Attacking the Dynamical Uniform Boundedness Conjecture

Robert L. Benedetto
Amherst College

Algebraic Dynamics Workshop, CUNY Grad Center

Friday, June 11, 2010

Dynamics on $\mathbb{P}^1$

Let K be a field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

[$\deg \phi := \max\{\deg f_1, \deg f_2\}$, where $\phi = f_1/f_2$ in lowest terms.]

Definition

A point $z \in \mathbb{P}^1(\overline{K}) = \overline{K} \cup \{\infty\}$ is called **preperiodic** if

$$\phi^n(z) = \phi^m(z) \quad \text{for some } n > m \geq 0.$$

Write $\text{Preper}(\phi, K) := \{z \in \mathbb{P}^1(K) : z \text{ is preperiodic under } \phi\}$.

The Dynamical Uniform Boundedness Conjecture

From now on, K **is a global field**.

Theorem (Northcott, 1950)

Let $\phi \in K(z)$ of degree $d \geq 2$. Then

$$\#Preper(\phi, K) < \infty.$$

Conjecture (Morton & Silverman, 1994)

For any integer $d \geq 2$, there is a constant $C = C(d, K)$ such that for any $\phi \in K(z)$ of degree d,

$$\#Preper(\phi, K) \leq C(d, K).$$

Example: Testing for Preperiodicity

$$\phi(z) = z^2 - \frac{133}{144}.$$

Is $z = 0$ preperiodic?

$$0 \mapsto -\frac{133}{144} \quad \mapsto -\frac{1463}{20736} \quad \mapsto -\frac{394995503}{429981696} \quad \mapsto \cdots$$

$$0 \mapsto -\frac{(*)}{2^4 \cdot 3^2} \quad \mapsto \frac{(*)}{2^8 \cdot 3^4} \quad \mapsto \frac{(*)}{2^{16} \cdot 3^8} \quad \mapsto \cdots$$

No, it's not preperiodic.

More Preperiodicity Testing

Still with $\phi(z) = z^2 - \dfrac{133}{144}$: is $\dfrac{17}{12}$ preperiodic?

$$\frac{17}{12} \mapsto \frac{13}{12} \mapsto \frac{1}{4} \mapsto -\frac{31}{36} \mapsto -\frac{59}{324} \mapsto \cdots$$

$$\frac{(*)}{2^2 \cdot 3} \mapsto \frac{(*)}{2^2 \cdot 3} \mapsto \frac{(*)}{2^2 \cdot 3^0} \mapsto \frac{(*)}{2^2 \cdot 3^2} \mapsto \frac{(*)}{2^2 \cdot 3^4} \mapsto \cdots$$

No. What about $\dfrac{43}{12}$?

$$\frac{43}{12} \mapsto \frac{143}{12} \mapsto \frac{1693}{12} \mapsto \frac{238843}{12} \mapsto \frac{4753831543}{12} \mapsto \cdots$$

No.

Quadratic Polynomial Record Holders Over $\mathbb{Q}$

$$\phi(z) = z^2 - \frac{133}{144}. \qquad \infty \to \infty$$

$$\frac{7}{12} \to -\frac{7}{12} \to -\frac{7}{12} \qquad -\frac{19}{12} \to \frac{19}{12} \to \frac{19}{12}$$

$$\frac{1}{12} \to -\frac{11}{12} \leftrightarrows -\frac{1}{12} \leftarrow \frac{11}{12}$$

$$\phi(z) = z^2 - \frac{29}{16}. \qquad \infty \to \infty$$

$$\begin{array}{ccccccc}
 & & -\frac{1}{4} & \longrightarrow & -\frac{7}{4} & \longrightarrow & \frac{5}{4} & \longrightarrow & -\frac{1}{4} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\pm\frac{3}{4} & \longrightarrow & -\frac{5}{4} & & \frac{1}{4} & & \frac{7}{4} & &
\end{array}$$

Cubic Polynomial Record Holders Over $\mathbb{Q}$

$$\phi(z) = -\frac{3}{2}z^3 + \frac{19}{6}z. \qquad \infty \to \infty \qquad 0 \to 0$$

$$\begin{array}{ccccccccccc}
\frac{1}{3} & \to & 1 & \to & \frac{5}{3} & \leftrightarrows & -\frac{5}{3} & \leftarrow & -1 & \leftarrow & -\frac{1}{3} \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & \frac{4}{3} & \to & \frac{2}{3} & & -\frac{2}{3} & \leftarrow & -\frac{4}{3} & &
\end{array}$$

$$\phi(z) = -\frac{3}{2}z^3 + \frac{73}{24}z. \qquad \infty \to \infty \qquad 0 \to 0$$

$$\frac{7}{6} \to \frac{7}{6} \qquad -\frac{7}{6} \to -\frac{7}{6}$$

$$\begin{array}{ccccccccccc}
-\frac{3}{2} & \to & \frac{1}{2} & \leftrightarrows & \frac{4}{3} & \quad & \frac{3}{2} & \to & -\frac{1}{2} & \leftrightarrows & -\frac{4}{3} \\
 & & \uparrow & & & & & & \uparrow & & \\
 & & \frac{1}{6} & & & & & & -\frac{1}{6} & &
\end{array}$$

Good Reduction

Definition

Let $v \in M_K$ be a non-archimedean place of K, and let $\phi \in K(z)$ a rational function.
We say ϕ has **good reduction** at v if ϕ makes sense modulo v, i.e., if ϕ may be written as

$$\phi\left(\frac{x}{y}\right) = \frac{F_1(x,y)}{F_2(x,y)}$$

with $F_1, F_2 \in \mathcal{O}_v[x,y]$ homogeneous of the same degree such that **the reductions $\overline{F}_1$ and $\overline{F}_2$ modulo v have no common zeros in $\overline{k}_v \times \overline{k}_v$ besides $(0,0)$.**

Definition

ϕ has **potentially good reduction** at v if $\theta \circ \phi \circ \theta^{-1}$ has good reduction at some place $w|v$, for some $\theta \in \mathrm{PGL}(2, \overline{K})$.
Otherwise, ϕ has **bad reduction** at v.

Example Revisited

$\phi(z) = z^2 - \dfrac{133}{144} \in \mathbb{Q}[z]$. We can write

$$\phi\left(\frac{x}{y}\right) = \frac{144x^2 - 133y^2}{144y^2}.$$

For $p \geq 5$, ϕ has good reduction at p.
But reducing mod 2 or 3 gives:

$$\overline{\phi} = \frac{0x^2 + y^2}{0y^2},$$

which is $0/0$ at $[x, y] = [1, 0]$.

So ϕ has bad reduction at $2, 3, \infty$.

Other Examples

At any non-archimedean place v of K, a polynomial

$$\phi(z) = a_d z^d + \cdots + a_1 z + a_0$$

has good reduction **if and only if**

$$|a_i|_v \le 1 \quad \text{for all } i, \quad \text{and} \quad |a_d|_v = 1.$$

$\phi(z) = \dfrac{3z^2 + 5z}{2z^2 - 7} \in \mathbb{Q}[z]$ has bad reduction at $7, 13, \infty$:

At $p = 7$, $\overline{\phi}(z) = \dfrac{3z(z+4)}{2z^2} = \dfrac{3(z+4)}{2z}$

At $p = 13$, $\overline{\phi}(z) = \dfrac{3z(z+6)}{2(z-6)(z+6)} = \dfrac{3z}{2(z-6)}$

The resultant $\mathrm{Res}(3z^2 + 5z, 2z^2 - 7)$ is $91 = 7 \cdot 13$, and so ϕ has good reduction at every other (finite) prime p.

A Bound Using One Good Prime

Theorem (Pezda, Morton & Silverman, Zieve, 1990s)

Let K be a number field, and let $\phi(z) \in K(z)$ be a rational function of degree $d \geq 2$, and suppose that ϕ has good reduction at a prime $\mathfrak{p}$. Then any K-rational periodic point of ϕ has period at most $O(eq^2)$, where $q = N\mathfrak{p}$ is the norm of $\mathfrak{p}$, and e is the ramification degree of $\mathfrak{p}$.

Corollary

For ϕ as in the theorem,

$$\#Preper(\phi, K) \leq O\left(d^{q^3}\right).$$

The proofs work entirely in the local field $K_{\mathfrak{p}}$.

Filled Julia Sets of Polynomials

Denote by $\mathbb{C}_v$ the completion of an algebraic closure of K_v.

Definition

The v-adic *filled Julia set* of a polynomial $\phi \in \mathbb{C}_v[z]$ of degree $d \geq 2$ is

$$\mathcal{K}_{\phi,v} = \{z \in \mathbb{C}_v : \{|\phi^n(z)|_v\}_{n\geq 0} \text{ is bounded}\}.$$

Note:

- All preperiodic points (besides ∞) lie in $\mathcal{K}_{\phi,v}$.
- $\phi(\mathcal{K}_{\phi,v}) = \phi^{-1}(\mathcal{K}_{\phi,v}) = \mathcal{K}_{\phi,v}$.
- If ϕ is monic, then the diameter $r_{\phi,v}$ of $\mathcal{K}_{\phi,v}$ satisfies $r_{\phi,v} \geq 1$, with equality **iff** ϕ is potentially good at v.

A Bound for Quadratic Polynomials Over $\mathbb{Q}$

Theorem (Call, Goldstine, 1997)

Let $\phi(z) \in \mathbb{Q}[z]$ be a quadratic polynomial, and let s be the number of places of bad reduction of ϕ. Then

$$\#Preper(\phi, \mathbb{Q}) \leq O\left(2^s\right).$$

Idea of Proof:

1. (Local Step):

At good places v, the filled Julia set $\mathcal{K}_{\phi,v}$ sits inside a unit disk.

At bad places v, prove that $\mathcal{K}_{\phi,v}$ sits inside a union of two unit disks. (With fudging at $v = 2, \infty$.)

2. (Global Step):

In each choice of one unit disk at each place, there is only one rational number.

Stronger Non-Uniform Bounds for Polynomials

Theorem (RB, 2004)

Let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$. Let s be the number of bad places of ϕ (including archimedean places). Then

$$\#\mathit{Preper}(\phi, K) \leq O\left(\frac{d^2}{\log d} \cdot s \log s\right).$$

For number fields, the big-Oh constant depends only on $[K : \mathbb{Q}]$. For function fields, the big-Oh constant is essentially 1.

Prelude to the Proof

Let $z_1, \ldots, z_N \in K$ be distinct preperiodic points. Then

$$\prod_{i \neq j} (z_i - z_j) \in K^\times.$$

Thus, by the product formula, $\prod_{v \in M_K} \prod_{i \neq j} |z_i - z_j|_v^{n_v} = 1$, i.e.,

$$\sum_{v \in M_K} n_v \sum_{i \neq j} - \log |z_i - z_j|_v = 0.$$

Recall $r_{\phi,v} \geq 1$ is the diameter of the filled Julia set $\mathcal{K}_{\phi,v}$. So

$$\sum_{i \neq j} - \log |z_i - z_j|_v \geq -N(N-1) \log r_{\phi,v}.$$

The key to the proof is to improve this naive bound.

Sketch of Proof of Theorem

(For ease, we omit archimedean fudge factors and also assume ϕ is monic.)

- At each $v \in M_K$, given $z_1, \dots, z_N \in \mathcal{K}_{\phi,v}$, prove **Lemma 1**:

$$-\sum_{i \neq j} \log |z_i - z_j|_v \geq -(d-1)(N \log_d N) \log r_{\phi,v}.$$

- At a place $w \in M_K$ where $r_{\phi,w} > 1$, partition $\mathcal{K}_{\phi,w} = U \sqcup V$. Given $z_1, \dots, z_N \in U$, prove **Lemma 2**, that

$$-\sum_{i \neq j} \log |z_i - z_j|_w \geq \left(\frac{N^2}{d-1} - (d-1)(N \log_d N) \right) \log r_{\phi,w},$$

(And similarly for $z_1, \dots, z_N \in V$.)

Finishing the Proof

Set $R_v := r_{\phi,v}^{n_v}$.

Let $w \in M_K$ be the place at which $R_w > 1$ is largest, and partition $\mathcal{K}_{\phi,w} = U \sqcup V$.

Given $z_1, \ldots, z_N \in K$ preperiodic and lying in U (resp., V) at w,

$$
\begin{aligned}
0 &= \sum_{v \in M_K} n_v \sum_{i \neq j} -\log|z_i - z_j|_v \geq \sum_{v \text{ bad}} n_v \sum_{i \neq j} -\log|z_i - z_j|_v \\
&\geq \frac{N^2}{d-1} \log R_w - \sum_{v \text{ bad}} (d-1)(N \log_d N) \log R_v \\
&\geq \left[\frac{N^2}{d-1} - s(d-1)(N \log_d N)\right] \log R_w .
\end{aligned}
$$

If $N \geq O_d(s \log s)$, we get $0 > 0$.

Proving the Lemmas: Lower Bounds for $-\sum_{i \neq j} \log |z_i - z_j|_v$

Key idea: if $\overline{D}(a, r_{\phi,v})$ is the smallest disk containing $\mathcal{K}_{\phi,v}$, then

$$|\phi^i(z) - a|_v \leq r_{\phi,v}$$

for all $i \geq 0$ and all $z \in \mathcal{K}_{\phi,v}$.

So we can write down monic polynomials

$$f_j(z) := \prod_{i=0}^{M} [\phi^i(z) - a]^{c_i}$$

of arbitrary degree $j \geq 0$ so that

$$\sup\{|f_j(z)|_v : z \in \mathcal{K}_{\phi,v}\}$$

is surprisingly small.

Completing the Proofs of the Lemmas

$\prod_{i \neq j}(z_i - z_j) = \pm(\det V)^2$, where V is the Vandermonde matrix

$$V = \begin{bmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{N-1} \\ 1 & z_2 & z_2^2 & \dots & z_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_N & z_N^2 & \dots & z_N^{N-1} \end{bmatrix}.$$

Since each f_j is monic, by column operations, $\det V = \det A$, where

$$A = \begin{bmatrix} 1 & f_1(z_1) & f_2(z_1) & \dots & f_{N-1}(z_1) \\ 1 & f_1(z_2) & f_2(z_2) & \dots & f_{N-1}(z_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_1(z_N) & f_2(z_N) & \dots & f_{N-1}(z_N) \end{bmatrix}.$$

Hadamard's inequality bounds $|\det A|_v$ above by the product of the norms of the columns.

Example: A Rational Function

$$\phi(z) = \frac{7z}{24} - \frac{7}{6z} = \frac{7(z^2-4)}{24z}. \qquad \text{Bad places: } v = 2, 3, 7, \infty.$$

---

12 $\mathbb{Q}$-rational preperiodic points:

$$\pm 2 \to 0 \to \infty \to \infty$$

$$\begin{array}{ccccccccc}
\frac{14}{5} & \longrightarrow & \frac{2}{5} & \longrightarrow & -\frac{14}{5} & \longrightarrow & -\frac{2}{5} & \longrightarrow & \frac{14}{5} \\
\Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\
10 & & -\frac{10}{7} & & -10 & & \frac{10}{7} & &
\end{array}$$

A non-preperiodic point:

$$1 \to -\frac{7}{8} \to \frac{69}{64} \to -\frac{81361}{105984} \to \cdots \qquad (\text{exploding at } v = 2, 3).$$

Baker and Rumely's Arakelov Green's Function

Definition

Given $\phi(z) \in \mathbb{C}_v(z)$ of degree $d \geq 2$, write $\phi\left(\frac{x}{y}\right) = \frac{F_1(x,y)}{F_2(x,y)}$, where $F_1, F_2 \in \mathbb{C}_v[x,y]$ are homogeneous polynomials of degree d. Define $\Phi : \mathbb{C}_v^2 \to \mathbb{C}_v^2$ by $\Phi(x,y) := (F_1(x,y), F_2(x,y))$. Given $z_1 = [x_1, y_1], z_2 = [x_2, y_2] \in \mathbb{P}^1(\mathbb{C}_v)$, lift them to $P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in \mathbb{C}_v^2$. Define

$$g_{\phi,v}(z_1, z_2) :=$$

$$-\log|x_1 y_2 - x_2 y_1|_v + \hat{\Lambda}_{\Phi,v}(P_1) + \hat{\Lambda}_{\Phi,v}(P_2) - \frac{\log|\mathrm{Res}(F_1, F_2)|_v}{d(d-1)},$$

where

$$\hat{\Lambda}_{\Phi,v}(x,y) := \lim_{n\to\infty} d^{-n} \log\|\Phi^n(x,y)\|_v.$$

Properties of $g_{\phi,v}$

$$-\log|P_1 \wedge P_2|_v + \hat{\Lambda}_{\Phi,v}(P_1) + \hat{\Lambda}_{\Phi,v}(P_2) - \frac{\log|\mathrm{Res}(F_1,F_2)|_v}{d(d-1)}$$

Idea: $\exp(-g_{\phi,v}(z_1,z_2))$ is a sort of (ϕ,v)-distance from z_1 to z_2.

- $g_{\phi,v}(z_1,z_2)$ is well-defined (i.e., independent of choices of Φ, P_1, P_2.)
- $g_{\phi,v}(z_1,z_2)$ is coordinate-independent: if $\psi = \theta \circ \phi \circ \theta^{-1}$ for some $\theta \in \mathrm{PGL}(2,\mathbb{C}_v)$, then $g_{\psi,v}(\theta(z_1),\theta(z_2)) = g_{\phi,v}(z_1,z_2)$.
- If ϕ has good reduction, then $g_{\phi,v} = -\log\Delta_v$, where $\Delta_v(z_1,z_2)$ is the spherical distance on $\mathbb{P}^1(\mathbb{C}_v)$.
- If ϕ is a monic polynomial and $z_1, z_2 \in \mathcal{K}_{\phi,v}$, then $g_{\phi,v}(z_1,z_2) = -\log|z_1 - z_2|_v$.

Lemma 1 for Rational Functions

Given $\phi \in \mathbb{C}_v(z)$, define

$$\rho_{\phi,v} := \inf\{g_{\phi,v}(z_1, z_2) : z_1, z_2 \in \mathbb{P}^1(\mathbb{C}_v)\}.$$

(For monic polynomials, $\rho_{\phi,v} = -\log r_{\phi,v}$.)

Proposition (Baker, 2005)

Given $\phi \in \mathbb{C}_v(z)$, we have $\rho_{\phi,v} \leq 0$, with equality **iff** *ϕ has potentially good reduction.*

Theorem (Baker, 2005)

Given $d \geq 2$, there is a constant C_d so that for any $\phi \in \mathbb{C}_v(z)$ of degree d and $z_1, \ldots, z_N \in \mathbb{P}^1(\mathbb{C}_v)$,

$$\sum_{i \neq j} g_{\phi,v}(z_i, z_j) \geq (C_d N \log N)\rho_{\phi,v}.$$

Idea of Baker's Proof

Essentially the same as before, only homogeneous:
$d^k g(z,0)$ can be a lot more negative than $g(\phi^k(z),0)$ can be.

Start with the homogeneous Vandermonde matrix

$$V = \begin{bmatrix} y_1^{N-1} & x_1 y_1^{N-2} & x_1^2 y_1^{N-3} & \dots & x_1^{N-1} \\ y_2^{N-1} & x_2 y_2^{N-2} & x_2^2 y_2^{N-3} & \dots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_N^{N-1} & x_N y_N^{N-2} & x_N^2 y_N^{N-3} & \dots & x_N^{N-1} \end{bmatrix}$$

and do column reduction to change powers of x and y to iterates of Φ.

The Global Step

Suppose we could also an analog of Lemma 2: $\mathbb{P}^1(\mathbb{C}_v)$ can be partitioned into $U \sqcup V$ such that for any $z_1, \ldots, z_N \in U$ (resp., V),

$$\sum_{i \neq j} g_{\phi,v}(z_i, z_j) \geq (C_d N \log N - C'_d N^2)\rho_{\phi,v}.$$

Then applying Lemma 2 at the place $w \in M_K$ minimizing $n_w \rho_{\phi,w}$, and Lemma 1 everywhere else, we get

$$\begin{aligned} 2(N-1)\sum_{i=1}^{N} \hat{h}_\phi(z_i) &= \sum_{v \in M_K} n_v \sum_{i \neq j} g_{\phi,v}(z_i, z_j) \\ &\geq [C_d s(N \log N) - C'_d N^2] n_w \rho_{\phi,w}. \end{aligned}$$

Thus, we get a contradiction if $N \gg s \log s$, assuming $z_1, \ldots z_N \in U$ (resp., V) all preperiodic, i.e., $\hat{h}_\phi(z_i) = 0$.

Note: In fact, we get a contradiction for $N \gg s \log s$ even if all the canonical heights $\hat{h}_\phi(z_i)$ are small enough.

Lower Bounds for Canonical Heights

Recall the canonical height is

$$\hat{h}_\phi(z) = \lim_{n\to\infty} \frac{1}{d^n} h(\phi^n(z)).$$

Conjecture (Silverman)

Let K be a number field and $d \geq 2$.
There is a constant $C = C(K, d)$ such that for any $\phi \in K(z)$ with $\deg \phi = d$, and for any non-preperiodic point $P \in \mathbb{P}^1(K)$,

$$\hat{h}_\phi(P) \geq Ch(\phi).$$

The preceding slides suggest that the same tools can be used to attack **both** the uniform boundedness and height lower bound conjectures.

But what does a point of small canonical height (i.e., for which $\hat{h}_\phi(P)/h(\phi)$ is small) look like?

A Quadratic Polynomial Example of A Small Point

$$\phi(z) = z^2 - \frac{181}{144}$$

Not small height: $0 \mapsto \frac{-181}{144} \mapsto \frac{6697}{20736} \mapsto -\frac{495613295}{429981696} \mapsto \cdots$

Small height:

$$\frac{7}{12} \mapsto -\frac{11}{12} \quad \mapsto -\frac{5}{12} \quad \mapsto -\frac{13}{12} \quad \mapsto -\frac{1}{12}$$

$$\mapsto -\frac{5}{4} \quad \mapsto \frac{11}{36} \quad \mapsto -\frac{377}{324} \quad \mapsto \cdots$$

$\hat{h}_\phi(7/12) = 2^{-5} \log 3 = 0.03433\ldots$, vs.

$h(\phi) = h(181/144) = \log 181 = 5.198\ldots$

Ratio is $\hat{h}_\phi(7/12)/h(\phi) = 0.00660\ldots$

Another Quadratic Polynomial Example

$$\phi(z) = z^2 - \frac{36989}{19600}$$

$$\frac{153}{140} \mapsto -\frac{97}{140} \mapsto -\frac{197}{140} \mapsto \frac{13}{140} \mapsto -\frac{263}{140}$$

$$\mapsto \frac{1609}{980} \mapsto \frac{38821}{48020} \mapsto \cdots$$

$\hat{h}_\phi(153/140) = 2^{-10}\log 5 + 2^{-4}\log 7 = 0.12319\ldots$, vs.
$h(\phi) = h(36989/19600) = \log 36989 = 10.518\ldots$
Ratio is $\hat{h}_\phi(153/140)/h(\phi) = 0.0117\ldots$

A Cubic Polynomial Example

$$\phi(z) = -\frac{1}{24}z^3 + \frac{97}{24}z + 5$$

$$-7 \mapsto 19 \mapsto -1 \mapsto 1 \mapsto 9$$

$$\mapsto 11 \mapsto -6 \mapsto -\frac{41}{4} \mapsto \frac{4323}{512} \mapsto \cdots$$

$\hat{h}_\phi(-7) = 0.0011\ldots$, vs.

$h(\phi) = \log(97) = 4.57\ldots$

Ratio is $\hat{h}_\phi(-7)/h(\phi) = 0.00025\ldots$