Isolation of postcritically finite parameters in $p$-adic dynamical moduli spaces

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Notation

- $K$ is an algebraically closed field, usually $\mathbb{C}$ or $\mathbb{C}_p$
- $f(z) \in K(z)$ is a separable rational function of degree $d \geq 2$ 
  \((\deg(g/h) = \max\{\deg g, \deg h\})\)
- $f^n = f \circ f \circ \cdots \circ f$ is the $n$-th iterate of $f$

We say $x \in \mathbb{P}^1(K)$ is \textit{preperiodic} if $f^n(x) = f^m(x)$ for some $n > m \geq 0$. 
Postcritically Finite Maps

Definition
We say a separable map $f \in K(z)$ is postcritically finite, or PCF, if every critical point $c \in \mathbb{P}^1(K)$ of $f$ is preperiodic under $f$.

Example. $f(z) = z^d$: $\infty \mapsto \infty$, $0 \mapsto 0$

Example. $f(z) = z^2 - 1$: $\infty \mapsto \infty$, $0 \mapsto -1 \mapsto 0$

Example. $f(z) = z^2 - 2$: $\infty \mapsto \infty$, $0 \mapsto -2 \mapsto 2 \mapsto 2$

Example. $f(z) = z^2 + i$:
$\infty \mapsto \infty$, $0 \mapsto i \mapsto i - 1 \mapsto -i \mapsto i - 1$

Example. $f(z) = -2z^3 + 3z^2$: $\infty \mapsto \infty$, $0 \mapsto 0$, $1 \mapsto 1$

Example. $f(z) = \frac{6z^2 + 16z + 16}{-3z^2 - 4z - 4}$:
$0 \mapsto -4 \mapsto -\frac{4}{3} \mapsto -\frac{4}{3}$, $-2 \mapsto -1 \mapsto -2$
Flexible Lattès Maps

Definition (Simplified)
Let $E/K$ be an elliptic curve in Weierstrass form, and let $m \geq 2$. Then there exists $f \in K(x)$ of degree $m^2$ such that

\[
\begin{array}{ccc}
E & \xrightarrow{[m]} & E \\
\downarrow x & & \downarrow x \\
\mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1
\end{array}
\]

commutes. We say $f$ is a (flexible) Lattès map.

Fact: Every Lattès map is PCF.
Why should we care about PCF maps?

Many reasons, including:

▶ Interesting complex Julia sets.
▶ Thurston rigidity.
▶ Tower of preimage fields $\cdots K_3/K_2/K_1/K$ is ramified over only finitely many primes. (Aitken, Hajir, Maire 2005).
▶ and much much more.

Idea: PCF maps are special points in moduli spaces of dynamical systems, analogous to CM elliptic curves.
The Quadratic Polynomial Family

Define \( f_c(z) = z^2 + c \). Critical points are \( \infty \) (fixed) and 0.

\[
0 \mapsto c \mapsto c^2 + c \mapsto (c^2 + c)^2 + c \mapsto \ldots
\]

We say \( c \) is a **PCF parameter** if \( f_c^n(0) = f_c^m(0) \) for some \( n > m \geq 0 \).

**Example:** \( f(z) = z^2 \) has \( m = 0, n = 1 \)

**Example:** \( f(z) = z^2 - 1 \) has \( m = 0, n = 2 \)

**Example:** \( f(z) = z^2 - 2 \) has \( m = 2, n = 3 \)

**Example:** \( f(z) = z^2 + i \) has \( m = 2, n = 4 \)
Lots of PCF parameters

Example. Fix a PCF map $\phi(z) \in K(z)$, let $h_t(z) \in \text{PGL}(2, K(t))$ be a one-parameter family of linear fractional transformations, and let $f_t = h_t \circ \phi \circ h_t^{-1}$.

Then $f_t$ is PCF for all parameters $t$.

Example. Let $E_t$ be a one-parameter family of elliptic curves, and let $g_t$ be the Lattès map for $[m] : E_t \to E_t$.

Then $g_t$ is PCF for all parameters $t$.

Example. Let $K = \mathbb{C}$ and let $f_c(z) = z^2 + c$.

Then the PCF parameters $c$ are dense in the boundary of the Mandelbrot set.
For prime $p \geq 2$, $\mathbb{C}_p =$ completion of algebraic closure of $\mathbb{Q}_p$.

For $r > 0$ and $b \in \mathbb{C}_p$, let $D(b, r) := \{x \in \mathbb{C}_p : |x - b|_p < r\}$.

Fix $d \geq 2$, $b \in \mathbb{C}_p$, and $S > 0$.

Consider a one-parameter family of rational function $f_t(z)$, with coefficients meromorphic in $t \in D(b, S)$, such that for all $t \in D(b, S)$,

- $f_t(z) \in \mathbb{C}_p(z)$ with $\text{deg}(f_t) = d$,
- $f_t$ has good reduction, and
- the critical points of $f_t$ are $\alpha_1(t), \ldots, \alpha_{2d-2}(t)$.

(Also meromorphic functions of $t \in D(b, S)$)

We call $f_t$ a meromorphic family of good reduction.

**Example:** Fix $d \geq 2$ and fix $b \in \mathbb{C}_p$ with $|b|_p \leq 1$.

Let $f_t(z) = z^d + t$ for $t \in D(b, 1)$,

with $\alpha_1 = \ldots = \alpha_{d-1} = 0$, and $\alpha_d = \ldots = \alpha_{2d-2} = \infty$. 
Theorem (B-Ih 2019)

Let $f_t(z)$ be a meromorphic family of good reduction on $t \in D(b, S)$. Then either

1. $f_t$ is conjugate to $f_b$ for all $t \in D(b, S)$, or
2. $f_t$ is flexible Lattès for all $t \in D(b, S)$, or
3. for any $0 < s < S$, there are only finitely many $t \in D(b, s)$ for which $f_t$ is PCF.

Corollary

Let $f_t(z) = z^d + t$. Let

$$T = \{ t \in \mathbb{C}_p | f_t \text{ is PCF} \}$$

Then every point of $T$ is isolated.
Sketch of proof: Setup

Let \( \alpha = \alpha(t) \) be a critical point of \( f_t \).
Replacing \( f_t \) by \( f_t^N \) and changing coordinates, we can assume that:

\[
 f_t(\alpha(t)) = 0, \quad \text{and} \quad f_t^2(\alpha(t)) \in D(0, 1) \quad \text{for all} \ t \in D(b, S).
\]

Note: this implies \( f_t(D(0, 1)) = D(0, 1) \).

We must show either

1. there are integers \( n > m \geq 0 \) such that \( f_t^n(0) = f_t^m(0) \) for all \( t \in D(b, S) \), (i.e., \( \alpha(t) \) is persistently preperiodic), or

2. for any \( 0 < s < S \), there are only finitely many \( t \in D(b, s) \) for which 0 and every critical point of \( f_t \) in \( D(0, 1) \) are all preperiodic.

**Case 1:** \( |f'_b(0)|_p < 1 \)

**Case 2:** \( |f'_b(0)|_p = 1 \)
Case 1: $|f'_b(0)|_p < 1$

Then we can show $f_t$ has an attracting fixed point $\beta(t) \in D(0, 1)$ for every $t \in D(b, S)$.

For any $0 < s < S$, then a $p$-adic analysis argument (similar to that in B-Ingram-Jones-Levy 2014) shows there is an integer $n \geq 0$ (independent of $t$) so that for all $t \in D(b, s)$, either

1. $f^n_t(0) = \beta(t)$, or
2. $f^n_t(0) \neq \beta(t)$ but is very close, or
3. $f^n_t(c_t) \neq \beta(t)$ but is very close, for some critical point $c_t$.

When (2) or (3) happens, either $\alpha(t)$ or $c_t$ has infinite forward orbit under $f_t$. Thus, $f_t$ is not PCF.

If (1) happens infinitely often on $D(b, s)$, then the power series $f^n_t(0) - \beta(t) \in \mathbb{C}_p[[t - b]]$ has infinitely many zeros in a proper subdisk of $D(b, S)$ and hence is trivial.

Thus, if (1) happens infinitely often on $D(b, s)$, then $\alpha(t)$ is persistently preperiodic on $D(b, S)$. 
Case 2: \( |f'_b(0)|_p = 1 \)

Choose \( e \geq 1 \) so that \( |f'_b(0)^e - 1|_p < 1 \).

Then we can show \( |(f^e_t)'(0) - 1|_p < 1 \), and \( f^e_t \) maps \( D(0, 1) \) bijectively onto itself, for every \( t \in D(b, S) \).

The *iterative logarithm* of \( f_t \) is

\[
\Lambda_t(z) := \lim_{n \to \infty} p^{-n}(f^{ep^n}_t(z) - z),
\]

which is a (two-variable) power series converging on \( (t, z) \in D(b, S) \times D(0, 1) \), following Rivera-Letelier 2003.

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**Idea:** \( \Lambda_t(z) \) measures how close \( f^{ep^n}_t(z) \) is to \( z \), relative to \( p^n \).

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Define \( F(t) := \Lambda_t(0) \in \mathbb{C}_p[[t - b]] \), which is a power series converging on \( D(b, S) \).
Case 2: $|f'_b(0)|_p = 1$: continued

$$\Lambda_t(z) = \lim_{n \to \infty} p^{-n}(f^{ep^n}(z) - z), \quad \text{and} \quad F(t) = \Lambda_t(0)$$

By results of Rivera-Letelier, *Astérisque* 2003 (Section 3.2) on the iterative logarithm,

$$F(t) = 0 \text{ iff } z = 0 \text{ is periodic under } f_t,$$

i.e., iff $\alpha(t)$ is preperiodic under $f_t$.

If $F$ is identically zero, then for each $t \in D(b, S)$, there are integers $n(t) > m(t) \geq 0$ so that $f_t^{n(t)}(\alpha(t)) = f_t^{m(t)}(\alpha(t))$.

Some such pair $n > m$ occurs uncountably often, so $f_t^n(\alpha(t)) = f_t^m(\alpha(t))$ for all $t \in D(b, S)$.

Otherwise, for any $0 < s < S$, there are only finitely many $t \in D(b, s)$ for which $\alpha(t)$ is preperiodic under $f_t$. 
Conclusion of the Proof

Applying the preceding arguments to each critical point $\alpha_i(t)$ of $f_t(z)$, then either

1. For every $i = 1, \ldots, 2d - 2$, there are integers $n_i > m_i \geq 0$ such that $f_t^{n_i}(\alpha_i(t)) = f_t^{m_i}(\alpha_i(t))$ for all $t \in D(b, S)$, or

2. For every $0 < s < S$, there are only finitely many $t \in D(b, s)$ for which $f_t$ is PCF.

If (1) happens, Thurston Rigidity (Douady and Hubbard, 1993) says that either

- Every $f_t$ is Lattès, or
- $f_t$ is conjugate to $f_u$ for uncountably many distinct $t, u$, and hence for all $t, u \in D(b, S)$.

(2) and the two above possibilities for (1) are the three outcomes stated in the Theorem. QED
Main Theorem, again

Theorem

Let $f_t(z)$ be a meromorphic family of good reduction on $t \in D(b, S)$. Then either

1. $f_t$ is conjugate to $f_b$ for all $t \in D(b, S)$, or
2. $f_t$ is flexible Latt`es for all $t \in D(b, S)$, or
3. for any $0 < s < S$, there are only finitely many $t \in D(b, s)$ for which $f_t$ is PCF.