Non-archimedean Dynamics in Dimension One: Lecture 1

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Non-archimedean Fields

Let K be a field with a non-archimedean absolute value $|\cdot| : K \to \mathbb{R}$.

That is, for all $x, y \in K$,

- $|x| \geq 0$, with equality **iff** $x = 0$,
- $|xy| = |x| \cdot |y|$,
- $|x + y| \leq \max\{|x|, |y|\}$.

We assume $|\cdot|$ is nontrivial; that is, $|K| \supsetneq \{0, 1\}$.

We usually assume K is *complete* w.r.t. $|\cdot|$.
(All Cauchy sequences converge).

Fun Fact: Let K be a complete non-archimedean field, and let $\{a_n\}_{n\geq 0}$ be a sequence in K. Then

$$\sum_{n\geq 0} a_n \text{ converges} \qquad \text{if and only if} \qquad \lim_{n\to\infty} a_n = 0.$$

The Residue Field and Value Group

Let K be a non-archimedean field.
The ring of integers and (unique) maximal ideal of K are

$$\mathcal{O}_K = \{x \in K : |x| \leq 1\} \quad \text{and} \quad \mathcal{M}_K = \{x \in K : |x| < 1\}.$$

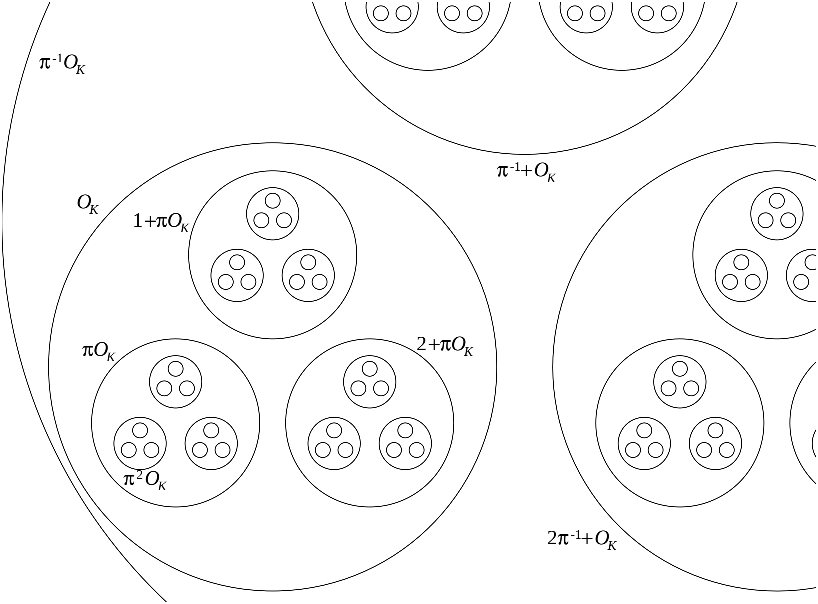
The *residue field* of K is

$$k := \mathcal{O}_K / \mathcal{M}_K.$$

The *value group* of K is

$$|K^\times| \subseteq (0, \infty).$$

A Sketch of a Non-archimedean Field with $k \cong \mathbb{F}_3$



Extension Fields

Let K be a **complete** non-archimedean field, and let L/K be an algebraic extension.

Then $|\cdot|$ extends uniquely to L.

The new residue field ℓ is an algebraic extension of k.

The new value group $|L^\times|$ contains $|K^\times|$ as a subgroup.

The algebraic closure $\overline{K}$ of K may **not** be complete.

But its completion $\mathbb{C}_K$ is both complete and algebraically closed.

Example: p-adic numbers

Fix $p \geq 2$ prime. The p-adic absolute value on $\mathbb{Q}$ is given by

$$\left|\frac{r}{s}p^n\right|_p = p^{-n} \qquad \text{for } r, s \in \mathbb{Z} \text{ not divisible by } p.$$

Idea: numbers divisible by large powers of p are "small".

$$\mathbb{Q}_p := \left\{ \sum_{n \geq n_0} a_n p^n : n_0 \in \mathbb{Z}, a_n \in \{0, 1, \ldots, p-1\} \right\}$$

is the completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$, with ring of integers

$$\mathbb{Z}_p := \mathcal{O}_{\mathbb{Q}_p} = \left\{ \sum_{n \geq 0} a_n p^n : a_n \in \{0, 1, \ldots, p-1\} \right\},$$

maximal ideal $\mathcal{M}_{\mathbb{Q}_p} := p\mathbb{Z}_p$, value group $|\mathbb{Q}_p^\times|_p = p^{\mathbb{Z}}$, and residue field $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.
The completion $\mathbb{C}_p$ of an algebraic closure $\overline{\mathbb{Q}}_p$ has residue field $\overline{\mathbb{F}}_p$ and value group $|\mathbb{C}_p^\times| = p^{\mathbb{Q}}$.

Example: Laurent and Puiseux Series

Fix $\mathbb{F}$ a field. The field of formal Laurent series

$$\mathbb{F}((t)) := \left\{ \sum_{n \geq n_0} a_n t^n : n_0 \in \mathbb{Z}, a_n \in \mathbb{F} \right\}$$

has a non-archimedean absolute value

$$|f| := \varepsilon^{\mathrm{ord}_{t=0} f},$$

where $0 < \varepsilon < 1$ is any (fixed) thing you want.
The ring of integers is the ring $\mathbb{F}[[t]]$ of power series, with maximal ideal $t\mathbb{F}[[t]]$, residue field

$$k = \mathbb{F}[[t]]/t\mathbb{F}[[t]] \cong \mathbb{F},$$

and value group $|\mathbb{F}((t))^\times| = \varepsilon^{\mathbb{Z}}$.
The completion $\mathbb{L}$ of an algebraic closure $\overline{\mathbb{F}((t))}$ is the field of formal *Puiseux series* over $\mathbb{F}$, with residue field $\overline{\mathbb{F}}$ and value group $|\mathbb{L}^\times| = \varepsilon^{\mathbb{Q}}$.

Disks

Given $a \in \mathbb{C}_K$ and $r > 0$,

$$D(a,r) := \{x \in \mathbb{C}_K : |x-a| < r\} \quad \text{and}$$
$$\overline{D}(a,r) := \{x \in \mathbb{C}_K : |x-a| \leq r\}$$

are the associated open disk and closed disk.

- if $r \notin |\mathbb{C}_K^\times|$, then $D(a,r) = \overline{D}(a,r)$ is an **irrational disk**
- if $r \in |\mathbb{C}_K^\times|$, then then $D(a,r) \subsetneq \overline{D}(a,r)$.
- $D(a,r)$ is a **rational open disk**
- $\overline{D}(a,r)$ is a **rational closed disk**

Note:

- All disks are (topologically) **both** open and closed
- Any disk is **exactly one** of: rational open, rational closed, or irrational (as a disk).

More about Disks

- Any point of a disk is a center:
  $D(a,r) = D(b,r)$ (resp., $\overline{D}(a,r) = \overline{D}(b,r)$)
  for all $b \in D(a,r)$ (resp., $b \in \overline{D}(a,r)$)
- Since our disks lie in $\mathbb{C}_K$, and $|\mathbb{C}_K^\times|$ is dense in $(0,\infty)$,
  the **radius** of a disk $D \subseteq \mathbb{C}_K$ is well-defined,
  and equal to the diameter $\sup\{|x-y| : x,y \in D\}$.
- Two disks intersect if and only if one contains the other.
- All non-archimedean fields are totally disconnected.
  (I.e., the only connected nonempty subsets are singletons.)
- $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$ are locally compact,
  but $\mathbb{C}_K$ is not locally compact.

(Power Series and) Polynomials on Disks

Theorem

Let $a \in \mathbb{C}_K$ and $r > 0$.
Let $g(z) = c_0 + c_1(z-a) + \cdots + c_M(z-a)^M \in \mathbb{C}_K[z]$ be a polynomial. (Or more generally, $g(z) \in \mathbb{C}_K[[z-a]]$ is a power series satisfying certain mild convergence conditions)
Let $s := \max_{n \geq 1}\{|c_n| r^n\}$, and

$$\begin{aligned} i &:= \text{ minimum } n \geq 1 \text{ for which } |c_n| r^n = s, \\ j &:= \text{ maximum } n \geq 1 \text{ for which } |c_n| r^n = s. \end{aligned}$$

Then g maps

$$\begin{aligned} &D(a,r) \quad i\text{-to-1 onto} \quad D(c_0, s), \qquad \text{and} \\ &\overline{D}(a,r) \quad j\text{-to-1 onto} \quad \overline{D}(c_0, s), \end{aligned}$$

counting multiplicity.

Example

$\mathbb{C}_K = \mathbb{C}_p$, and $g(z) = p^4 z^5 + p^2 z^3 + z^2 + pz + p^3$.
Then for any $r > 0$, $g\big(\overline{D}(0,r)\big) = \overline{D}(p^3, s)$, where

$$s = \begin{cases} |p|_p r = p^{-1} r & \text{if } 0 < r \le |p|_p = \dfrac{1}{p}, \\ r^2 & \text{if } \dfrac{1}{p} = |p|_p < r \le |p|_p^{-4/3} = p^{4/3}, \\ |p^4|_p r^5 = p^{-4} r^5 & \text{if } r \ge |p|_p^{-4/3} = p^{4/3}. \end{cases}$$

[Note: $\overline{D}(p^3, s) = \overline{D}(0, s)$ for $s \ge |p|_p^3 = p^{-3}$.]

The mapping is 1-1 for $r < |p|_p$,
2-1 for $|p|_p \le r < |p|_p^{-4/3}$,
5-1 for $r \ge |p|_p^{-4/3}$.

$\mathbb{P}^1(\mathbb{C}_K)$-Disks

Recall $\mathbb{P}^1(\mathbb{C}_K) = \mathbb{C}_K \cup \{\infty\}$.

Definition

A $\mathbb{P}^1(\mathbb{C}_K)$-*disk* is either

- a disk $D \subseteq \mathbb{C}_K$, or
- the complement $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus D$ of a disk $D \subseteq \mathbb{C}_K$.

We can attach the adjectives *rational open*, *rational closed*, or *irrational* in the obvious way.

Theorem

Let $g(z) \in \mathbb{C}_K(z)$ be a non-constant rational function, and let $D \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a $\mathbb{P}^1(\mathbb{C}_K)$-disk. Then $g(D)$ is either

- *all of $\mathbb{P}^1(\mathbb{C}_K)$, or*
- *a $\mathbb{P}^1(\mathbb{C}_K)$-disk of the same type as D.*

Connected Affinoids

Definition

A *connected affinoid* in $\mathbb{P}^1(\mathbb{C}_K)$ is a nonempty intersection of finitely many $\mathbb{P}^1(\mathbb{C}_K)$-disks. Equivalently, a connected affinoid is $\mathbb{P}^1(\mathbb{C}_K)$ with finitely many $\mathbb{P}^1(\mathbb{C}_K)$-disks removed.

We can attach the adjectives *rational open*, *rational closed*, or *irrational* in the obvious way.

Theorem

Let $g(z) \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 1$, and let $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ be a connected affinoid. Then

- *$g(U)$ is either $\mathbb{P}^1(\mathbb{C}_K)$ or a connected affinoid of the same type as U.*
- *$g^{-1}(U)$ is a union of $1 \leq \ell \leq d$ connected affinoids $V_1, \ldots, V_\ell$ of the same type, and $g : V_i \to U$ is d_i-to-1, where $1 \leq d_i \leq d$, and $\sum_{i=1}^{\ell} d_i = d$.*

A Polynomial Example

$\mathbb{C}_K = \mathbb{C}_p$, and $g(z) = pz^3 - z^2 + z$. Then

- Let U be the rational closed annulus $\overline{D}(0,1) \setminus D(0,1)$. Then $g(U) = \overline{D}(0,1)$.

  [**Note:** some points map 1-to-1, but others map 2-to-1.]

- $g^{-1}(\overline{D}(0,1)) = \overline{D}(0,1) \cup \overline{D}(1/p, |p|_p)$, with
  - $g : \overline{D}(0,1) \to \overline{D}(0,1)$ mapping 2-to-1, and
  - $g : \overline{D}(1/p, |p|_p) \to \overline{D}(0,1)$ mapping 1-to-1.

- $g^{-1}(\overline{D}(0, |p|_p^{-3})) = \overline{D}(0, |p|_p^{-4/3})$, mapping 3-to-1.

A Rational Example

$\mathbb{C}_K$ is any complete, algebraically closed non-archimedean field, and $h(z) = z - \frac{1}{z} = \frac{z^2 - 1}{z}$.

- $h^{-1}(D(0,1)) = D(1,1) \cup D(-1,1)$, with
  - each of $D(\pm 1, 1)$ mapping 1-1 onto $D(0,1)$ if the residue characteristic **is not** 2, or
  - $D(-1,1) = D(1,1)$ mapping 2-1 onto $D(0,1)$ if the residue characteristic **is** 2.
- $h^{-1}(\overline{D}(0,1))$ is the annulus $\overline{D}(0,1) \setminus D(0,1)$, which maps 2-to-1 onto $\overline{D}(0,1)$.

Dynamics on $\mathbb{P}^1(\mathbb{C}_K)$: Classifying Periodic Points

Fix a rational function $\phi(z) \in \mathbb{C}_K(z)$ of degree $d \geq 2$.

If $x \in \mathbb{P}^1(\mathbb{C}_K)$ is periodic of exact period n, then $\lambda := (\phi^n)'(x)$ is the **multiplier** of x. We say x is

- **attracting** if $|\lambda| < 1$.
- **repelling** if $|\lambda| > 1$.
- **indifferent** (or **neutral**) if $|\lambda| = 1$.

Note:

- The multiplier is the the same for all points in the periodic cycle of x.
- The multiplier is coordinate-independent.

The Spherical Metric on $\mathbb{P}^1(\mathbb{C}_K)$

There is a spherical metric on $\mathbb{P}^1(\mathbb{C}_K)$ analogous to that on $\mathbb{P}^1(\mathbb{C})$:

$$\Delta(z_1, z_2) := \frac{|z_1 - z_2|}{\max\{1, |z_1|\} \max\{1, |z_2|\}}$$

More precisely, to allow the point at ∞,
in homogeneous coordinates we write:

$$\Delta([x_1, y_1], [x_2, y_2]) := \frac{|x_1 y_2 - x_2 y_1|}{\max\{|x_1|, |y_1|\} \max\{|x_2|, |y_2|\}}$$

Fatou and Julia Sets

Definition

Let $\phi \in \mathbb{C}_K(z)$ be a rational function of degree $d \geq 2$.

The **(classical) Fatou set** $\mathcal{F} = \mathcal{F}_\phi$ of ϕ is

$$\mathcal{F} = \left\{x \in \mathbb{P}^1 : \{\phi^n\}_{n\geq 0} \text{ is equicontinuous on a neighborhood of } x\right\}$$
$$= \left\{x \in \mathbb{P}^1 : \text{for all } n \geq 1 \text{ and } y \in \mathbb{P}^1(\mathbb{C}_K) \text{ s.t. } \Delta(x,y) \text{ is small, } \Delta(\phi^n(x), \phi^n(y)) \text{ is also small.}\right\}$$

The **(classical) Julia set** $\mathcal{J} = \mathcal{J}_\phi$ is ϕ is $\mathcal{J} = \mathbb{P}^1(\mathbb{C}_K) \setminus \mathcal{F}$.

Idea:

- In the Fatou set, small errors stay small under iteration.
- In the Julia set, small errors may become large.

Basic Properties of Fatou and Julia Sets

For both $\mathbb{C}$ and $\mathbb{C}_K$:

- $\mathcal{F}$ is open, and $\mathcal{J}$ is closed.
- $\mathcal{F}_{\phi^n} = \mathcal{F}_\phi$, and $\mathcal{J}_{\phi^n} = \mathcal{J}_\phi$.
- $\phi(\mathcal{F}) = \mathcal{F} = \phi^{-1}(\mathcal{F})$, and $\phi(\mathcal{J}) = \mathcal{J} = \phi^{-1}(\mathcal{J})$.
- All attracting periodic points are Fatou.
- All repelling periodic points are Julia.

An equivalent definition for $\mathbb{C}_K$:

Theorem

Let $\phi \in \mathbb{C}_K(z)$, and let $x \in \mathbb{P}^1(\mathbb{C}_K)$. Then $x \in \mathcal{F}_\phi$ if and only if there is a $\mathbb{P}^1(\mathbb{C}_K)$-disk $D \ni x$ such that

$$\#\mathbb{P}^1(\mathbb{C}_K) \setminus \Big[\bigcup_{n \geq 0} \phi^n(D) \Big] \geq 2.$$

A Quadratic Example

$\phi(z) = z^2 + az \in \mathbb{C}_K[z]$.

- If $|a| \leq 1$, then $\phi(\overline{D}(0,1)) \subseteq \overline{D}(0,1)$,
  and $\phi(\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \overline{D}(0,1)) \subseteq \mathbb{P}^1(\mathbb{C}_K) \smallsetminus \overline{D}(0,1)$.

  So $\mathcal{F}_\phi = \mathbb{P}^1(\mathbb{C}_K)$, and $\mathcal{J}_\phi = \varnothing$.

- If $|a| = R > 1$, set $U_0 = \overline{D}(0,R)$.
  Then $\phi(\mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_0) \subseteq \mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_0$, so $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_0 \subseteq \mathcal{F}_\phi$.

  For all $n \geq 1$, set $U_n := \phi^{-n}(U_0)$.
  Then U_n is a disjoint union of 2^n closed disks of radius R^{1-n}.

  $\mathcal{J}_\phi = \bigcap_{n \geq 0} U_n$ is a Cantor set, and all points of
  $\mathcal{F}_\phi = \mathbb{P}^1(\mathbb{C}_K) \smallsetminus \mathcal{J}_\phi$ are attracted to ∞ under iteration.

Similarly: Over $\mathbb{C}_p$, Smart and Woodcock showed
$\phi(z) = (z^p - z)/p$ has $\mathcal{J}_\phi = \mathbb{Z}_p$.

A Cubic Example (due to Hsia)

Assume the residue characteristic is not 2, and set

$$\phi(z) = az^3 + z^2 + bz + c, \quad \text{where } 0 < |a| < 1, \text{ and } |b|, |c| \leq 1.$$

Then $\phi(\overline{D}(0,1)) \subseteq \overline{D}(0,1)$, so $\overline{D}(0,1) \subseteq \mathcal{F}_\phi$.

But ϕ has a repelling fixed point α with $|\alpha| = |a|^{-1} > 1$.

For all $n \geq 1$, there is a point $\beta_n \in \phi^{-n}(\alpha)$ s.t. $|\beta_n| = |a|^{-1/2^n}$.
Since $\beta_n \in \mathcal{J}_\phi$, the set $\mathcal{J}_\phi$ is not compact!!!

Note: if we set $U_0 = \overline{D}(0, |a|^{-1})$, then

$$\phi(\mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_0) \subseteq \mathbb{P}^1(\mathbb{C}_K) \smallsetminus U_0$$

as before, and $U_n := \phi^{-n}(U_0)$ is a disjoint union of many disks.

In fact, $\mathcal{F}_\phi$ is the union of $\mathbb{P}^1(\mathbb{C}_K) \smallsetminus \bigcap_{n\geq 1} U_n$ and all preimages of $\overline{D}(0,1)$.

Contrasts with $\mathbb{C}$

$\mathbb{C}$	$\mathbb{C}_K$
Some indifferent points are Fatou, and some are Julia.	**All** indifferent points are Fatou
$\mathcal{J}$ is compact	$\mathcal{J}$ may not be compact
$\mathcal{J}$ is nonempty	$\mathcal{J}$ may be empty
$\mathcal{F}$ may be empty	$\mathcal{F}$ is nonempty
$\mathcal{J}$ is the closure of the set of repelling periodic points	??? (see Project $\#$ 1)

A Quick Technical Note

The field $\mathbb{C}_K$ is complete, but it is usually not **spherically complete**.

That is, it is possible to have a decreasing chain of disks

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots$$

in a (**not** spherically complete field) $\mathbb{C}_K$ such that

$$\bigcap_{n \geq 1} D_n = \varnothing.$$

In this case, the disks D_n must have radius bounded below by some $R > 0$.

For example, $\mathbb{C}_p$ and the Puiseux series field $\mathbb{L}$ are **not** spherically complete.