Non-archimedean Dynamics in Dimension One: Lecture 2

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Problems with $\mathbb{P}^1(\mathbb{C}_K)$

- $\mathbb{P}^1(\mathbb{C}_K)$ is not compact, or even locally compact.
- $\mathbb{P}^1(\mathbb{C}_K)$ is totally disconnected.

  That makes it hard to study "components" of the Fatou set in a meaningful way.

  There are ways to get around that (see Section 5 of the lecture notes), but there is a better way.

There is a nicer space $\mathbb{P}^1_{\text{Ber}}$ that:

- contains $\mathbb{P}^1(\mathbb{C}_K)$ as a subspace,
- is compact,
- is (still) Hausdorff, and
- is path-connected.

The Gauss Norm

$\overline{\mathcal{A}}(0,1) = \mathbb{C}_K\langle\langle z\rangle\rangle$ is the ring of all power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathbb{C}_K[[z]] \qquad \text{such that } \lim_{n\to\infty} c_n = 0,$$

i.e., the ring of power series converging on $\overline{D}(0,1)$.

The **Gauss norm** on $\overline{\mathcal{A}}(0,1)$ is $\|\cdot\|_{\zeta(0,1)} : \overline{\mathcal{A}}(0,1) \to [0,\infty)$, by

$$\Big\| \sum_{n=0}^{\infty} c_n z^n \Big\|_{\zeta(0,1)} := \max\{|c_n| : n \geq 0\}.$$

Equivalently, for all $f \in \overline{\mathcal{A}}(0,1)$,

$$\begin{aligned} \|f\|_{\zeta(0,1)} &:= \sup\{|f(x)| : x \in \overline{D}(0,1)\} \\ &= \max\{|f(x)| : x \in \overline{D}(0,1)\} \end{aligned}$$

Bounded Multiplicative Seminorms

Definition

A **bounded multiplicative seminorm** on $\overline{\mathcal{A}}(0,1)$ is a function $\zeta = \|\cdot\|_\zeta : \overline{\mathcal{A}}(0,1) \to [0,\infty)$ such that

- $\|0\|_\zeta = 0$ and $\|1\|_\zeta = 1$,
- $\|fg\|_\zeta = \|f\|_\zeta \cdot \|g\|_\zeta$ for all $f, g \in \overline{\mathcal{A}}(0,1)$,
- $\|f+g\|_\zeta \leq \|f\|_\zeta + \|g\|_\zeta$ for all $f, g \in \overline{\mathcal{A}}(0,1)$, and
- $\|f\|_\zeta \leq \|f\|_{\zeta(0,1)}$ for all $f \in \overline{\mathcal{A}}(0,1)$.

Note: We do **not** require that $\|f\|_\zeta = 0$ implies $f = 0$.

By the way: we get $\|f+g\|_\zeta \leq \max\{\|f\|_\zeta, \|g\|_\zeta\}$ for free.

Examples of Bounded Multiplicative Seminorms

1. For any $x \in \overline{D}(0,1)$, define $\| \cdot \|_x$ by $\|f\|_x := |f(x)|$.
2. For any disk $D \subseteq \overline{D}(0,1)$, define $\| \cdot \|_D$ by

$$\|f\|_D := \sup\{|f(x)| : x \in D\}.$$

If $D = \overline{D}(a,r)$ or $D = D(a,r)$, and $f(z) = \sum c_n(z-a)^n$, then

$$\|f\|_D = \max\{|c_n|r^n : n \geq 0\}.$$

If D is rational closed, then $\|f\|_D = \max\{|f(x)| : x \in D\}$.

Since $\| \cdot \|_{\overline{D}(a,r)} = \| \cdot \|_{D(a,r)}$, we can denote both by $\| \cdot \|_{\zeta(a,r)}$.

The Berkovich Disk

Definition

The **Berkovich unit disk** $\overline{D}_{\text{Ber}}(0, 1)$ is the set of all bounded multiplicative seminorms on $\overline{\mathcal{A}}(0, 1)$.

As a topological space, $\overline{D}_{\text{Ber}}(0, 1)$ is equipped with the **Gel'fand topology**.

This is the weakest topology such that for every $f \in \overline{\mathcal{A}}(0, 1)$, the map $\overline{D}_{\text{Ber}}(0, 1) \to \mathbb{R}$ given by

$$\zeta \mapsto \|f\|_\zeta$$

is continuous.

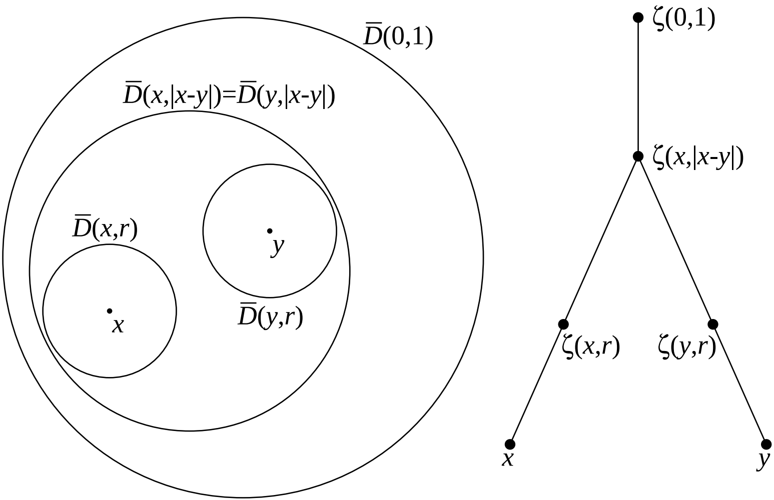
Berkovich's Classification of Points

There are four kinds of points in $\overline{D}_{\text{Ber}}(0,1)$.

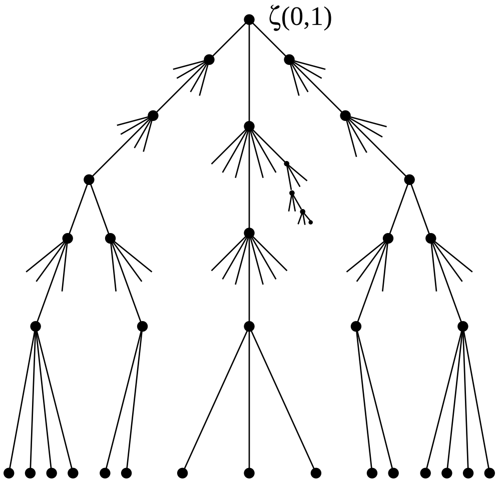
1. Type I: seminorms $\|\cdot\|_x$ corresponding to (classical) points $x \in \overline{D}(0,1)$.
2. Type II: norms $\|\cdot\|_{\zeta(a,r)}$ corresponding to **rational** closed disks $\overline{D}(a,r) \subseteq \overline{D}(0,1)$.
3. Type III: norms $\|\cdot\|_{\zeta(a,r)}$ corresponding to **irrational** disks $\overline{D}(a,r) \subset \overline{D}(0,1)$.
4. Type IV: norms $\|\cdot\|_{\zeta}$ corresponding to (equivalence classes of) decreasing chains $D_1 \supseteq D_2 \supseteq \cdots$ of disks with **empty intersection**.

Chains of disks as in Type IV must have radius **bounded below**.

Path-connectedness, intuitively

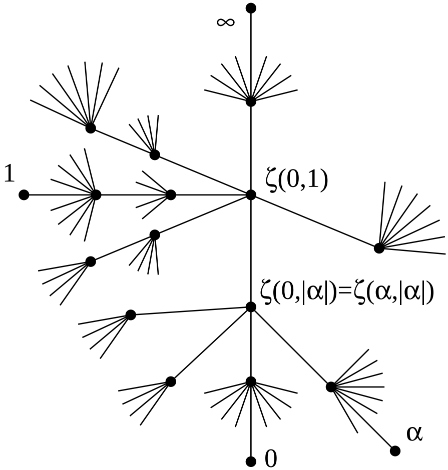


$\overline{D}_{\mathrm{Ber}}(0,1)$ as an $\mathbb{R}$-tree



The Berkovich Projective Line $\mathbb{P}^1_{\mathrm{Ber}}$

Glue two copies of $\overline{D}_{\mathrm{Ber}}(0,1)$ along $|z| = 1$ via $z \mapsto 1/z$.



Berkovich Disks

Definition

Let $a \in \mathbb{C}_K$ and $r > 0$.

- The **closed Berkovich disk** $\overline{D}_{\text{Ber}}(a, r)$ is the set of all $\zeta \in \mathbb{P}^1_{\text{Ber}}$ corresponding to a point/disk/chain of disks contained in $\overline{D}(a, r)$.
- The **open Berkovich disk** $D_{\text{Ber}}(a, r)$ is the set of all $\zeta \in \mathbb{P}^1_{\text{Ber}}$ corresponding to a point/disk/chain of disks contained in $D(a, r)$, **except** $\zeta(a, r)$ itself.

Fact:

$$D_{\text{Ber}}(a, r) \quad \text{is open,} \quad \text{and} \quad \overline{D}_{\text{Ber}}(a, r) \quad \text{is closed.}$$

Moreover:

The open Berkovich disks and the complements of closed Berkovich disks together form a **subbasis** for the Gel'fand topology.

More on the Gel'fand Topology

Definition

An **(open) connected Berkovich affinoid** is the intersection of finitely many (open) Berkovich disks and complements of (closed) Berkovich disks.

Theorem

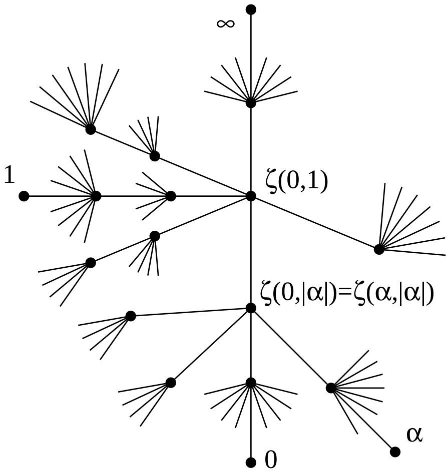
- *The open connected Berkovich affinoids form a basis for the Gel'fand topology.*
- $\mathbb{P}^1_{\mathrm{Ber}}$ *is uniquely path-connected.*

For any $\zeta \in \mathbb{P}^1_{\mathrm{Ber}}$, the complement $\mathbb{P}^1_{\mathrm{Ber}} \setminus \{\zeta\}$ consists of

1. one component if ζ is type I or type IV,
2. infinitely many components if ζ is type II,
3. two components if ζ is type III.

The components of $\mathbb{P}^1_{\mathrm{Ber}} \setminus \{\zeta\}$ are called the **directions** at ζ.

Recall: The Berkovich Projective Line $\mathbb{P}^1_{\mathrm{Ber}}$



Rational Functions Acting on $\mathbb{P}^1_{\mathrm{Ber}}$

Let $\phi(z) \in \mathbb{C}_K(z)$. Then for each point $\zeta \in \mathbb{P}^1_{\mathrm{Ber}}$, there is a unique point $\phi(\zeta) \in \mathbb{P}^1_{\mathrm{Ber}}$ such that

$$\|h\|_{\phi(\zeta)} = \|h \circ \phi\|_{\zeta}$$

for all $h \in \mathbb{C}_K(z)$.

If ζ is type I, then $\phi(\zeta)$ is what you think.

Then $\phi : \mathbb{P}^1_{\mathrm{Ber}} \to \mathbb{P}^1_{\mathrm{Ber}}$ is the unique continuous extension of $\phi : \mathbb{P}^1(\mathbb{C}_K) \to \mathbb{P}^1(\mathbb{C}_K)$.

Understanding degree one maps on $\mathbb{P}^1_{\text{Ber}}$

- $\phi(z) = cz$ maps $\zeta(a, r)$ to $\zeta(ca, |c|r)$.
- $\phi(z) = z + b$ maps $\zeta(a, r)$ to $\zeta(a + b, r)$.
- $\phi(z) = 1/z$ maps $\zeta(a, r)$ to $\begin{cases} \zeta(0, 1/r) & \text{if } 0 \in \overline{D}(a, r), \\ \zeta(1/a, r/|a|^2) & \text{if } 0 \notin \overline{D}(a, r). \end{cases}$
- So for any $\phi \in \mathrm{PGL}(2, \mathbb{C}_K)$, i.e., $\phi(z) = \dfrac{az + b}{cz + d}$ with $ad - bc \neq 0$, you can figure out what $\phi(\zeta)$ is for any $\zeta \in \mathbb{P}^1_{\text{Ber}}$.
- Given $\phi \in \mathrm{PGL}(2, \mathbb{C}_K)$, then

$$\phi(\zeta(0, 1)) = \zeta(0, 1) \quad \textbf{if and only if} \quad \phi \in \mathrm{PGL}(2, \mathcal{O}),$$

i.e., $\phi(z) = \dfrac{az + b}{cz + d}$ with $|a|, |b|, |c|, |d| \leq 1$ and $|ad - bc| = 1$.

Reduction of $\phi \in \mathbb{C}_K(z)$

For more general $\phi \in \mathbb{C}_K(z)$, when does $\phi(\zeta(0,1)) = \zeta(0,1)$?

Write $\phi(z) = \dfrac{a_d z^d + \cdots + a_1 z + a_0}{b_d z^d + \cdots + b_1 z + b_0}$,
with $a_i, b_i \in \mathcal{O}$ and **some** $|a_i| = 1$ and/or **some** $|b_j| = 1$.

Then $\overline{\phi}(z) := \dfrac{\overline{a}_d z^d + \cdots + \overline{a}_1 z + \overline{a}_0}{\overline{b}_d z^d + \cdots + \overline{b}_1 z + \overline{b}_0} \in \overline{k}(z)$.

But we might have cancellation in $\overline{\phi}$.

If $\deg \overline{\phi} = \deg \phi$, we say ϕ has **good reduction**.
If $\deg \overline{\phi} \geq 1$, we say ϕ has **nonconstant reduction**.

Fact: $\phi(\zeta(0,1)) = \zeta(0,1)$ if and only if ϕ has nonconstant reduction.

Understanding $\phi \in \mathbb{C}_K(z)$ at type II points

- For any type II point $\zeta \in \mathbb{P}^1_{\text{Ber}}$, there is some $\eta \in \mathrm{PGL}(2, \mathbb{C}_K)$ such that $\eta(\zeta) = \zeta(0,1)$.
- Given $\phi \in \mathbb{C}_K(z)$ nonconstant and $\zeta \in \mathbb{P}^1_{\text{Ber}}$ of type II, choose $\eta \in \mathrm{PGL}(2, \mathbb{C}_K)$ for ζ as above. Then there is some $\theta \in \mathrm{PGL}(2, \mathbb{C}_K)$ such that the rational function

  $$\theta \circ \phi \circ \eta^{-1}(z) \in \mathbb{C}_K(z)$$

  has nonconstant reduction.
- Then $\phi(\zeta) = \theta^{-1}(\zeta(0,1))$.
- $\eta, \theta \in \mathrm{PGL}(2, \mathbb{C}_K)$ are not unique, but the cosets $\mathrm{PGL}(2, \mathcal{O})\eta$ and $\mathrm{PGL}(2, \mathcal{O})\theta$ are unique.

Example

$\mathbb{C}_K = \mathbb{C}_p$, $\zeta = \zeta(0, |p|_p)$, and $\phi(z) = \dfrac{z^3 - z^2 + z + p^2}{z}$.

What is $\phi(\zeta)$?

$\eta(z) = z/p$ maps ζ to $\zeta(0,1)$, and

$$\phi \circ \eta^{-1}(z) = \phi(pz) = \frac{p^2 z^3 - pz^2 + z + p}{z}.$$

Note $\overline{\phi \circ \eta^{-1}} = z/z = 1$ is constant.

So let $\theta(z) = (z-1)/p$.

Then $\theta \circ \phi \circ \eta^{-1}(z) = \dfrac{pz^3 - z^2 + 1}{z}$, and so $\overline{\theta \circ \phi \circ \eta^{-1}}(z) = (1 - z^2)/z$ is nonconstant.

So $\phi(\zeta) = \theta^{-1}(\zeta(0,1)) = \zeta(1, |p|_p)$.

Dynamics on $\mathbb{P}^1_{\mathrm{Ber}}$: Classifying Periodic Points

Definition

If ζ and ξ are type II points and $\phi(\zeta) = \xi$, then the **local degree** or **multiplicity** of ϕ at ζ is

$$\deg_\zeta \phi := \deg \overline{\theta \circ \phi \circ \eta^{-1}},$$

where $\eta(\zeta) = \zeta(0,1)$ and $\theta(\xi) = \zeta(0,1)$.

If ζ is type II and periodic of exact period n, we say ζ is

- **indifferent** (or **neutral**) if $\deg_\zeta \phi^n = 1$.
- **repelling** if $\deg_\zeta \phi^n \geq 2$.

Warning: Repelling type II points (usually) do not actually repel in most directions.

Note: Periodic type III and IV points are always indifferent.

Berkovich Fatou and Julia Sets

Definition

An open set $U \subseteq \mathbb{P}^1_{\mathrm{Ber}}$ is **dynamically stable** under $\phi \in \mathbb{C}_K(z)$ if $\bigcup_{n \geq 0} \phi^n(U)$ omits infinitely many points of $\mathbb{P}^1_{\mathrm{Ber}}$.

The **(Berkovich) Fatou set of** ϕ is the set $\mathcal{F}_{\mathrm{Ber}} = \mathcal{F}_{\phi,\mathrm{Ber}}$ given by

$$\mathcal{F}_{\mathrm{Ber}} := \{x \in \mathbb{P}^1_{\mathrm{Ber}} : x \text{ has a dynamically stable neighborhood}\}.$$

The **(Berkovich) Julia set of** ϕ is the set

$$\mathcal{J}_{\mathrm{Ber}} = \mathcal{J}_{\phi,\mathrm{Ber}} := \mathbb{P}^1_{\mathrm{Ber}} \smallsetminus \mathcal{F}_{\phi,\mathrm{Ber}}.$$

Basic Properties of Berkovich Fatou and Julia Sets

- $\mathcal{F}_{\text{Ber}}$ is open, and $\mathcal{J}_{\text{Ber}}$ is closed.
- $\mathcal{F}_{\phi^n,\text{Ber}} = \mathcal{F}_{\phi,\text{Ber}}$, and $\mathcal{J}_{\phi^n,\text{Ber}} = \mathcal{J}_{\phi,\text{Ber}}$
- $\phi(\mathcal{F}_{\text{Ber}}) = \mathcal{F}_{\text{Ber}} = \phi^{-1}(\mathcal{F}_{\text{Ber}})$, and $\phi(\mathcal{J}_{\text{Ber}}) = \mathcal{J}_{\text{Ber}} = \phi^{-1}(\mathcal{J}_{\text{Ber}})$.
- $\mathcal{F} = \mathcal{F}_{\text{Ber}} \cap \mathbb{P}^1(\mathbb{C}_K)$, and $\mathcal{J} = \mathcal{J}_{\text{Ber}} \cap \mathbb{P}^1(\mathbb{C}_K)$.
- All attracting periodic points are Fatou.
- All repelling periodic points are Julia.
- Indifferent periodic type II points are Fatou if the residue field is algebraic over a finite field, but they can be Julia otherwise.

  In general, if $\zeta(0,1)$ is fixed by ϕ,
  and if $\overline{\phi}^m(z) = z$ for some $m \geq 1$,
  then $\zeta(0,1)$ is Fatou.

$\mathbb{P}^1(\mathbb{C})$, $\mathbb{P}^1(\mathbb{C}_K)$, and $\mathbb{P}^1_{\mathrm{Ber}}$

$\mathbb{P}^1(\mathbb{C})$	$\mathbb{P}^1(\mathbb{C}_K)$	$\mathbb{P}^1_{\mathrm{Ber}}$
Some indifferent points are Fatou, and some are Julia	**All** indifferent points are Fatou	**Most** indifferent points are Fatou.
$\mathcal{J}$ is compact	$\mathcal{J}$ may not be compact	$\mathcal{J}_{\mathrm{Ber}}$ is compact
$\mathcal{J}$ is nonempty	$\mathcal{J}$ may be empty	$\mathcal{J}_{\mathrm{Ber}}$ is nonempty
$\mathcal{F}$ may be empty	$\mathcal{F}$ is nonempty	$\mathcal{F}_{\mathrm{Ber}}$ is nonempty
$\mathcal{J}$ is the closure of the set of repelling periodic points	??? (see Project #1)	$\mathcal{J}_{\mathrm{Ber}}$ is the closure of the set of repelling periodic (Type I & II) points