Isolation of postcritically finite parameters in p-adic dynamical moduli spaces

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Notation

- K is an algebraically closed field, usually $\mathbb{C}$ or $\mathbb{C}_p$
- $f(z) \in K(z)$ is a separable rational function of degree $d \geq 2$ $(\deg(g/h) = \max\{\deg g, \deg h\})$
- $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n}$ is the n-th iterate of f

We say $x \in \mathbb{P}^1(K)$ is *preperiodic* if $f^n(x) = f^m(x)$ for some $n > m \geq 0$.

Postcritically Finite Maps

Definition

We say a separable map $f \in K(z)$ is *postcritically finite*, or *PCF*, if every critical point $c \in \mathbb{P}^1(K)$ of f is preperiodic under f.

Example. $f(z) = z^d$: $\quad \infty \mapsto \infty \quad 0 \mapsto 0$

Example. $f(z) = z^2 - 1$: $\quad \infty \mapsto \infty \quad 0 \mapsto -1 \mapsto 0$

Example. $f(z) = z^2 - 2$: $\quad \infty \mapsto \infty \quad 0 \mapsto -2 \mapsto 2 \mapsto 2$

Example. $f(z) = z^2 + i$:

$$\infty \mapsto \infty \qquad 0 \mapsto i \mapsto i - 1 \mapsto -i \mapsto i - 1$$

Example. $f(z) = -2z^3 + 3z^2$: $\quad \infty \mapsto \infty \quad 0 \mapsto 0 \quad 1 \mapsto 1$

Example. $f(z) = \dfrac{6z^2 + 16z + 16}{-3z^2 - 4z - 4}$:

$$0 \mapsto -4 \mapsto -\frac{4}{3} \mapsto -\frac{4}{3} \qquad -2 \mapsto -1 \mapsto -2$$

Flexible Lattès Maps

Definition (Simplified)

Let E/K be an elliptic curve in Weierstrass form, and let $m \geq 2$. Then there exists $f \in K(x)$ of degree m^2 such that

$$\begin{array}{ccc} E & \xrightarrow{[m]} & E \\ \Big\downarrow x & & \Big\downarrow x \\ \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

commutes. We say f is a (*flexible*) *Lattès map*.

Fact: Every Lattès map is PCF.

Why should we care about PCF maps?

Many reasons, including:

- Interesting complex Julia sets.
- Thurston rigidity.
- Tower of preimage fields $\cdots K_3/K_2/K_1/K$ is ramified over only finitely many primes. (Aitken, Hajir, Maire 2005).
- and much much more.

Idea: PCF maps are special points in moduli spaces of dynamical systems, analogous to CM elliptic curves.

The Quadratic Polynomial Family

Define $f_c(z) = z^2 + c$. Critical points are ∞ (fixed) and 0.

$$0 \mapsto c \mapsto c^2 + c \mapsto (c^2 + c)^2 + c \mapsto \cdots$$

We say c is a **PCF parameter** if $f_c^n(0) = f_c^m(0)$ for some $n > m \geq 0$.

Example: $f(z) = z^2$ has $m = 0,\ n = 1$

Example: $f(z) = z^2 - 1$ has $m = 0,\ n = 2$

Example: $f(z) = z^2 - 2$ has $m = 2,\ n = 3$

Example: $f(z) = z^2 + i$ has $m = 2,\ n = 4$

Lots of PCF parameters

Example. Fix a PCF map $\phi(z) \in K(z)$, let $h_t(z) \in \mathrm{PGL}(2, K(t))$ be a one-parameter family of linear fractional transformations, and let $f_t = h_t \circ \phi \circ h_t^{-1}$.

Then f_t is PCF for all parameters t.

Example. Let E_t be a one-parameter family of elliptic curves, and let g_t be the Lattès map for $[m] : E_t \to E_t$.

Then g_t is PCF for all parameters t.

Example. Let $K = \mathbb{C}$ and let $f_c(z) = z^2 + c$.

Then the PCF parameters c are dense in the boundary of the Mandelbrot set.

p-adic Meromorphic Families of Good Reduction

For prime $p \geq 2$, $\mathbb{C}_p$ =completion of algebraic closure of $\mathbb{Q}_p$.

For $r > 0$ and $b \in \mathbb{C}_p$, let $D(b, r) := \{x \in \mathbb{C}_p : |x - b|_p < r\}$.

Fix $d \geq 2$, $b \in \mathbb{C}_p$, and $S > 0$.

Consider a one-parameter family of rational function $f_t(z)$, with coefficients meromorphic in $t \in D(b, S)$, such that for all $t \in D(b, S)$,

- $f_t(z) \in \mathbb{C}_p(z)$ with $\deg(f_t) = d$,
- f_t has good reduction, and
- the critical points of f_t are $\alpha_1(t), \ldots, \alpha_{2d-2}(t)$. (also meromorphic functions of $t \in D(b, S)$)

We call f_t a *meromorphic family of good reduction*.

Example: Fix $d \geq 2$ and fix $b \in \mathbb{C}_p$ with $|b|_p \leq 1$.
Let $f_t(z) = z^d + t$ for $t \in D(b, 1)$,
with $\alpha_1 = \ldots = \alpha_{d-1} = 0$, and $\alpha_d = \ldots = \alpha_{2d-2} = \infty$.

p-adic PCF parameters

Theorem (B-Ih 2019)

Let $f_t(z)$ be a meromorphic family of good reduction on $t \in D(b, S)$. Then either

1. *f_t is conjugate to f_b for all $t \in D(b, S)$, or*
2. *f_t is flexible Lattès for all $t \in D(b, S)$, or*
3. *for any $0 < s < S$, there are only finitely many $t \in D(b, s)$ for which f_t is PCF.*

Corollary

Let $f_t(z) = z^d + t$. Let

$$T = \left\{ t \in \mathbb{C}_p \middle| f_t \text{ is PCF } \right\}$$

Then every point of T is isolated.

Sketch of proof: Setup

Let $\alpha = \alpha(t)$ be a critical point of f_t.

Replacing f_t by f_t^N and changing coordinates, we can assume that:

$$f_t(\alpha(t)) = 0, \quad \text{and} \quad f_t^2(\alpha(t)) \in D(0,1) \quad \text{for all } t \in D(b,S).$$

Note: this implies $f_t(D(0,1)) = D(0,1)$.

We must show either

1. there are integers $n > m \geq 0$ such that $f_t^n(0) = f_t^m(0)$ for all $t \in D(b,S)$, (i.e., $\alpha(t)$ is *persistently preperiodic*), or
2. for any $0 < s < S$, there are only finitely many $t \in D(b,s)$ for which 0 and every critical point of f_t in $D(0,1)$ are all preperiodic.

Case 1: $|f_b'(0)|_p < 1$

Case 2: $|f_b'(0)|_p = 1$

Case 1: $|f_b'(0)|_p < 1$

Then we can show f_t has an attracting fixed point $\beta(t) \in D(0,1)$ for every $t \in D(b,S)$.

For any $0 < s < S$, then a p-adic analysis argument (similar to that in B-Ingram-Jones-Levy 2014) shows there is an integer $n \geq 0$ (**independent of** t) so that for all $t \in D(b,s)$, either

1. $f_t^n(0) = \beta(t)$, or
2. $f_t^n(0) \neq \beta(t)$ but is very close, or
3. $f_t^n(c_t) \neq \beta(t)$ but is very close, for some critical point c_t.

When (2) or (3) happens, either $\alpha(t)$ or c_t has infinite forward orbit under f_t. Thus, f_t **is not PCF**.

If (1) happens infinitely often on $D(b,s)$, then the power series $f_t^n(0) - \beta(t) \in \mathbb{C}_p[[t-b]]$ has infinitely many zeros in a proper subdisk of $D(b,S)$ and hence is trivial.
Thus, if (1) happens infinitely often on $D(b,s)$, then $\alpha(t)$ is persistently preperiodic on $D(b,S)$.

Case 2: $|f_b'(0)|_p = 1$

Choose $e \geq 1$ so that $|f_b'(0)^e - 1|_p < 1$.

Then we can show $|(f_t^e)'(0) - 1|_p < 1$, and f_t^e maps $D(0,1)$ bijectively onto itself, for **every** $t \in D(b,S)$.

The *iterative logarithm* of f_t is

$$\Lambda_t(z) := \lim_{n \to \infty} p^{-n}\left(f_t^{ep^n}(z) - z\right),$$

which is a (two-variable) power series converging on $(t,z) \in D(b,S) \times D(0,1)$, following Rivera-Letelier 2003.

Idea: $\Lambda_t(z)$ measures how close $f_t^{ep^n}(z)$ is to z, relative to p^n.

Define $F(t) := \Lambda_t(0) \in \mathbb{C}_p[[t-b]]$, which is a power series converging on $D(b,S)$.

Case 2: $|f_b'(0)|_p = 1$: continued

$$\Lambda_t(z) = \lim_{n\to\infty} p^{-n}\left(f_t^{ep^n}(z) - z\right), \quad \text{and} \quad F(t) = \Lambda_t(0)$$

By results of Rivera-Letelier, *Astérisque* 2003 (Section 3.2) on the iterative logarithm,

$$F(t) = 0 \text{ iff } z = 0 \text{ is periodic under } f_t,$$
$$\text{i.e., iff } \alpha(t) \text{ is preperiodic under } f_t.$$

If F is identically zero, then for each $t \in D(b, S)$, there are integers $n(t) > m(t) \geq 0$ so that $f_t^{n(t)}(\alpha(t)) = f_t^{m(t)}(\alpha(t))$.

Some such pair $n > m$ occurs uncountably often, so $f_t^n(\alpha(t)) = f_t^m(\alpha(t))$ for **all** $t \in D(b, S)$.

Otherwise, for any $0 < s < S$, there are only finitely many $t \in D(b, s)$ for which $\alpha(t)$ is preperiodic under f_t.

Conclusion of the Proof

Applying the preceding arguments to each critical point $\alpha_i(t)$ of $f_t(z)$, then either

1. For every $i = 1, \ldots, 2d-2$, there are integers $n_i > m_i \geq 0$ such that $f_t^{n_i}(\alpha_i(t)) = f_t^{m_i}(\alpha_i(t))$ for all $t \in D(b, S)$, or
2. For every $0 < s < S$, there are only finitely many $t \in D(b, s)$ for which f_t is PCF.

If (1) happens, Thurston Rigidity (Douady and Hubbard, 1993) says that either

- Every f_t is Lattès, or
- f_t is conjugate to f_u for uncountably many distinct t, u, and hence for all $t, u \in D(b, S)$.

(2) and the two above possibilities for (1) are the three outcomes stated in the Theorem. QED

Main Theorem, again

Theorem

Let $f_t(z)$ be a meromorphic family of good reduction on $t \in D(b, S)$. Then either

1. *f_t is conjugate to f_b for all $t \in D(b, S)$, or*
2. *f_t is flexible Lattès for all $t \in D(b, S)$, or*
3. *for any $0 < s < S$, there are only finitely many $t \in D(b, s)$ for which f_t is PCF.*