Non-archimedean Dynamics in Dimension One: Lecture 4

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Tuesday, $\sqrt{10}$ Day, 2010

The Equilibrium Measure

Today's Goal: Develop the theory to show that there is a Borel probability measure μ on $\mathbb{P}^1_{\mathsf{Ber}}$, supported on $\mathcal{J}_{\phi,\mathsf{Ber}}$, and invariant under ϕ.

To do so, we'll need to present material on:

- measures,
- potential theory, and
- a little ergodic theory.

Pushforward Measures

Let X be a topological space (think $\mathbb{P}^1_{\text{Ber}}$ or $\mathbb{P}^1(\mathbb{C})$), let μ be a signed Borel measure on X, and let $\phi : X \to X$ be a measurable function (think $\phi \in \mathbb{C}_K(z)$ or $\phi \in \mathbb{C}(z)$).

(Recall: "Borel" means using the Borel σ-algebra, generated from the open sets by allowing complements and **countable** unions. "Signed" means $\mu = \mu_+ - \mu_-$, where $\mu_\pm$ both take nonnegative **finite** values.)

The **pushforward measure** $\phi_*(\mu)$ is the signed Borel measure

$$\phi_*(\mu)(U) := \mu(\phi^{-1}(U)).$$

- If $f : X \to \mathbb{R}$ is measurable, then
$$\int_X f(x)\, d\big(\phi_*(\mu)\big)(x) = \int_X f\big(\phi(x)\big)\, d\mu(x).$$
- If μ is a probability measure (i.e., if $\mu \geq 0$ and $\mu(X) = 1$), then so is $\phi_*(\mu)$.
- $\big(\phi \circ \psi\big)_*(\mu) = \phi_*\big(\psi_*(\mu)\big)$.

Pullback Measures

Now specifically set $X = \mathbb{P}^1(\mathbb{C})$ or $X = \mathbb{P}^1_{\text{Ber}}$, and $\phi \in \mathbb{C}(z)$ or $\phi \in \mathbb{C}_K(z)$ nonconstant.

Given a signed Borel measure μ, the **pullback measure** $\phi^*(\mu)$ is the signed Borel measure such that

$$\int_X f(x)\, d\big(\phi^*(\mu)\big)(x) = \int_X \sum_{y \in \phi^{-1}(x)} (\deg_y \phi) \cdot f(y)\, d\mu(x).$$

That is,

$$\phi^*(\mu)(U) := \int_X \sum_{y \in \phi^{-1}(x)} (\deg_y \phi) \cdot \mathbf{1}_U(y)\, d\mu(x).$$

- If μ is a probability measure, then so is $\dfrac{1}{\deg \phi}\phi^*(\mu)$.
- $(\phi \circ \psi)^*(\mu) = \psi^*(\phi^*(\mu))$.

Some Properties

- $\text{supp}\left(\phi_*(\mu)\right) = \phi\left(\text{supp}(\mu)\right)$
- $\text{supp}\left(\phi^*(\mu)\right) = \phi^{-1}\left(\text{supp}(\mu)\right)$
- More specifically, for any $a \in \mathbb{P}^1_{\text{Ber}}$,

$$\phi_*\left(\delta_a\right) = \delta_{\phi(a)} \qquad \text{and} \qquad \phi^*\left(\delta_a\right) = \sum_{b \in \phi^{-1}(a)} (\deg_b \phi)\delta_b.$$

- $\phi_*\left(\phi^*(\mu)\right) = (\deg \phi)\mu.$
- If $\phi^*(\mu) = (\deg \phi)\mu$, then we get $\phi_*(\mu) = \mu$ for free, from the previous line.

  (A pair (T, μ) where $T : X \to X$ satisfies $T_*\mu = \mu$, i.e., $\mu(T^{-1}(U)) = \mu(U)$, is a central object of study in ergodic theory. We say T is *measure-preserving* w.r.t. μ, and μ is *invariant* w.r.t. T.)

The Invariant/Canonical/Equilibrium Measure

Theorem (Favre & Rivera-Letelier; Baker & Rumely; Autissier, Chambert-Loir, & Thuiller, mid 2000s)

Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field, and let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$. Then there is a unique probability measure $\mu = \mu_\phi$ on $\mathbb{P}^1_{\mathrm{Ber}}$ with the following properties:

- $\phi^*(\mu) = d \cdot \mu$, *and*
- $\mu(E_\phi) = 0$,

where $E_\phi \subseteq \mathbb{P}^1(\mathbb{C}_K)$ is the (type I) exceptional set of ϕ.

More Properties of the Equilibrium Measure

In fact, $\mathrm{supp}(\mu) = \mathcal{J}_{\phi,\mathrm{Ber}}$, and μ is (strong) mixing and hence ergodic.

"Recall" that μ ergodic means for any measurable $A \subseteq \mathbb{P}^1_{\mathrm{Ber}}$,

$$\text{if } \phi^{-1}(A) = A, \qquad \text{then either } \mu(A) = 0 \quad \text{or} \quad \mu(A) = 1.$$

Also "recall" that μ (strongly) mixing means for any measurable $A, B \subseteq \mathbb{P}^1_{\mathrm{Ber}}$,

$$\lim_{n\to\infty} \mu\big(A \cap \phi^{-n}(B)\big) = \mu(A)\mu(B).$$

In addition, there is a point $\zeta \in \mathbb{P}^1_{\mathrm{Ber}}$ for which $\mu(\{\zeta\}) > 0$ **if and only if** ζ is type II, $\mathcal{J}_{\phi,\mathrm{Ber}} = \{\zeta\}$, and ϕ has potentially good reduction, attained by moving ζ to $\zeta(0,1)$.

Weak Convergence of Measures

Definition

Let X be a topological space, and let $\{\mu_n\}_{n\geq 1}$ be a sequence of signed Borel measures on X. We say that $\{\mu_n\}$ **converges weakly** to a signed Borel measure μ if for every continuous function $f : X \to \mathbb{R}$ with compact support,

$$\lim_{n\to\infty} \int_X f(x)\, d\mu_n(x) = \int_X f(x)\, d\mu(x).$$

Example

$X = \mathbb{R}$. Then $\{\delta_{1/n}\}_{n\geq 1}$ converges weakly to δ_0, because for any $f \in C_c(\mathbb{R})$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, d\delta_{1/n}(x) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0) = \int_{\mathbb{R}} f(x)\, d\delta_0(x).$$

Note: The assumption that f is continuous is **crucial** here.

Definition of the Equilibrium Measure

Given $\phi \in \mathbb{C}_K(z)$ of degree $d \geq 2$, choose any point $\xi \in \mathbb{P}^1_{\mathrm{Ber}}$ that is **not** an exceptional point in $\mathbb{P}^1(\mathbb{C}_K)$.

Set $\mu_0 := \delta_\xi$, and for each $n \geq 1$, set

$$\mu_n := \frac{1}{d}\phi^*(\mu_{n-1}) = \frac{1}{d^n}(\phi^n)^*(\delta_\xi).$$

Thus, μ_n is a probability measure with support $\mathrm{supp}(\mu_n) = \phi^{-n}(\xi)$.

Then the sequence $\{\mu_n\}_{n\geq 0}$ converges weakly to a measure, and this measure is the equilibrium measure $\mu = \mu_\phi$.

Complex Potential Theory in One Slide

Given a real-valued function $f \in C^2(\mathbb{C})$, recall the Laplacian of f is

$$\Delta f = \partial_x^2 f + \partial_y^2 f = 4\partial_z \partial_{\bar{z}} f.$$

Fact: $$\int_{\mathbb{C}} g\, \Delta f\, dA = \int_{\mathbb{C}} f\, \Delta g\, dA \quad \text{for all} \quad g \in C_c^\infty(\mathbb{C}).$$

(By Green's Theorem. Intuition: $\int gf'' = -\int f'g' = \int fg''$.)

For more general (nice enough) f, there is a unique (Radon) **measure** Δf such that

$$\int_{\mathbb{C}} g\, \Delta f = \int_{\mathbb{C}} f\, \Delta g\, dA \quad \text{for all} \quad g \in C_c^\infty(\mathbb{C}).$$

Examples:

$\Delta(\log|z|) = 2\pi\delta_0$.

$\Delta(\log \max\{|z|, 1\})$ is Lebesgue measure on the unit circle.

The Laplacian on $\mathbb{P}^1_{\mathrm{Ber}}$

There is a metric on $\mathbb{P}^1_{\mathrm{Ber}} \setminus \mathbb{P}^1(\mathbb{C}_K)$ as suggested by the tree: the distance from $\zeta(a, r)$ to $\zeta(a, s)$ is $|\log s - \log r|$.

Given a (nice enough) function $f : \mathbb{P}^1_{\mathrm{Ber}} \to [-\infty, \infty]$, here's the **idea** of how to define Δf.

At a point ζ, for each direction $\vec{u}$ emanating from ζ, we can define the directional derivative

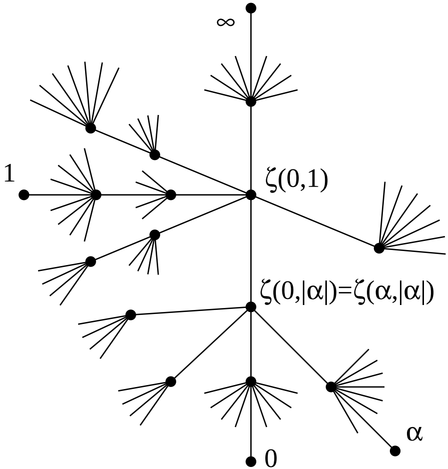
$$D_{\vec{u}} f(\zeta) \text{ ``}=\text{''} \lim_{h \to 0^+} \frac{f(\zeta + h\vec{u}) - f(\zeta)}{h}.$$

If S is the set of directions at ζ, then we define Δf to have a δ mass at ζ of mass $\sum_{\vec{u} \in S} D_{\vec{u}} f(\zeta)$.

Equip a segment $[\zeta(a, r), \zeta(a, s)]$, which is isometric to $[\log r, \log s] \subseteq \mathbb{R}$, with Lebesgue measure λ.

Define Δf along $[\log r, \log s]$ to be just $f''(x)\lambda$.

$\mathbb{P}^1_{\text{Ber}}$



Think of Δf as the description of where the electrical charges are if the voltage at each $\zeta \in \mathbb{P}^1_{\text{Ber}}$ is $f(\zeta)$.

Note: $\Delta f(\mathbb{P}^1_{\text{Ber}}) = 0$.

Examples

Let $f(\zeta) =$ "$\log\max\{|\zeta|, 1\}$" $= \log\max\{\|z\|_\zeta, 1\}$.

- f has constant value 0 on $\overline{D}_{\mathrm{Ber}}(0,1)$.
- $f\big(\zeta(0,t)\big) = \log t$ for $t \geq 1$, so by isometrically identifying the segment from $\zeta(0,1)$ to ∞ with the real interval $[0,\infty]$, we have $f(x) = x$.
- f is constant (with value $\log t$) on any side branch emanating from $\zeta(0,t)$.

So $\Delta f = \delta_{\zeta(0,1)} - \delta_\infty$.

Fix $s \in (0,1)$. Let $g\big(\zeta(0,t)\big) = \begin{cases} \frac{1}{2}\log s < 0 & \text{if } t < s, \\ -\frac{(\log t)^2}{2\log s} + \log t & \text{if } s \leq t \leq 1, \\ \log t > 0 & \text{if } t > 1, \end{cases}$

and extend g to be constant along any side branch emanating from $\zeta(0,t)$.

Then $\Delta g = (-1/\log s)\lambda - \delta_\infty$, where λ is Lebesgue measure on the segment from $\zeta(0,s)$ to $\zeta(0,1)$.

Some Properties of the Laplacian

- Δ is linear.
- $\Delta f = 0$ iff f is constant.
- $\Delta f = \Delta g$ iff $f - g$ is constant.
- $\displaystyle\int_{\mathbb{P}^1_{\mathrm{Ber}}} f \Delta g = \int_{\mathbb{P}^1_{\mathrm{Ber}}} g \Delta f.$
- $\phi^*(\Delta f) = \Delta(f \circ \phi)$.
- If $f_n \to f$ uniformly on $\mathbb{P}^1_{\mathrm{Ber}}$ (and $|\Delta f_n|$ is uniformly bounded on $\mathbb{P}^1_{\mathrm{Ber}}$), then $\Delta f_n \to \Delta f$ weakly.

  (In fact, stronger such convergence results are true, but the above one is all we'll need.)

Proving Convergence to the Equilibrium Measure

Recall: We chose $\xi \in \mathbb{P}^1_{\mathrm{Ber}}$ non-exceptional.
Assume for ease of exposition that ξ is **not** of type I.

We set $\mu_n = \dfrac{1}{d^n}(\phi^n)^*(\delta_\xi)$.

We claimed that $\{\mu_n\}_{n\geq 0}$ converges weakly to a probability measure $\mu = \mu_\phi$ satisfying $\phi^*\mu = d \cdot \mu$.

Choose $u : \mathbb{P}^1_{\mathrm{Ber}} \to [-\infty, \infty]$ such that $\Delta u = \mu_1 - \delta_\xi$.

For each $n \geq 1$, set $f_n = \displaystyle\sum_{i=0}^{n-1} d^{-i} \cdot u \circ \phi^i$.

Finishing the Proof

$$\mu_n = d^{-n}(\phi^n)^*(\delta_\xi), \qquad \Delta u = d^{-1}\phi^*(\delta_\xi) - \delta_\xi,$$
$$f_n = \sum_{i=0}^{n-1} d^{-i} \cdot u \circ \phi^i$$

So

$$\Delta f_n = \sum_{i=0}^{n-1} d^{-i} \Delta\left(u \circ \phi^i\right) = \sum_{i=0}^{n-1} d^{-i}(\phi^i)^*(\Delta u)$$
$$= \sum_{i=0}^{n-1} \left[d^{-i-1}(\phi^{i+1})^*(\delta_\xi) - d^{-i}(\phi^i)^*(\delta_\xi)\right]$$
$$= d^{-n}(\phi^n)^*(\delta_\xi) - \delta_\xi = \mu_n - \delta_\xi$$

On the other hand, u is bounded (since ξ is not of type I), and hence $\{f_n\}$ converges uniformly to some function f.

Thus, set $\mu = \mu_\phi := \Delta f + \delta_\xi$.

We immediately obtain $\mu_n \to \mu$ weakly and $\phi^* \mu = d \cdot \mu$.

The Invariant Potential

Let's consider $\xi = \infty$, assuming $\infty \notin E_\phi$.
Thus, $\Delta u = d^{-1}\phi^*(\delta_\infty) - \delta_\infty$.
What is this potential function f that gives us $\Delta f = \mu_\phi - \delta_\infty$?

Since $f = \sum_{n \geq 0} d^{-n} \cdot u \circ \phi^n$, we have

$$f \circ \phi = \sum_{n \geq 0} d^{-n} u \circ \phi^{n+1} = d \cdot \sum_{n \geq 1} d^{-n} u \circ \phi^n = d \cdot (f - u).$$

Write $\phi(z) = F(z)/G(z)$. Since $\xi = \infty$, we can choose

$$u(x) = d^{-1} \log |G(x)|.$$

Then $f(\phi(x)) = d \cdot f(x) - \log |G(x)|$.

The Punchline

Let K be a **number** field, with set of places M_K.

Let $\phi(z) = F(z)/G(z) \in K(z)$ be a rational function of degree $d \geq 2$.

For each $v \in M_K$, let f_v be the v-adic invariant potential on $\mathbb{P}^1_{\mathrm{Ber},v}$.

That is, Δf_v is the equilibrium measure $\mu_{\phi,v}$ on $\mathbb{P}^1_{\mathrm{Ber},v}$, and

$$f_v(\phi(x)) = d \cdot f_v(x) - \log|G(x)|_v.$$

Then $\hat{h}_\phi(x) := [K:\mathbb{Q}]^{-1} \sum_{v \in M_K} n_v f_v(x)$ satisfies $\hat{h}_\phi \circ \phi = d \cdot \hat{h}_\phi$, since $\sum_v n_v \log|G(x)|_v = 0$.

Fact: the individual functions f_v are the local canonical height functions $\hat{\lambda}_{\phi,v,(\infty)}$ for ϕ, and $\hat{h}_\phi$ **is** the canonical height function for ϕ.