

Non-archimedean Dynamics in Dimension One: Lecture 3

Robert L. Benedetto
Amherst College

Arizona Winter School

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Revisiting the Quadratic Polynomial Example

$$\phi(z) = z^2 + az \in \mathbb{C}_K[z].$$

- ▶ If $|a| \leq 1$, then $\phi(\zeta(0, 1)) = \zeta(0, 1)$, with $\bar{\phi} = z^2 + \bar{a}z$, and hence with $\deg_{\zeta(0,1)} \phi = 2$.

Then ϕ has **good reduction**, because $\deg \bar{\phi} = \deg \phi$.

Each residue class \bar{x} is mapped to the residue class $\bar{\phi}(\bar{x})$.

So $\mathcal{F}_{\phi, \text{Ber}} = \mathbb{P}_{\text{Ber}}^1 \setminus \{\zeta(0, 1)\}$, and $\mathcal{J}_{\phi, \text{Ber}} = \{\zeta(0, 1)\}$.

- ▶ If $|a| > 1$, then $\mathcal{J}_{\phi, \text{Ber}} = \mathcal{J}_{\phi} \subseteq \mathbb{P}^1(\mathbb{C}_K)$ is the same Cantor set as before.

Then $\mathcal{F}_{\phi, \text{Ber}} = \mathbb{P}_{\text{Ber}}^1 \setminus \mathcal{J}_{\phi}$, all points of which are attracted to ∞ under iteration.

Revisiting Hsia's Cubic Polynomial Example

$\phi(z) = az^3 + z^2 + bz + c$, where $0 < |a| < 1$, and $|b|, |c| \leq 1$.

Then $\bar{\phi}(z) = z^2 + \bar{b}z + \bar{c}$, so that ϕ maps $\zeta(0, 1)$ to itself with multiplicity 2.

So $\zeta(0, 1) \in \mathcal{J}_{\phi, \text{Ber}}$ is a (Type II) repelling fixed point.

ϕ maps each residue class \bar{x} **other than** $\bar{\infty}$ to its image under $\bar{\phi}$. Since none of them ever hits $\bar{\infty}$, they are all contained in \mathcal{F}_{Ber} . However, ϕ maps the residue class $\bar{\infty}$ onto all of $\mathbb{P}_{\text{Ber}}^1$. The Julia set $\mathcal{J}_{\phi, \text{Ber}}$ is scattered through this residue class.

Recall that the classical Julia set \mathcal{J}_{ϕ} was not compact; but of course the Berkovich Julia set $\mathcal{J}_{\phi, \text{Ber}}$ must be compact.

In particular, that sequence β_1, β_2, \dots (of preimages of the repelling fixed point α) accumulates at $\zeta(0, 1) \in \mathcal{J}_{\phi, \text{Ber}}$.

Components of the (Berkovich) Fatou Set

Theorem

Let $\phi(z) \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$, with (Berkovich) Fatou set $\mathcal{F}_{\phi, \text{Ber}}$ and Julia set $\mathcal{J}_{\phi, \text{Ber}}$.

Let $U \subseteq \mathcal{F}_{\phi, \text{Ber}}$ be a connected component of $\mathcal{F}_{\phi, \text{Ber}}$, and let $x \in U$. Then

- ▶ U is the union of all connected Berkovich affinoids containing x and contained in $\mathcal{F}_{\phi, \text{Ber}}$.
- ▶ $\phi(U)$ is a connected component of $\mathcal{F}_{\phi, \text{Ber}}$.
- ▶ $\phi^{-1}(U)$ is a disjoint union of at most d connected components V_1, \dots, V_ℓ of $\mathcal{F}_{\phi, \text{Ber}}$.
Each V_i maps d_i -to-1 onto U , for some $d_i \geq 1$, and $d_1 + \dots + d_\ell = d$.

Periodic Fatou Components

Definition

Let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Fatou set $\mathcal{F}_{\phi, \text{Ber}}$.

Let $U \subseteq \mathcal{F}_{\phi, \text{Ber}}$ be a connected component of the Fatou set, and suppose that $\phi^m(U) = U$ for some (minimal) integer $m \geq 1$.

- ▶ We say U is an *indifferent component* if the mapping $\phi^m : U \rightarrow U$ is one-to-one.
- ▶ We say U is an *attracting component* if there is an attracting periodic point $x \in U$ of period m , and if $\lim_{n \rightarrow \infty} \phi^{mn}(\zeta) = x$ for all $\zeta \in U$.

A connected component of $\mathcal{F}_{\phi, \text{Ber}}$ that is not preperiodic is called a *wandering domain*.

Rivera-Letelier's Classification Theorem

Theorem (Rivera-Letelier, 2000)

Let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Fatou set $\mathcal{F}_{\phi, \text{Ber}}$.

Let $U \subseteq \mathcal{F}_{\phi, \text{Ber}}$ be a connected component of the Fatou set.

Then exactly one of the following three possibilities occurs.

1. Some iterate $\phi^n(U)$ is an indifferent periodic component.
2. Some iterate $\phi^n(U)$ is an attracting periodic component.
3. U is a wandering domain.

Example: Good Reduction

Recall that ϕ has **good reduction** if when we write $\phi(z) = \frac{f(z)}{g(z)}$

where $f, g \in \mathcal{O}[z]$ satisfy

- ▶ $(f, g) = 1$,
- ▶ at least one coefficient of f and/or g is a unit (i.e., $|a| = 1$),

then $\deg(\bar{f}/\bar{g}) = \deg \phi$.

In that case, $\mathcal{J}_{\phi, \text{Ber}} = \{\zeta(0, 1)\}$, and each residue class is a Fatou component.

The components map to each other as dictated by $\bar{\phi} := \bar{f}/\bar{g}$ acting on $\mathbb{P}^1(\bar{k})$.

An n -periodic residue class $D_{\text{Ber}}(a, 1)$ is attracting if and only if $\bar{\phi}$ has a critical point among $\{\bar{a}, \bar{\phi}(\bar{a}), \dots, \bar{\phi}^{n-1}(\bar{a})\}$.

Example: Polynomials

If $\phi(z) \in \mathbb{C}_v[z]$ with $\deg \phi \geq 2$ is a polynomial, then the Fatou component W containing ∞ is fixed and attracting.

If ϕ is not of *potentially good reduction*, then W is **not a disk**. Instead, it is of **Cantor type**.

That is, let $V_0 = \mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}_{\text{Ber}}(a, r) \subseteq \mathcal{F}_\phi$ be the largest open $\mathbb{P}_{\text{Ber}}^1$ -disk containing ∞ .

$V_1 := \phi^{-1}(V_0) \supsetneq V_0$ is a non-disk open affinoid, with at least two ends outside (the unique) end of V_0 .

$V_2 := \phi^{-1}(V_1) \supsetneq V_1$ is a non-disk open affinoid; with at least two ends outside each end of V_1 .

etc. In the end, $W = \bigcup_{n \geq 0} V_n$.

Rivera-Letelier's Classification: Continued

Theorem (Rivera-Letelier, 2000)

Let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$ with Fatou set $\mathcal{F}_{\phi, \text{Ber}}$, and let $U \subseteq \mathcal{F}_{\phi, \text{Ber}}$ be a **periodic** connected component of the Fatou set.

1. If U is indifferent, then U is a rational open connected affinoid, and ϕ permutes the (finitely many) boundary points of U . The boundary points are all type II periodic Julia points.
2. If U is attracting, then U is either a rational open disk or a domain of Cantor type.

For an open disk, the unique boundary point is a type II repelling periodic (Julia) point.

For Cantor type, the boundary is uncountable and contained in the Julia set. The boundary can include points of type I, type II, or type IV.

(Maybe also type III? Requires a wandering domain with certain properties.)

Example: A Non-disk Indifferent Component

$$\phi(z) = \frac{1}{1-z} + \pi^2 z = \frac{\pi^2 z^2 - \pi^2 z - 1}{z-1} \in \mathbb{C}_K[z], \text{ with } 0 < |\pi| < 1.$$

Then ϕ has a repelling fixed point at ∞ ($|\text{multiplier}| = |\pi|^{-2} > 1$), but $\zeta(0, 1)$ is an indifferent fixed point, with $\bar{\phi}(z) = \frac{1}{1-z}$.

Note that $\bar{\phi}^3(z) = z$, with $\overline{\infty} \mapsto \bar{0} \mapsto \bar{1} \mapsto \overline{\infty}$.

It is not hard to check that

$$\phi \text{ maps } \begin{cases} \zeta(0, |\pi|^{-1}) \mapsto \zeta(0, |\pi|) & \text{with multiplicity 2,} \\ \zeta(0, |\pi|) \mapsto \zeta(1, |\pi|) & \text{with multiplicity 1,} \\ \zeta(1, |\pi|) \mapsto \zeta(0, |\pi|^{-1}) & \text{with multiplicity 1,} \end{cases}$$

so these three type II points form a repelling cycle of period 3.

It's also easy to check that ϕ maps the open connected affinoid

$$U := D_{\text{Ber}}(0, |\pi|^{-1}) \setminus (\bar{D}_{\text{Ber}}(0, |\pi|) \cup \bar{D}_{\text{Ber}}(1, |\pi|))$$

bijjectively onto itself.

No Wandering Domains

Definition

Let $\phi \in \mathbb{C}_K(z)$ be a rational function, and let $x \in \mathbb{P}^1(\mathbb{C}_K)$.

- ▶ x is **recurrent** under ϕ if $x \in \overline{\bigcup_{n \geq 1} \phi^n(x)}$.
- ▶ x is a **wild** critical point of ϕ if the multiplicity $\deg_x \phi$ of ϕ at x is divisible by the residue characteristic of \mathbb{C}_K .

Theorem (RB, 1998)

Let K be a locally compact non-archimedean field, with \mathbb{C}_K the completion of an algebraic closure of K . (Note: $\text{char } k = p > 0$.)

Let $\phi \in K(z)$ be a rational function of degree $d \geq 2$ with classical Julia set $\mathcal{J}_{\phi,1}$ and Berkovich Fatou set $\mathcal{F}_{\phi,\text{Ber}}$.

Suppose that either

- ▶ $\text{char } K = p$ and $\mathcal{J}_{\phi,1}$ contains no wild critical points, or
- ▶ $\text{char } K = 0$ and $\mathcal{J}_{\phi,1}$ contains no wild recurrent critical points.

Then $\mathcal{F}_{\phi,\text{Ber}}$ has no wandering domains.

A Power Series Lemma

Lemma

Let $a \in \mathbb{C}_K^\times$, set $r = |a|$, and let $0 < s < r$.

Let $f(z) = c_0 + c_d z^d + \cdots \in \mathbb{C}_K[[z]]$ converge on $\overline{D}(0, r)$, and assume that $|c_n| r^n < |dc_d| r^d$ for all $n > d \geq 1$.

(In particular, f has no critical points in $\overline{D}(0, r)$ except maybe at $z = 0$; and in positive characteristic, $z = 0$ is **not** wild.)

If $p|d$, assume $s < |p|^{1/(p-1)} r$. Then
$$\frac{\text{diam}(f(\overline{D}(a, s)))}{\text{diam}(f(\overline{D}(0, r)))} = |d| \frac{s}{r}.$$

Idea of Proof. Rewrite $f(z)$ as a power series $\sum_{n=0}^{\infty} b_n(z-a)^n$ centered at a , and use $|c_n| r^n < |dc_d| r^d$ to show that

$$|b_1|s = |dc_d a^{d-1}|s > |b_n|s^n \quad \text{for all } n \geq 2.$$

So $\text{diam}(f(\overline{D}(a, s))) = |dc_d| r^{d-1} s$, and $\text{diam}(f(\overline{D}(0, r))) = |c_d| r^d$.

Sketch of Proof of No Wandering Domains

Change coordinates so that $\mathcal{J}_\phi \subseteq \overline{D}(0, 1)$.

Extend K to include all critical points of ϕ and some point of a supposed wandering domain U .

Note that $\phi^n(U)$ is a disk, and does not contain any critical points, for all n big enough.

For any $n \geq 0$, let $V_n \supsetneq \phi^n(U)$ be a slightly larger disk.

Then V_n intersects $\mathcal{J}_{\phi, \text{Ber}}$, so the forward iterates of V_n get big.

By the no wild (recurrent) Julia critical hypothesis, there is a radius $R > 0$ so that $\phi^m(V_n)$ has to get up to radius at least R before it can contain any (or more than M) wild critical points.

By the power series lemma, $\phi^{m+n}(U)$ has to have radius at least about R (or $|p|^{M'} R$).

So U has infinitely many non-overlapping iterates of radius bounded below and intersecting the compact set \mathcal{O}_K , a contradiction.

There **Can** be Wandering Domains

One of the hypotheses of the No Wandering Domains result is that ϕ is defined over a locally compact subfield of \mathbb{C}_K .

But if we relax that condition, we can find wandering domains.

Theorem

Let \mathbb{C}_K have residue field \bar{k} that is **not** algebraic over a finite field. Then any $\phi(z) \in \mathbb{C}_K(z)$ with a type II Julia periodic point ζ has wandering domains “in the basin of attraction” of ζ .

The wandering domains in question are just wandering residue classes of ζ whose iterates avoid “bad” residue classes.

No **Other** Wandering Domains

Theorem (RB, 2005)

Let K be a complete **discretely valued** non-archimedean field of **residue characteristic zero**, let \mathbb{C}_K be the completion of an algebraic closure of K , and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

Then ϕ has no wandering domains besides those in attracting basins of periodic type II points.

Theorem (Trucco, 2009)

Let \mathbb{L} be the field of Puiseux series over $\overline{\mathbb{Q}}$, and let $\phi \in \mathbb{L}[z]$ be a **polynomial** of degree $d \geq 2$.

Then ϕ has no wandering domains besides those in attracting basins of periodic type II points.

Subtler Wandering Domains

Even for \mathbb{C}_p and other fields with residue field $\overline{\mathbb{F}}_p$, there can be wandering domains not associated with type II periodic points.

Theorem (RB, 2002)

Let \mathbb{C}_K have residue characteristic $p > 0$.

Then there is a parameter $a \in \mathbb{C}_K$ (in fact, a dense set of such parameters in $\mathbb{C}_K \setminus \overline{D}(0, 1)$) such that

$$\phi_a(z) := (1 - a)z^{p+1} + az^p$$

has a wandering domain not in the attracting basin of a periodic type II point.

(Idea of Proof: see Project #4)

Other Ways to Break the Hypotheses

What if we stick to K locally compact?

It is easy to force a wild critical point into the Julia set.

Example. $0 < |\pi| < 1$, and $\phi(z) = \pi^{-1}(z^{p+1} - z^p) + 1$, which maps $0 \mapsto 1 \mapsto 1$, with 0 wild critical and 1 repelling fixed.

What about wild **recurrent** Julia critical points?

Theorem (Rivera-Letelier, 2005)

Let K be a complete non-archimedean field of residue characteristic p . Then there are polynomials $\phi \in K[z]$ with wild recurrent Julia critical points.

Proof. See project #4.

In both cases ($\text{char } K = p > 0$ with wild Julia critical points, or $\text{char } K = 0$ with wild recurrent Julia critical points), we don't know whether there can be wandering domains.