Computing Small Canonical Heights in Arithmetic Dynamics

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Some Students and Former Students

2003–04 senior thesis: Andrew Gillette

2007 REU: With Ben Dickman, Sasha Joseph, Dan Rubin,
plus Ben Krause and Xinwen Zhou

2011 REU: With Trevor Hyde and Colin White,
plus Ruqian Chen and Yordanka Kovacheva

For the raw data: see `http://www.cs.amherst.edu/~rlb`

Dynamics on $\mathbb{P}^1$

Let K be a field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

[$\deg \phi := \max\{\deg f_1, \deg f_2\}$, where $\phi = f_1/f_2$ in lowest terms.]

Definition

A point $P \in \mathbb{P}^1(\overline{K}) = \overline{K} \cup \{\infty\}$ is called **preperiodic** if

$$\phi^n(P) = \phi^m(P) \quad \text{for some } n > m \geq 0.$$

Write $\text{Preper}(\phi, K) := \{P \in \mathbb{P}^1(K) : P \text{ is preperiodic under } \phi\}$.

The Dynamical Uniform Boundedness Conjecture

From now on, K **is a global field**.

Theorem (Northcott, 1950)

Let $\phi \in K(z)$ *of degree* $d \geq 2$. *Then*

$$\#\mathit{Preper}(\phi, K) < \infty.$$

Conjecture (Morton & Silverman, 1994)

For any integer $d \geq 2$, *there is a constant* $C = C(d, K)$ *such that for any* $\phi \in K(z)$ *of degree* d,

$$\#\mathit{Preper}(\phi, K) \leq C(d, K).$$

Example: Testing for Preperiodicity

$$\phi(z) = z^2 - \frac{133}{144}.$$

Is $z = 0$ preperiodic?

$$0 \mapsto -\frac{133}{144} \quad \mapsto -\frac{1463}{20736} \quad \mapsto -\frac{394995503}{429981696} \quad \mapsto \cdots$$

$$0 \mapsto -\frac{(*)}{2^4 \cdot 3^2} \quad \mapsto \frac{(*)}{2^8 \cdot 3^4} \quad \mapsto \frac{(*)}{2^{16} \cdot 3^8} \quad \mapsto \cdots$$

No, it's not preperiodic.

More Preperiodicity Testing

Still with $\phi(z) = z^2 - \frac{133}{144}$: is $\frac{17}{12}$ preperiodic?

$$\frac{17}{12} \mapsto \frac{13}{12} \mapsto \frac{1}{4} \mapsto -\frac{31}{36} \mapsto -\frac{59}{324} \mapsto \cdots$$
$$\frac{(*)}{2^2 \cdot 3} \mapsto \frac{(*)}{2^2 \cdot 3} \mapsto \frac{(*)}{2^2 \cdot 3^0} \mapsto \frac{(*)}{2^2 \cdot 3^2} \mapsto \frac{(*)}{2^2 \cdot 3^4} \mapsto \cdots$$

No. What about $\frac{43}{12}$?

$$\frac{43}{12} \mapsto \frac{143}{12} \mapsto \frac{1693}{12} \mapsto \frac{238843}{12} \mapsto \frac{4753831543}{12} \mapsto \cdots$$

No.

Quadratic Polynomial Record Holders Over $\mathbb{Q}$

$$\phi(z) = z^2 - \frac{133}{144}. \qquad \infty \to \infty$$

$$\frac{7}{12} \to -\frac{7}{12} \to -\frac{7}{12} \qquad -\frac{19}{12} \to \frac{19}{12} \to \frac{19}{12}$$

$$\frac{1}{12} \to -\frac{11}{12} \leftrightarrows -\frac{1}{12} \leftarrow \frac{11}{12}$$

$$\phi(z) = z^2 - \frac{29}{16}. \qquad \infty \to \infty$$

$$\begin{array}{ccccccccc} & & -\frac{1}{4} & \longrightarrow & -\frac{7}{4} & \longrightarrow & \frac{5}{4} & \longrightarrow & -\frac{1}{4} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \pm\frac{3}{4} & \longrightarrow & -\frac{5}{4} & & \frac{1}{4} & & \frac{7}{4} & & \end{array}$$

Lower Bounds for Canonical Heights

The canonical height of $P \in \mathbb{P}^1(K)$ is

$$\hat{h}_\phi(P) = \lim_{n\to\infty} \frac{1}{d^n} h(\phi^n(P)),$$

where

$$h(a/b) = \log \max\{|a|, |b|\}.$$

Conjecture (Silverman)

Let K be a number field and $d \geq 2$.
There is a constant $C = C(K, d)$ such that for any $\phi \in K(z)$ with $\deg \phi = d$, and for any non-preperiodic point $P \in \mathbb{P}^1(K)$,

$$\hat{h}_\phi(P) \geq Ch(\phi).$$

But what does a point of small canonical height (i.e., for which $\hat{h}_\phi(P)/h(\phi)$ is small) look like?

A Quadratic Polynomial Example of a Small Point

$$\phi(z) = z^2 - \frac{181}{144}$$

Not small height: $0 \mapsto \frac{-181}{144} \mapsto \frac{6697}{20736} \mapsto -\frac{495613295}{429981696} \mapsto \cdots$

Small height:

$$\frac{7}{12} \mapsto -\frac{11}{12} \mapsto -\frac{5}{12} \mapsto -\frac{13}{12} \mapsto -\frac{1}{12}$$

$$\mapsto -\frac{5}{4} \mapsto \frac{11}{36} \mapsto -\frac{377}{324} \mapsto \cdots$$

$\hat{h}_\phi(7/12) = 2^{-5}\log 3 = 0.03433\ldots$, vs.
$h(\phi) = h(181/144) = \log 181 = 5.198\ldots$
Ratio is $\hat{h}_\phi(7/12)/h(\phi) = 0.00660\ldots$

Another Quadratic Polynomial Example

$$\phi(z) = z^2 - \frac{36989}{19600}$$

$$\frac{153}{140} \mapsto -\frac{97}{140} \mapsto -\frac{197}{140} \mapsto \frac{13}{140} \mapsto -\frac{263}{140}$$

$$\mapsto \frac{1609}{980} \mapsto \frac{38821}{48020} \mapsto \cdots$$

Ratio is $\hat{h}_\phi(153/140)/h(\phi) = 0.0117\ldots$

Some Quadratic Polynomial Observations

For $\phi(z) = z^2 + \frac{m}{n}$ to have *any* $\mathbb{Q}$-rational preperiodic points (besides ∞), we must have:

1. $n = k^2$ is a perfect square,
2. $-m$ is a square modulo k,
3. $\frac{m}{k^2} \leq \frac{1}{4}$.

In fact, if we want more than just a few $\mathbb{Q}$-rational preperiodic points, we must also have $4 \mid k$.

Of course, if you're looking for small height points, some of those conditions may be violated.

A Simple Search Algorithm: Quadratic Polynomials over $\mathbb{Q}$

1. List as many $c \in \mathbb{Q}$ fitting the previous slide's restrictions as you have the CPU time for.
   Let $\phi_c(z) = z^2 + c$.
2. For each such $c = \frac{m}{k^2}$, list all $P = \frac{y}{k}$ with $|y| \approx \sqrt{|m|}$.
   Perhaps also restrict y so that $y^2 \equiv -m \pmod{k}$.
   Of course, you might need to relax these conditions if you're looking for non-preperiodic small height points and you want to be absolutely thorough.
3. For each such P, keep applying ϕ_c until you either
   - see that P has large height,
   - see a repeat, or
   - become convinced P has small height.

An Alternate Search Algorithm

1. List as many $P \in \mathbb{Q}$ of the form $P = \dfrac{y}{4j}$ as you have the CPU time for.
2. For each such P, let $k = 2j$, and list all $m \in -y^2 + k\mathbb{Z}$ with $-y^2 - k|y| \leq m \leq -y^2 + k|y|$. (Approximately.) Probably throw out any m with $\gcd\left(k, \dfrac{m}{k}\right) \neq 1$.
3. For each such m, let $c = \dfrac{m}{k^2}$. Now start taking iterates of P under ϕ_c as before.

Note:

- Not good for finding *all* preperiodic points of a given ϕ_c.
- May miss some small height points, in theory.

One Result of the Alternate Search Algorithm

$$\phi(z) = z^2 - \frac{931161001}{476985600} \qquad [476985600 = (2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13)^2]$$

Small height:

$$\frac{30379}{21840} \mapsto -\frac{379}{21840} \mapsto -\frac{42629}{21840} \mapsto \frac{40571}{21840} \mapsto \frac{32731}{21840}$$

$$\mapsto \frac{27809}{94640} \mapsto -\frac{76737829}{41127840} \mapsto -\frac{2534854375585937}{16576692386042880} \mapsto \cdots$$

$\hat{h}_\phi(30379/28140) = 0.28564\ldots$,
Ratio is $\hat{h}_\phi(7/12)/h(\phi) = 0.013831\ldots$

Cubic Polynomials

Proposition

Suppose char $K \neq 3$, *and let* $\phi(z) = Az^3 + Bz^2 + Cz + D \in K[z]$ *be a cubic polynomial.*
Then ϕ *is conjugate over* K *to a polynomial either of the form*

$$az^3 + bz + 1$$

or

$$az^3 + bz$$

with $a, b \in K$.
For the first family, the parameters a, b *are unique; for the second,* b *is unique, and* a *is unique up to multiplication by a square in* $K^\times$.

Filled Julia Sets of Polynomials

Let v be a place of K.
Denote by $\mathbb{C}_v$ the completion of an algebraic closure of K_v.

Definition

The v-adic *filled Julia set* of a polynomial $\phi \in \mathbb{C}_v[z]$ of degree $d \geq 2$ is

$$\mathcal{K}_{\phi,v} = \{z \in \mathbb{C}_v : \{|\phi^n(z)|_v\}_{n \geq 0} \text{ is bounded}\}.$$

If $P \in K$ lies outside $\mathcal{K}_{\phi,v}$ for any v, then P is **not preperiodic**.

Moreover, we get a positive lower bound for $\hat{h}_\phi(P)$, based on how far outside $\mathcal{K}_{\phi,v}$ the point P lies.

Approximating the Filled Julia Set

Fix ϕ.

At all but finitely many places v, we have $\mathcal{K}_{\phi,v} = \overline{D}(0,1)$.

For the finitely many other v, we can easily compute the smallest disk $\overline{D}(Q,r) \subseteq \mathbb{C}_v$ containing $\mathcal{K}_{\phi,v}$.

Then $\phi^{-1}(\overline{D}(Q,r))$ is a union of up to $\deg \phi$ much smaller disks.

A Search Algorithm: Cubic Polynomials over $\mathbb{Q}$

1. List as many $a, b \in \mathbb{Q}$ as you have the CPU time for.
   Let $\phi_{a,b}(z) = az^3 + bz + 1$ OR $\psi_{a,b}(z) = az^3 + bz$.
   [Discard $\psi_{a,b}$ if you already did a conjugate of it.]
2. At each bad v, compute the 1–3 small disks whose union contains $\mathcal{K}_{\phi,v}$.
3. List the (finitely many) $P \in \mathbb{Q}$ lying in $\overline{D}(0,1)$ at each good prime and in one of the disks at each bad prime.
4. For each such P, keep applying $\phi_{a,b}$ [OR $\psi_{a,b}$] until you either
   - see that P has large height,
   - see a repeat, or
   - become convinced P has small height.

Cubic Polynomial Records Over $\mathbb{Q}$: Preperiodic Points

$\phi(z) = -\frac{3}{2}z^3 + \frac{19}{6}z.$ $\infty \to \infty$ $0 \to 0$

$$
\begin{array}{ccccccccccc}
\frac{1}{3} & \to & 1 & \to & \frac{5}{3} & \leftrightarrows & -\frac{5}{3} & \leftarrow & -1 & \leftarrow & -\frac{1}{3} \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & \frac{4}{3} & \to & \frac{2}{3} & & -\frac{2}{3} & \leftarrow & -\frac{4}{3} & &
\end{array}
$$

$\phi(z) = -\frac{3}{2}z^3 + \frac{73}{24}z.$ $\infty \to \infty$ $0 \to 0$

$$
\frac{7}{6} \to \frac{7}{6} \qquad -\frac{7}{6} \to -\frac{7}{6}
$$

$$
\begin{array}{ccccccccccc}
-\frac{3}{2} & \to & \frac{1}{2} & \leftrightarrows & \frac{4}{3} & \qquad & \frac{3}{2} & \to & -\frac{1}{2} & \leftrightarrows & -\frac{4}{3} \\
 & & \uparrow & & & & & & \uparrow & & \\
 & & \frac{1}{6} & & & & & & -\frac{1}{6} & &
\end{array}
$$

Some Small Heights Discovered

$$\phi(z) = -\frac{3}{4}z^3 + \frac{25}{12}z + 1$$

$$-1 \mapsto -\frac{1}{3} \mapsto \frac{1}{3} \mapsto \frac{5}{3} \mapsto 1 \mapsto \frac{7}{3} \mapsto -\frac{11}{3} \mapsto \frac{91}{3} \mapsto -\frac{62605}{3} \mapsto \frac{20447763377503}{3} \mapsto \cdots$$

Height ratio is $\hat{h}_\phi(-1)/h(\phi) = 0.0004642\ldots$

$$\phi(z) = -\frac{1}{48}z^3 + \frac{31}{12}z + 1$$

$$4 \mapsto 10 \mapsto 6 \mapsto 12 \mapsto -4 \mapsto -8 \mapsto -9 \mapsto -\frac{113}{16} \mapsto -\frac{649189}{65536} \mapsto \cdots$$

Height ratio is $\hat{h}_\phi(4)/h(\phi) = 0.0003945\ldots$

The Cubic Polynomial Record Holder over $\mathbb{Q}$: Small Height

$$\phi(z) = -\frac{25}{24}z^3 + \frac{97}{24}z + 1$$

$$-\frac{7}{5} \mapsto -\frac{9}{5} \mapsto -\frac{1}{5} \mapsto \frac{1}{5} \mapsto \frac{9}{5}$$

$$\mapsto \frac{11}{5} \mapsto -\frac{6}{5} \mapsto -\frac{41}{20} \mapsto \frac{4323}{2560} \mapsto \cdots$$

$\hat{h}_\phi(-7) = 0.0011\ldots$, vs.

$h(\phi) = \log(97) = 4.57\ldots$

Ratio is $\hat{h}_\phi(-7)/h(\phi) = 0.00025\ldots$

A Search Algorithm: Cubic Polynomials over Quadratic Number Fields

1. List as many $a, b \in \overline{\mathbb{Q}}$ with $[\mathbb{Q}(a, b) : \mathbb{Q}] \leq 2$ as you have the CPU time for.
   Let $\phi_{a,b}(z) = az^3 + bz + 1$ OR $\psi_{a,b}(z) = az^3 + bz$.
   [Discard $\psi_{a,b}$ if you already did a conjugate of it.]
2. At each bad v, compute the 1–3 small disks whose union contains $\mathcal{K}_{\phi,v}$.
   [Note: if $a, b \in \mathbb{Q}$, we have to decide which quadratic fields K to consider.]
3. List the (finitely many) $P \in K$ in $\overline{D}(0, 1)$ at each good prime and in one of the disks at each bad prime.
4. For each such P, keep applying $\phi_{a,b}$ [OR $\psi_{a,b}$] until you either
   - see that P has large height,
   - see a repeat, or
   - become convinced P has small height.

A preliminary finding over $\mathbb{Q}(i)$

$$\phi(z) = \Big(-\frac{1}{4} - \frac{3}{5}i\Big)z^3 + \Big(\frac{7}{10} - 2i\Big)z + 1$$

Small height:

$$\frac{6}{13} - \frac{4}{13}i \mapsto \frac{8}{13} - \frac{14}{13}i \mapsto -\frac{3}{13} - \frac{11}{13}i \mapsto -\frac{9}{13} - \frac{7}{13}i$$
$$\mapsto -1 + i \mapsto 3 + i \mapsto -\frac{81}{5} - \frac{113}{5}i$$
$$\mapsto -\frac{849899}{625} + \frac{8660743}{625}i \mapsto \cdots$$

$\hat{h}_\phi\Big(\frac{6}{13} - \frac{4}{13}i\Big) = 0.00647\ldots,$

Ratio is $\hat{h}_\phi\Big(\frac{6}{13} - \frac{4}{13}i\Big)/h(\phi) = 0.00173\ldots$

Preliminary findings over $\mathbb{Q}(\sqrt{2})$

$$\phi(z) = \left(\frac{1}{4} - \frac{\sqrt{2}}{4}\right)z^3 + \left(\frac{1}{2} + 2\sqrt{2}\right)z + 1$$

Small height:

$$-2 \mapsto -2-2\sqrt{2} \mapsto -2-\sqrt{2} \mapsto -2-3\sqrt{2} \mapsto 4+\sqrt{2} \mapsto 4-\sqrt{2} \mapsto 46-27\sqrt{2} \mapsto 179932-127247\sqrt{2}$$

Ratio is $\hat{h}_\phi(-2)/h(\phi) = 0.001375\ldots$

$$\phi(z) = (-2-\sqrt{2})z^3 + \left(3 + \frac{\sqrt{2}}{2}\right)z + 1$$

Small height:

$$-\frac{\sqrt{2}}{2} \mapsto 1-\sqrt{2} \mapsto -1+\frac{\sqrt{2}}{2} \mapsto 0 \mapsto 1 \mapsto 2-\frac{\sqrt{2}}{2} \mapsto -3+\sqrt{2} \mapsto 25-\frac{23\sqrt{2}}{2} \mapsto -21652+13724\sqrt{2}$$

Ratio is $\hat{h}_\phi(-\sqrt{2}/2)/h(\phi) = 0.0010\ldots$

And beyond that ...

... nothing interesting yet.

There are a **LOT** of pairs (a, b) to try.
Even for $K = \mathbb{Q}(i)$ and $h(a), h(b) \leq \log 20$ (or so), there are **hundreds of billions** of polynomials to test.
[Recall the quadratic record-holder for $\mathbb{Q}$ appeared at height $\log 181$, and the cubic one for $\log 97$.]

Also, for each cubic polynomial, there are more candidate points P to test.

We seem to need better ways to discard many polynomials before we even look at them, as we do for quadratic polynomials.

Any ideas?