algebraically closed field containing $\mathbb{Q}$ but equipped with **the p-adic absolute value** $|\cdot|_p$.

$$\left|\frac{r}{s}p^n\right|_p = p^{-n} \quad \text{for } r, s \in \mathbb{Z} \text{ not divisible by } p.$$

In particular,

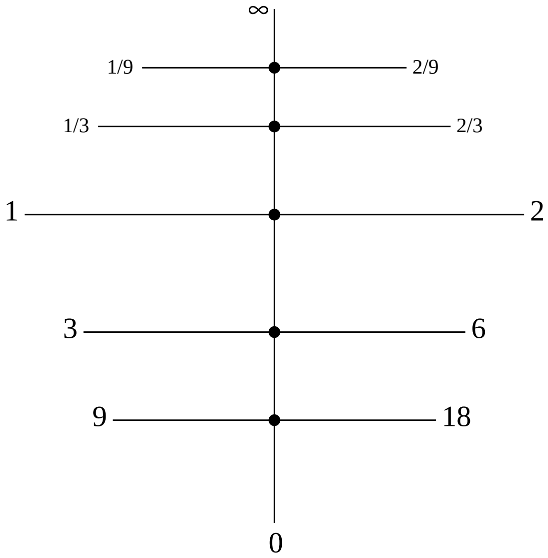
- all integers have $|n|_p \leq 1$,
- factors of p in your numerator make you small,
- factors of p in your denominator make you big.

Goal: Do dynamics, and in particular ergodic theory, over $\mathbb{C}_p$.

Problem: The naive analog of the Riemann sphere, $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$, is not a good setting for ergodic theory. (Mainly: not compact.)

Idea: Make a bigger space $\mathbb{P}^1_{\text{Ber}}$ containing $\mathbb{P}^1(\mathbb{C}_p)$, but **compact and path-connected**.

Building the Berkovich Projective Line



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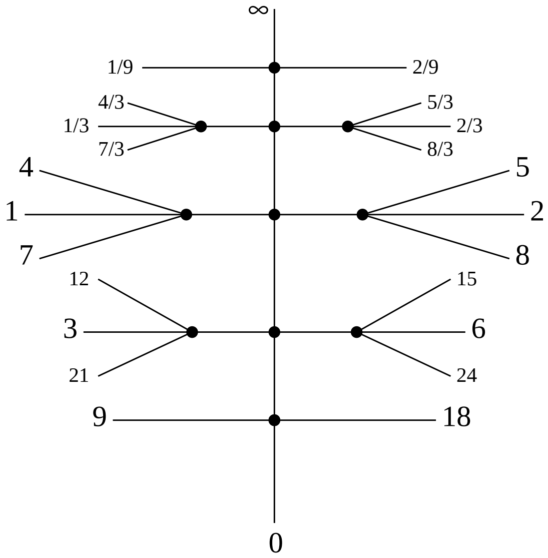

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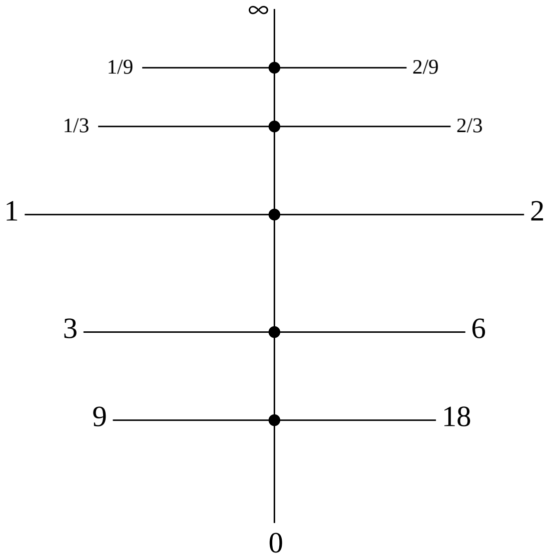

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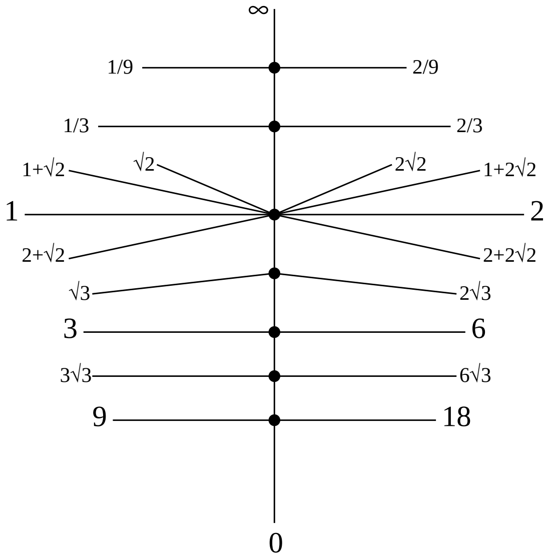

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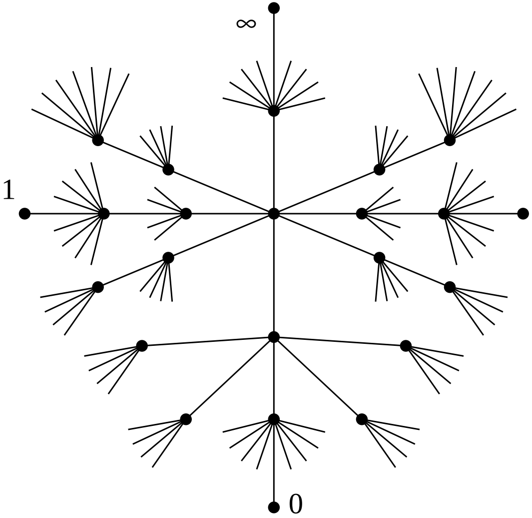


Dynamics on the Berkovich Projective Line

Let $\phi(z) = \dfrac{f(z)}{g(z)} \in \mathbb{C}_p(z)$ be a rational function of degree $d = \deg \phi := \max\{\deg f, \deg g\}$. Then $\phi : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p)$, and this action extends continuously to $\phi : \mathbb{P}^1_{\mathrm{Ber}} \to \mathbb{P}^1_{\mathrm{Ber}}$.

Associated to ϕ are:

- A **Julia set** $\mathcal{J} = \mathcal{J}_\phi \subseteq \mathbb{P}^1_{\mathrm{Ber}}$.
  - $\mathcal{J}$ is compact.
  - $\mathcal{J}$ is invariant under ϕ, i.e., $\phi^{-1}(\mathcal{J}) = \mathcal{J}$.
- A Borel probability measure $\mu = \mu_\phi$.
  - $\mathrm{supp}(\mu) = \mathcal{J}$.
  - μ is invariant under ϕ, i.e., $\mu(\phi^{-1}(E)) = \mu(E)$.

Entropy in $\mathbb{P}^1_{\text{Ber}}$

Theorem (Favre & Rivera-Letelier,2010)

Let $\phi \in \mathbb{C}_p(z)$ be a rational function of degree $d \geq 2$, with associated Julia set $\mathcal{J} \subseteq \mathbb{P}^1_{\text{Ber}}$ and invariant measure μ. Then

- *ϕ is ergodic with respect to μ, and*
- $0 \leq h_\mu(\phi) \leq h_{\text{top}}(\phi) \leq \log d.$

Example. $\phi(z) = z^2 + c$ with $|c|_p > 1$. Then $\mathcal{J}$ is a Cantor set, and the action of ϕ on $\mathcal{J}$ is symbolic dynamics on two symbols. So

$$h_\mu(\phi) = h_{\text{top}}(\phi) = \log 2.$$

Example. $\phi(z) = z^2 + c$ with $|c|_p \leq 1$. Then $\mathcal{J}$ is a single point. So

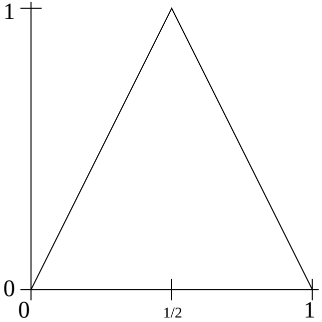
$$h_\mu(\phi) = h_{\text{top}}(\phi) = 0.$$

Lattès Maps

If ϕ is a Lattès map (arising from an elliptic curve E), there are two possibilities.

Case 1. If E has "(potential) good reduction at p," then the Julia set $\mathcal{J}$ of ϕ is again a single point, so $h_\mu(\phi) = h_{\mathrm{top}}(\phi) = 0$.

Case 2. Otherwise, E has "(potential) multiplicative reduction at p." Then $(\mathcal{J}, \mu)$ is homeomorphic to $([0, 1], \lambda)$, with ϕ given by:

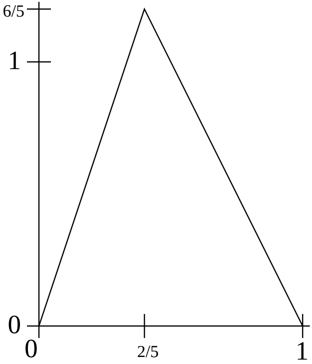


(assuming $\deg \phi = 4$). So $h_\mu(\phi) = h_{\mathrm{top}}(\phi) = \log 2$.

Non-Maximal Entropy

Favre and Rivera-Letelier gave examples where $h_\mu(\phi) < h_{\text{top}}(\phi)$.

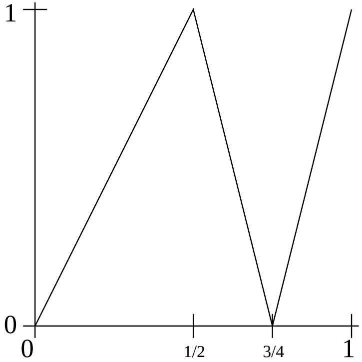
Example. Fix $a \in \mathbb{C}_p$ with $0 < |a|_p < 1$. Let $\phi(z) = \dfrac{z^3}{1 + az^5}$.
Then $\mathcal{J}$ is homeomorphic to a Cantor set in $[0, 1]$ with ϕ given by:



So $h_\mu(\phi) = \log 5 - \frac{2}{5} \log 2 - \frac{3}{5} \log 3 < \log 2 = h_{\text{top}}(\phi)$.

Non-Maximal Entropy and Connected Julia Set

Example. Fix $a \in \mathbb{C}_p$ with $0 < |a|_p < 1$. Let $\phi(z) = \dfrac{z^2(1 + a^2 z^8)}{1 + az^6}$.
Then $(\mathcal{J}, \mu)$ is homeomorphic to $([0,1], \lambda)$, with ϕ given by:



So $h_\mu(\phi) = \log 5 - \frac{4}{5} \log 2 < \log 3 = h_{\text{top}}(\phi)$.

Lower Degree?

Motivated by these examples, Favre and Rivera-Letelier ask:

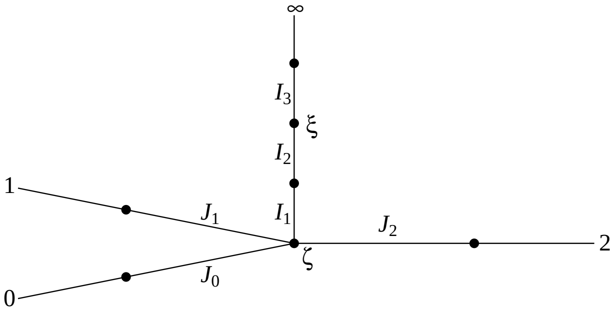
Question: Is there a rational function ϕ of degree ≤ 9 with connected Julia set $\mathcal{J}$ and with $h_\mu(\phi) < h_{\text{top}}(\phi)$?

Our Answer: For $p = 3$, the answer is **yes**.

Fix $a \in \mathbb{C}_3$ with $|3|_3 \leq |a|_3 < 1$, e.g. $a = 3$.

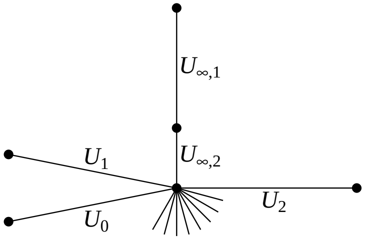
Let $\phi(z) = \dfrac{az^6 + 1}{az^6 + z^3 - z} = 1 + \dfrac{-z^3 + z + 1}{az^6 + z^3 - z}$.

Dynamics of $\phi(z) = \dfrac{az^6 + 1}{az^6 + z^3 - z} = 1 + \dfrac{-z^3 + z + 1}{az^6 + z^3 - z}$



- $\zeta \mapsto \zeta$, and $J_0, J_1, J_2 \to I := I_1 \cup I_2 \cup I_3$.
- $\xi \mapsto \zeta$, and $I_1, I_2 \to J_0$, and $I_3 \to J_1$.
- The (countably many) other branches off ξ and ζ map to the various branches off ζ (including J_1 and J_2).

A Countable Generator for ϕ



Partition $\mathcal{J}$ into $U_{\infty,1}$, $U_{\infty,2}$, U_0, U_1, $U_2, \ldots$.

- $U_0, U_1, U_2 \to (U_{\infty,1} \cup U_{\infty,2})$
- $U_{\infty,2} \to U_0$, and $U_{\infty,1} \to (U_{\infty,1} \cup U_{\infty,2} \cup U_0 \cup U_1 \cup U_2 \cup \cdots)$
- for each $c \neq (\infty, 1), (\infty, 2), 0$, there are three U_d's with $U_d \to U_c$.

It can be shown that $\mathcal{P} = \{U_{\infty,1}, U_{\infty,2}, U_0, U_1, U_2, \ldots\}$ is a countable generator for ϕ of finite entropy.

The Shift on Countably Many Symbols

The symbolic dynamics on the symbols $U_{\infty,1}, U_{\infty,2}, U_0, U_1, U_2, \ldots$ is given by the (transpose of the) transition matrix

$$\begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

Computing $h_{\text{top}}(\phi)$ and $h_\mu(\phi)$

Truncating to an $n \times n$ matrix, finding the maximal real eigenvalue, and taking the limit as $n \to \infty$, we get $h_{\text{top}}(\phi) = \log \beta$, where $\beta \approx 3.8558$ is the largest real root of $t^3 - 4t^2 - t + 6$.

Meanwhile, Favre and Rivera-Letelier give a simple expression for Jac_ϕ, so we can invoke a formula of Rokhlin to compute $h_\mu(\phi)$. (Alternately, one can work out the $\infty \times \infty$ Markov matrix based on the previous slide.) The upshot:

$$\begin{aligned} h_\mu(\phi) &= (\log 2)\mu(U_{\infty,1} \cup U_{\infty,2}) + (\log 6)\mu(U_0 \cup U_1 \cup U_2 \cup \cdots) \\ &= \frac{6}{11} \log 2 + \frac{5}{11} \log 6 \approx \log 3.2954 \end{aligned}$$

Thus, $0 < h_\mu(\phi) < h_{\text{top}}(\phi) < \log 6$.

References

D. Bajpai, R. Benedetto, R. Chen, E. Kim, O. Marschall, D. Onul, Y. Xiao, Non-archimedean connected Julia sets with branching, `arXiv:1410.0591`

C. Favre, J. Rivera-Letelier, Théorie ergodique des fractions rationnelles sur un corps ultramétrique, *Proc. Lond. Math. Soc. (3)* **100** (2010), 116–154.

B. Gurevich, Topological entropy of enumerable Markov chains, *Dokl. Akad. Nauk SSSR* **187** (1969), 715–718; English translation: *Soviet Math. Dokl.* **10** (1969), 911–915.

O. Sarig, Thermodynamic formalism for countable Markov shifts, *Ergod. Theor. Dynam. Syst.* **19** (1999), 1565–1593.