Computing New Quadratic Rational Examples in Arithmetic Dynamics

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Dynamics on $\mathbb{P}^1$

Let K be a field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

[$\deg \phi := \max\{\deg f_1, \deg f_2\}$, where $\phi = f_1/f_2$ in lowest terms.]

Definition

A point $P \in \mathbb{P}^1(\overline{K}) = \overline{K} \cup \{\infty\}$ is called **preperiodic** if

$$\phi^n(P) = \phi^m(P) \quad \text{for some } n > m \geq 0.$$

Write $\text{Preper}(\phi, K) := \{P \in \mathbb{P}^1(K) : P \text{ is preperiodic under } \phi\}$.

The Dynamical Uniform Boundedness Conjecture

From now on, K **is a global field**.

Theorem (Northcott, 1950)

Let $\phi \in K(z)$ of degree $d \geq 2$. Then

$$\#\mathit{Preper}(\phi, K) < \infty.$$

Conjecture (Morton & Silverman, 1994)

For any integer $d \geq 2$, there is a constant $C = C(d, K)$ such that for any $\phi \in K(z)$ of degree d,

$$\#\mathit{Preper}(\phi, K) \leq C(d, K).$$

Uniform Boundedness in Degree Two over $\mathbb{Q}$

Conjecture (Poonen, 2000)

Let $\phi \in \mathbb{Q}[z]$ be a quadratic polynomial. Then

$$\#\mathit{Preper}(\phi, \mathbb{Q}) \leq 9.$$

Question (Manes, 2007)

Let $\phi \in \mathbb{Q}(z)$ be a rational function of degree 2. Is it always true that

$$\#\mathit{Preper}(\phi, \mathbb{Q}) \leq 12?$$

Quadratic Polynomial Record Holders Over $\mathbb{Q}$

$$\phi(z) = z^2 - \frac{133}{144}. \qquad \infty \to \infty$$

$$\frac{7}{12} \to -\frac{7}{12} \to -\frac{7}{12} \qquad -\frac{19}{12} \to \frac{19}{12} \to \frac{19}{12}$$

$$\frac{1}{12} \to -\frac{11}{12} \leftrightarrows -\frac{1}{12} \leftarrow \frac{11}{12}$$

$$\phi(z) = z^2 - \frac{29}{16}. \qquad \infty \to \infty$$

$$\begin{array}{ccccccc} & & -\frac{1}{4} & \longrightarrow & -\frac{7}{4} & \longrightarrow & \frac{5}{4} & \longrightarrow & -\frac{1}{4} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \pm\frac{3}{4} & \longrightarrow & -\frac{5}{4} & & \frac{1}{4} & & \frac{7}{4} & & \end{array}$$

Quadratic Rationals Over $\mathbb{Q}$ with 12 Points [Manes]

$$\phi(z) = -\frac{25}{24}z + \frac{1}{6z} \qquad \pm\frac{2}{5} \to 0 \to \infty \to \infty$$

$$\frac{14}{25} \to -\frac{2}{7} \to -\frac{2}{7} \qquad -\frac{14}{25} \to \frac{2}{7} \to \frac{2}{7}$$

$$\frac{2}{25} \to -2 \leftrightarrows -2 \leftarrow -\frac{2}{25}$$

$$\phi(z) = \frac{7}{24}z - \frac{7}{6z} \qquad \pm 2 \to 0 \to \infty \to \infty$$

$$\begin{array}{ccccccc}
10 & \longrightarrow & \frac{14}{5} & \longrightarrow & \frac{2}{5} & \longleftarrow & -\frac{10}{7} \\
 & & \uparrow & & \downarrow & & \\
\frac{10}{7} & \longrightarrow & -\frac{2}{5} & \longleftarrow & -\frac{14}{5} & \longleftarrow & -10
\end{array}$$

Lower Bounds for Canonical Heights

The canonical height of $P \in \mathbb{P}^1(K)$ is

$$\hat{h}_\phi(P) = \lim_{n\to\infty} \frac{1}{d^n} h(\phi^n(P)),$$

where

$$h(a/b) = \log \max\{|a|, |b|\}.$$

Conjecture (Silverman)

Let K be a number field and $d \geq 2$.
There is a constant $C = C(K, d)$ such that for any $\phi \in K(z)$ with $\deg \phi = d$, and for any non-preperiodic point $P \in \mathbb{P}^1(K)$,

$$\hat{h}_\phi(P) \geq Ch(\phi).$$

But what does a point of small canonical height (i.e., for which $\hat{h}_\phi(P)/h(\phi)$ is small) look like?

A Quadratic Polynomial Example of a Small Point

$$\phi(z) = z^2 - \frac{181}{144}$$

Not small height: $0 \mapsto \frac{-181}{144} \mapsto \frac{6697}{20736} \mapsto -\frac{495613295}{429981696} \mapsto \cdots$

Small height:

$$\frac{7}{12} \mapsto -\frac{11}{12} \mapsto -\frac{5}{12} \mapsto -\frac{13}{12} \mapsto -\frac{1}{12}$$

$$\mapsto -\frac{5}{4} \mapsto \frac{11}{36} \mapsto -\frac{377}{324} \mapsto \cdots$$

$\hat{h}_\phi(7/12) = 2^{-5}\log 3 = 0.03433\ldots$, vs.

$h(\phi) = h(181/144) = \log 181 = 5.198\ldots$

Ratio is $\hat{h}_\phi(7/12)/h(\phi) = 0.00660\ldots$

Other Quadratic Polynomial Examples

$$\phi(z) = z^2 - \frac{36989}{19600}$$

$$\frac{153}{140} \mapsto -\frac{97}{140} \quad \mapsto -\frac{197}{140} \quad \mapsto \frac{13}{140} \quad \mapsto -\frac{263}{140}$$

$$\mapsto \frac{1609}{980} \quad \mapsto \frac{38821}{48020} \quad \mapsto \cdots$$

Ratio is $\hat{h}_\phi(153/140)/h(\phi) = 0.0117\ldots$

$$\phi(z) = z^2 - \frac{931161001}{476985600} \qquad [476985600 = (2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13)^2]$$

Small height:

$$\frac{30379}{21840} \mapsto -\frac{379}{21840} \mapsto -\frac{42629}{21840} \mapsto \frac{40571}{21840} \mapsto \frac{32731}{21840}$$

$$\mapsto \frac{27809}{94640} \mapsto -\frac{76737829}{41127840} \mapsto -\frac{2534854375585937}{1657669238604288\,0} \mapsto \cdots$$

Ratio is $\hat{h}_\phi(30379/28140)/h(\phi) = 0.013831\ldots$

Moduli Spaces

The moduli space $\mathrm{Rat}_d(K)$ is $(2d+1)$-dimensional:
$\phi = f/g$, where $f, g \in K[z]$ and $\deg f, \deg g \le d$.
[Remove degenerate ϕ, and mod out by constant multiples.]

The moduli space $\mathcal{M}_d := \mathrm{Rat}_d/\mathrm{PGL}_2$ of rational maps up to conjugacy is $(2d-2)$-dimensional.

Let's consider the moduli space $\mathcal{MP}_d(K)$ of

$$\textbf{pairs } (\phi, P) \in K(z) \times \mathbb{P}^1(K)$$

where $\deg \phi = d$, up to conjugacy. Note $\mathcal{MP}_d$ is $(2d-1)$-dimensional.

Note: The subspace of $\mathcal{MP}_d$ consisting of conjugacy classes (ϕ, P) for which the orbit of P has a particular preperiodic structure would be expected to have dimension $2d-2$.

Searching for Interesting Dynamical Examples

A Common Strategy:

1. List elements of the moduli space $\mathcal{M}_d(K)$ (say, by height).
2. For each such map ϕ, compute the K-rational preperiodic points and/or the points of small canonical height.

An Alternate Strategy:

1. List elements (ϕ, P) of the moduli space $\mathcal{MP}_d(K)$.
2. For each pair, estimate $\hat{h}_\phi(P) = \lim_{n\to\infty} d^{-n} h(\phi^n(P))$ by computing a few iterates.
3. If the estimated height is too large, discard (ϕ, P). Otherwise, look for preperiodicity or at least compute the height more precisely.

Searching for Quadratic Rational Examples

Every element of $\mathcal{MP}_2(K)$, other than those conjugacy classes (ϕ, P) where P has a very short preperiodic orbit, has a unique representative of the form (ϕ, ∞) where

$$\infty \mapsto 1 \mapsto 0 \mapsto x_3 \mapsto x_4 \mapsto x_5,$$

with $x_3, x_4, x_5 \in \mathbb{P}^1(K)$. Specifically,

$$\phi(z) = \frac{(a_1 z + a_0)(z-1)}{b_2 z^2 + b_1 z + b_0},$$

where

$$\begin{aligned}
a_1 &= b_2 = x_4(x_3^2 x_4 - x_3^2 x_5 - x_3^2 + 2x_3 x_5 - x_4 x_5),\\
a_0 &= -x_3 b_0 = x_3^2 x_4 (x_4 - 1)(x_5 - x_4 + x_4 x_5 - x_3 x_5),\\
b_1 &= x_3^2 x_4^2 x_5 - x_3^3 x_4^2 + 2x_3^3 x_4 - x_3^2 x_4^2 + x_4^3 x_5 - x_3^2 x_4 x_5 - x_3 x_4^2 x_5\\
&\quad - x_3^3 + x_3 x_4^2 - x_4^3 + x_3^2 x_5 + x_3^2 - x_3 x_4 + x_4^2 - x_3 x_5
\end{aligned}$$

The Algorithm for Quadratic Rationals Over $\mathbb{Q}$

1. List triples $(x_3, x_4, x_5) \in \mathbb{Q}^3$, by height.
2. For each such triple, discard if the orbit is illegal or too simple, e.g. any of x_3, x_4, x_5 equal each other or 0, 1, ∞. (Incidentally, the moduli space for which $\phi^2(x_4) = x_4$ is birational to $\mathbb{P}^2$, so we also discard those triples.)
3. Check up to $\phi^8(\infty)$ and then $\phi^{11}(\infty)$, looking for repeats or, if not, using those iterates to estimate $\hat{h}_\phi(\infty)$. Discard if height is too large.
4. Compute a few more iterates and record both $\hat{h}_\phi(\infty)$ and the height ratio $\hat{h}_\phi(\infty)/h(\phi)$.
   Warning: $h(\phi) \neq h(x_3, x_4, x_5)$. Instead, use the parametrization of $\mathcal{M}_2$ by multipliers of the fixed points.

Some 6-cycles

$$\phi(z) = \frac{66z^2 + 130z - 196}{66z^2 + 133z + 42}$$

$$\infty \to 1 \to 0 \to -\frac{14}{3} \to \frac{2}{3} \to -\frac{6}{11} \to \infty$$

$$\phi(z) = \frac{124z^2 - 130z + 6}{124z^2 - 247z - 72}$$

$$\infty \to 1 \to 0 \to -\frac{1}{12} \to -\frac{7}{20} \to \frac{9}{4} \to \infty$$

A 7-cycle, and $\#\mathrm{Preper}(\phi, \mathbb{Q}) = 14$

$$\phi(z) = \frac{570z^2 - 2533z - 2704}{95z^2 - 963z - 260}$$

$$\begin{array}{ccccccccccccccc}
\infty & \to & 6 & \to & -1 & \to & \frac{1}{2} & \to & \frac{16}{3} & \to & 0 & \to & \frac{52}{5} & \to & \infty \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
-\frac{5}{19} & & \frac{104}{295} & & -\frac{26}{35} & & \frac{468}{95} & & -\frac{208}{5} & & -\frac{169}{190} & & \frac{179}{10} & &
\end{array}$$

Other maps with $\#\mathrm{Preper}(\phi, \mathbb{Q}) = 14$

$$\phi(z) = \frac{30z^2 - 10z - 20}{30z^2 + 7z - 30}$$

$$\begin{array}{ccccccccccccccc}
\infty \to & 1 & \to & 0 & \to & \frac{2}{3} & \to & \frac{10}{9} & \to & \frac{2}{5} & \to & \frac{6}{7} & \leftrightarrow & \frac{10}{3} \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
 & \frac{10}{17} & & -\frac{2}{3} & & \frac{22}{15} & & -6 & & -\frac{2}{5} & & & & -\frac{4}{3}
\end{array}$$

$$\phi(z) = \frac{33z^2 - 429z + 396}{33z^2 - 197z + 132}$$

$$\begin{array}{cccccccccccccc}
\infty \to & 1 & \to & 0 & \to & 3 & \to & \frac{11}{3} & \to & 5 & \to & 33 & \leftrightarrow & \frac{3}{4} \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
 & \frac{33}{29} & & 12 & & \frac{27}{11} & & \frac{1}{3} & & \frac{6}{11} & & & & \frac{12}{11}
\end{array}$$

. . .and many others.

Some Very Small Canonical Heights

$$\phi(z) = \frac{7z^2 - 15z + 2}{49z^2 - 39z + 2}$$

$$\infty \mapsto \frac{1}{7} \mapsto 0 \mapsto 1 \mapsto -\frac{1}{2} \mapsto \frac{1}{3} \mapsto \frac{2}{5} \mapsto \frac{1}{2} \mapsto \frac{5}{7} \mapsto 6$$

$$\mapsto \frac{41}{383} \mapsto -\frac{2320}{7889} \mapsto \frac{3465767}{8746087} \mapsto \frac{5181181753447}{10490726400146} \mapsto \cdots$$

$\hat{h}_\phi(\infty) = 0.003606\ldots$, $h(\phi) = \log(2299) = 7.7402\ldots$.

Ratio is $\hat{h}_\phi(\infty)/h(\phi) = 0.0004659\ldots$

$$\phi(z) = \frac{91z^2 + 399z - 490}{91z^2 - 16z - 350}$$

$$\infty \mapsto 1 \mapsto 0 \mapsto \frac{7}{5} \mapsto -\frac{14}{11} \mapsto \frac{14}{3} \mapsto \frac{28}{13} \mapsto 21 \mapsto \frac{28}{23}$$

$$\mapsto -\frac{329}{591} \mapsto \frac{975016}{446071} \mapsto \frac{331725423603}{20220662173} \mapsto \cdots$$

$\hat{h}_\phi(\infty) = 0.01221\ldots$, $h(\phi) = \log(1314021) = 14.0886\ldots$.

Ratio is $\hat{h}_\phi(\infty)/h(\phi) = 0.0008668\ldots$