Integrality and p-adic discreteness results for postcritically finite parameters

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Quick Facts on p-adic Numbers

Fix $p \geq 2$ a prime number. $\mathbb{C}_p$ is an algebraically closed field that contains $\mathbb{Q}$ and is equipped with an absolute value $|\cdot|_p$.

- $\left|\frac{r}{s}p^e\right|_p = \frac{1}{p^e}$ for $r, s, e \in \mathbb{Z}$ with $p \nmid rs$.
- $\mathbb{C}_p$ has characteristic zero, but $|p|_p < 1$.
- $|\cdot|_p$ is non-archimedean, meaning:

$$|x+y|_p \leq \max\{|x|_p, |y|_p\} \quad \text{for all } x, y \in \mathbb{C}_p.$$

  This implies:
  - $|x+y|_p = |x|_p$ if $|x|_p > |y|_p$,
  - all disks are topologically both open and closed
  - any point of a disk is its center
  - $\mathbb{Z}$ is contained in the closed unit disk $\overline{D}(0,1)$
- $\mathbb{C}_p$ is complete with respect to $|\cdot|_p$.

Local p-adic Dynamics

Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in \mathbb{C}_p[[z]]$, with $|a_i|_p \leq 1$ for all i, and $|a_0|_p < 1$.

Assume $|a_m|_p = 1$ for some minimal $m \geq 1$, called the **Weierstrass degree** of f on $D(0,1)$.

Then f maps the open unit disk $D(0,1)$ onto itself m-to-1.

Conversely, if $f : D(0,1) \to D(0,1)$ is (rigid) analytic, surjective, and finite-to-one,

then f is of the form above.

There are two main cases:

Case 1: $|a_1|_p < 1$. Equivalently, $m \geq 2$.

Case 2: $|a_1|_p = 1$. Equivalently, $m = 1$.

Local p-adic Dynamics: Attracting Case (no big surprises)

$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in \mathbb{C}_p[[z]]$,
with $|a_i|_p \leq 1$ for all i, and $|a_0|_p < 1$.
and $|a_m|_p = 1$ for some minimal $m \geq 1$,
so f maps the open unit disk $D(0,1)$ onto itself m-to-1.

If $|a_1|_p < 1$, then $D(0,1)$ contains an attracting fixed point β.

- All points of $D(0,1)$ are attracted to β under iteration of f.
- Unless β is a totally ramified fixed point, there will be infinitely many preperiodic points in $D(0,1)$, all eventually mapping to β.

Mild surprise for complex dynamicists: If $p|m$, there might not be any critical points in this basin of attraction.

Local p-adic Dynamics: Indifferent Case (some surprises!)

$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in \mathbb{C}_p[[z]]$,
with $|a_i|_p \le 1$ for all i, and $|a_0|_p < 1$.
and $|a_m|_p = 1$ for some minimal $m \ge 1$,
so f maps the open unit disk $D(0,1)$ onto itself m-to-1.

If $|a_1|_p = 1$, (and assuming a modest technical assumption),

then f has **infinitely** many indifferent periodic points in $D(0,1)$.

Why? Consider the coefficients "mod p"

If $g(z) = z + bz^n + O(z^{n+1})$ and $h(z) = z + cz^n + O(z^{n+1})$,

then $g \circ h(z) = z + (b+c)z^n + O(z^{n+1})$.

So $g^p(z) = z + O(z^{n+1})$.

So p-adic Siegel disks:

- contain infinitely many indifferent periodic points.
- are stable in moduli space: if f has a Siegel disk and $g \approx f$, then g also has a Siegel disk.

Dynamics of p-adic Rational Functions

If $f(z) \in \mathbb{C}_p(z)$ is a rational function of degree $d \geq 2$, then f has a Fatou set and a Julia set.

Theorem (Rivera-Letelier, 2003)

If U is a periodic component of the Fatou set, with period n, then either:

- *U is attracting: there is a unique attracting periodic point $\beta \in U$ (of period n), and all points of U are attracted to β under iteration of f^n, or*
- *U is indifferent: U is an open disk with finitely many closed disks removed. f^n maps U bijectively onto itself.*

- In the attracting case, U (probably) contains infinitely many preperiodic points, all of which eventually land on β.
- In the indifferent case, U contains infinitely many preperiodic points, all of which are periodic.

Postcritically Finite Maps

Definition

A rational function f is *postcritically finite*, or *PCF*, if every critical point c of f is preperiodic under f.

Example. $f(z) = z^d$: $\quad \infty \mapsto \infty \quad 0 \mapsto 0$

Example. $f(z) = z^2 - 1$: $\quad \infty \mapsto \infty \quad 0 \mapsto -1 \mapsto 0$

Example. $f(z) = z^2 - 2$: $\quad \infty \mapsto \infty \quad 0 \mapsto -2 \mapsto 2 \mapsto 2$

Example. $f(z) = z^2 + i$:

$$\infty \mapsto \infty \qquad 0 \mapsto i \mapsto i-1 \mapsto -i \mapsto i-1$$

Example. $f(z) = -2z^3 + 3z^2$: $\quad \infty \mapsto \infty \quad 0 \mapsto 0 \quad 1 \mapsto 1$

Example. $f(z) = \dfrac{6z^2+16z+16}{-3z^2-4z-4}$:

$$0 \mapsto -4 \mapsto -\frac{4}{3} \mapsto -\frac{4}{3} \qquad -2 \mapsto -1 \mapsto -2$$

Lots of PCF Parameters

Let K be an algebraically closed field (like $\mathbb{C}$ or $\mathbb{C}_p$)

Example. Fix a PCF map $\phi(z) \in K(z)$, let $h_t(z) \in \mathrm{PGL}(2, K(t))$ be a one-parameter family of linear fractional transformations, and let $f_t = h_t \circ \phi \circ h_t^{-1}$.

Then f_t is PCF for all parameters t. (Isotrivial)

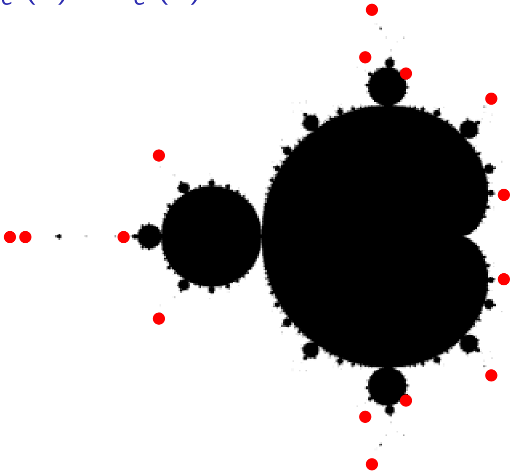
Example. Let E_t be a one-parameter family of elliptic curves, and let g_t be the Lattès map for $[m] : E_t \to E_t$.

Then g_t is PCF for all parameters t. (Flexible Lattès)

Example. Let $K = \mathbb{C}$ and let $f_c(z) = z^2 + c$.

Then the PCF parameters c are dense in the boundary of the Mandelbrot set.

Example: $f_c^7(0) = f_c^5(0)$



$f_c(z) = z^2 + c$. These are the roots of the Misiurewicz polynomial

$$\frac{\left(f_c^6(0) + f_c^4(0)\right) \cdot f_c(0)}{\left(f_c^5(0) + f_c^4(0)\right) \cdot f_c^2(0)}$$

A Stability Condition for One-Parameter p-adic Families

Fix $d \geq 2$, $b \in \mathbb{C}_p$, and $S > 0$.

Let $f_c(z)$ be a one-parameter family of rational functions, with coefficients meromorphic in the parameter $c \in D(b, S) \subseteq \mathbb{C}_p$.

Consider the following stability condition: for all $c \in D(b, S)$,

- $f_c(z) \in \mathbb{C}_p(z)$ with $\deg(f_c) = d$,
- the critical points of f_c are $\alpha_1(c), \ldots, \alpha_{2d-2}(c)$ (also meromorphic functions of c), and
- for each $i = 1, \ldots, 2d-2$, there are open disks $U_{i,j}$ with

$$\alpha_i(c) \in U_{i,0} \xrightarrow{f_c} U_{i,1} \xrightarrow{f_c} U_{i,2} \xrightarrow{f_c} \cdots \xrightarrow{f_c} U_{i,N_i}$$

  with $U_{i,N_i} \subseteq U_{i,M_i}$, where $N_i > M_i \geq 0$.

Example: Fix $d \geq 2$ and fix $b \in \mathbb{C}_p$ with $|b|_p \leq 1$.
Let $f_c(z) = z^d + c$ for $c \in D(b, 1)$,
with $\alpha_1 = \ldots = \alpha_{d-1} = 0$, and $\alpha_d = \ldots = \alpha_{2d-2} = \infty$.

p-adic PCF Parameters

Theorem (B-Ih)

Let $f_c(z)$ be a one-parameter family satisfying our stability condition on $D(b, S)$. Then either

1. *f_c is conjugate to f_b for all $c \in D(b, S)$, or*
2. *f_c is flexible Lattès for all $c \in D(b, S)$, or*
3. *for any $0 < s < S$, there are only finitely many $c \in D(b, s)$ for which f_c is PCF.*

Corollary

Let $f_c(z) = z^d + c$. Let

$$T = \left\{ c \in \mathbb{C}_p \middle| f_c \text{ is PCF} \right\}$$

Then the set T has no accumulation points in $\mathbb{P}^1(\mathbb{C}_p)$.

Some Examples

Example: $p = 2$, $d = 2$: $f_c(z) = z^2 + c$ over $\mathbb{C}_2$. Note:

$$c = 0, -2 \in D(0,1) \quad \text{and} \quad c = -1, i, -i \in D(1,1)$$

are all PCF parameters. More generally, for any $n \geq 1$, the roots of $f_c^n(0) + f_c^{n-1}(0) \in \mathbb{C}_2[c]$ all lie in $D(0,1)$.

Example: $p = 3$, $d = 2$: $f_c(z) = z^2 + c$ over $\mathbb{C}_3$. One can show $f_c^{1+3^n}(0) + f_c(0) \in \mathbb{C}_3[c]$ has (many) roots in $D(1,1)$.

Example: $f_c(z) := c(pz^{p+1} - (p+1)z^p + 1)$ has critical points at $z = 0, 1, \infty$, with

$$\infty \mapsto \infty, \qquad 1 \mapsto 0 \mapsto c.$$

So f_c is PCF if and only if c is preperiodic.
BUT infinitely many PCF parameters accumulate at $c = 1 + p^{-1}$.

This is not a stable family; for $c = 1 + p^{-1}$, f_c has a repelling fixed point at $z = c$.

Integrality

Let k be a number field with algebraic closure $\overline{k}$, and let $S \subseteq M_k$ be a finite set of places of k, including all the archimedean places.

Definition

Let D be a divisor on $\mathbb{P}^1(\overline{k})$ defined over k. A point $x \in \mathbb{P}^1(\overline{k})$ is (D, S)-**integral** if for any place $v \notin S$ of k, and for any $\sigma \in \mathrm{Gal}(\overline{k}/k)$, the reduction of x^σ modulo v is disjoint from the support of D modulo v.

In particular, if $\alpha \in k$, then $x \in \mathbb{P}^1(\overline{k})$ is $((\alpha), S)$-integral iff

$$x^\sigma \not\equiv \alpha \pmod{v} \quad \text{for all } v \in M_k \smallsetminus S \text{ and all } \sigma \in \mathrm{Gal}(\overline{k}/k).$$

For example, if $k = \mathbb{Q}$ and $S = \{\infty\}$, then

$$\{x \in \mathbb{P}^1(\mathbb{Q}) : x \text{ is } ((\infty), S)\text{-integral}\} = \mathbb{Z}.$$

A Result on Integrality of PCF Parameters

Let k be a number field, and let $S \subseteq M_k$ be a finite set of places of k, including all the archimedean places.

Corollary

Let $d \geq 2$, and let $f_c(z) = z^d + c$.
Let $\alpha \in k$, and suppose that f_α is **not** *PCF.*
Suppose also that for any archimedean place v of k, α does not lie in the boundary of the multibrot set

$$\mathbf{M}_{d,v} := \{c \in \mathbb{C} : |f_c^n(0)|_v \text{ is bounded as } n \to \infty\}.$$

Then there are only finitely many parameters $c \in \overline{k}$ which are $((\alpha), S)$-integral and for which f_c is PCF.

Proving the Corollary

Corollary

Let $d \geq 2$, and let $f_c(z) = z^d + c$. Let $\alpha \in k$ with f_α **not** *PCF.*

Suppose that for any archimedean place v of k, $\alpha \notin \partial \mathbf{M}_{d,v}$

Then there are only finitely many parameters $c \in \overline{k}$ which are $((\alpha), S)$-integral and for which f_c is PCF.

Sketch of Proof: Suppose $\{x_n\}_{n\geq 1}$ is a sequence of **distinct** PCF parameters in $\overline{k}$ that are $((\alpha), S)$-integral.

$$0 < \hat{h}_{f_\alpha}(0) = \sum_{v \in M_k} \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \int_{\mathbb{P}^1_{\text{an},v}} \log|x - \alpha|_v \, d\mu_{\text{Bif},v}(x)$$

where $\mu_{\text{Bif},v}$ is the bifurcation measure for the family f_c at v.

We'd like to apply equidistribution results
[Yuan (2008) and Ghioca-Krieger-Nguyen-Ye (2015)]
but $x \mapsto \log|x - \alpha|_v$ is not continuous.

Proof of Corollary, continued

For each $v \in M_k$, there is $r_v > 0$ so that $D(\alpha, r_v) \subseteq \mathbb{C}_v$ contains no PCF parameters and does not intersect $\partial \mathbf{M}_{d,v}$.

For v archimedean, this is by hypothesis.
For v non-archimedean, this is by our Theorem.

Let $F_v(x) := \log \max\{r_v, |x - \alpha|_v\}$. Then

$$\int_{\mathbb{P}^1_{\mathrm{an},v}} \log |x - \alpha|_v \, d\mu_{\mathrm{Bif},v}(x) = \int_{\mathbb{P}^1_{\mathrm{an},v}} F_v(x) \, d\mu_{\mathrm{Bif},v}(x)$$

$$= \lim_{n\to\infty} \frac{1}{[k(x_n) : k]} \sum_{\sigma} F_v(x_n^\sigma) = \lim_{n\to\infty} \frac{1}{[k(x_n) : k]} \sum_{\sigma} \log |x_n^\sigma - \alpha|_v,$$

where the sum is over all field embeddings $\sigma : k(x_n) \hookrightarrow \mathbb{C}_v$ fixing k.

Finishing the Proof of the Corollary

By integrality, we have $|x_n^\sigma - \alpha|_v = 1$ for all $v \in M_k \setminus S$ and all x_n and σ. Thus,

$$\begin{aligned}
0 < \hat{h}_{f_\alpha}(0) &= \sum_{v \in M_k} \lim_{n \to \infty} \frac{[k_v : \mathbb{Q}_v]}{[k(x_n) : \mathbb{Q}]} \sum_\sigma \log |x_n^\sigma - \alpha|_v \\
&= \sum_{v \in S} \lim_{n \to \infty} \frac{[k_v : \mathbb{Q}_v]}{[k(x_n) : \mathbb{Q}]} \sum_\sigma \log |x_n^\sigma - \alpha|_v \\
&= \lim_{n \to \infty} \sum_{v \in S} \frac{[k_v : \mathbb{Q}_v]}{[k(x_n) : \mathbb{Q}]} \sum_\sigma \log |x_n^\sigma - \alpha|_v \\
&= \lim_{n \to \infty} \sum_{v \in M_k} \frac{[k_v : \mathbb{Q}_v]}{[k(x_n) : \mathbb{Q}]} \sum_\sigma \log |x_n^\sigma - \alpha|_v = 0
\end{aligned}$$

Contradiction! QED

Recall Stability Condition and Main Theorem

[Modified to replace $D(b, S)$ by $D(0, S)$.]

Stability Condition: for all $c \in D(0, S)$,

- $f_c(z) \in \mathbb{C}_p(z)$ with $\deg(f_c) = d$,
- the critical points of f_c are $\alpha_1(c), \ldots, \alpha_{2d-2}(c)$, and
- for each $i = 1, \ldots, 2d-2$, there are open disks $U_{i,j}$ with
$$\alpha_i(c) \in U_{i,0} \xrightarrow{f_c} U_{i,1} \xrightarrow{f_c} U_{i,2} \xrightarrow{f_c} \cdots \xrightarrow{f_c} U_{i,N_i}$$
with $U_{i,N_i} \subseteq U_{i,M_i}$, where $N_i > M_i \geq 0$.

Theorem (B-Ih)

Let $f_c(z)$ be a one-parameter family satisfying our stability condition on $D(0, S)$. Then either

1. *f_c is conjugate to f_0 for all $c \in D(0, S)$, or*
2. *f_c is flexible Lattès for all $c \in D(0, S)$, or*
3. *for any $0 < s < S$, there are only finitely many $c \in D(0, s)$ for which f_c is PCF.*

Sketch of Proof: Setup

Let $\alpha = \alpha(c)$ be a critical point of f_c.

Replacing f_c by f_c^N and changing coordinates, we can assume that:

$$f_c(\alpha(c)) = 0, \quad \text{and} \quad f_c(D(0,1)) \subseteq D(0,1)$$

for all $c \in D(0,S)$.

We must show either

1. there are integers $n > m \geq 0$ such that $f_c^n(0) = f_c^m(0)$ for all $c \in D(0,S)$, (i.e., $\alpha(c)$ is *persistently preperiodic*), or
2. for any $0 < s < S$, there are only finitely many $c \in D(0,s)$ for which 0 and every critical point of f_c in $D(0,1)$ are all preperiodic.

Case 1: $|f_0'(0)|_p < 1$ (Attracting component)

Case 2: $|f_0'(0)|_p = 1$ (Siegel disk)

Case 1: $|f_0'(0)|_p < 1$

Then we can show f_c has an attracting fixed point $\beta(c) \in D(0,1)$ for every $c \in D(0,S)$.

For any $0 < s < S$, a p-adic analysis argument (similar to that in B-Ingram-Jones-Levy 2014) shows there is an integer $n = n(s) \geq 0$ (**independent of** c) so that for all $c \in D(0,s)$, either

1. $f_c^n(0) = \beta(c)$, or
2. $f_c^n(0) \neq \beta(c)$ but is very close, or
3. $f_c^n(\gamma_c) \neq \beta(c)$ but is very close, for some critical point γ_c.

When (2) or (3) happens, either $\alpha(c)$ or γ_c has infinite forward orbit under f_c. Thus, f_c **is not PCF**.

If (1) happens infinitely often on $D(0,s)$, then the power series $f_c^n(0) - \beta(c) \in \mathbb{C}_p[[c]]$ has infinitely many zeros in a proper subdisk of $D(0,S)$ and hence is trivial.
Thus, if (1) happens infinitely often on $D(0,s)$, then $\alpha(c)$ is persistently preperiodic on $D(0,S)$.

Case 2: $|f_0'(0)|_p = 1$

Choose $e \geq 1$ so that $|f_0'(0)^e - 1|_p < 1$.

Then we can show $|(f_c^e)'(0) - 1|_p < 1$ for **every** $c \in D(0, S)$.

The *iterative logarithm* of f_c is

$$\Lambda_c(z) := \lim_{n\to\infty} p^{-n}\left(f_c^{ep^n}(z) - z\right),$$

which is a (two-variable) power series converging on $(c, z) \in D(0, S) \times D(0, 1)$, following Rivera-Letelier 2003.

Idea: $\Lambda_c(z)$ measures how close $f_c^{ep^n}(z)$ is to z, relative to p^n.

Define $F(c) := \Lambda_c(0) \in \mathbb{C}_p[[c]]$, which is a power series converging on $D(0, S)$.

Intuitive Digression on the Iterative Logarithm

Given $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in \mathbb{C}_p[[z]]$,
with $|a_i|_p \le 1$ for all i, and $|a_0|_p < 1$.
Assume $|a_1|_p = 1$. In fact, assume $|a_1 - 1|_p < 1$. Define

$$\Lambda(z) := \lim_{n\to\infty} p^{-n}\big(f^{p^n}(z) - z\big).$$

Example. $f(z) = z + 1$. Then $f^{p^n}(z) = z + p^n$, so

$$\Lambda(z) = \lim_{n\to\infty} p^{-n}\big(z + p^n - z\big) = 1.$$

Example. $f(z) = (1+z)^d - 1$, where $d \equiv 1 \pmod{p}$. Then

$$\Lambda(z) = \lim_{n\to\infty} \frac{1}{p^n}\Big((1+z)^{d^{p^n}} - 1 - z\Big) = (\log_p d)(1+z)\log_p(1+z).$$

Note: For $z \in D(0,1)$,
$\Lambda(z) = 0 \Leftrightarrow (1+z)$ is a p-power root of unity $\Leftrightarrow z$ is periodic.

Back to the Proof: Case 2: $|f_0'(0)|_p = 1$: continued

$$\Lambda_c(z) = \lim_{n\to\infty} p^{-n}\left(f_c^{ep^n}(z) - z\right), \quad \text{and} \quad F(c) = \Lambda_c(0)$$

By results of Rivera-Letelier, *Astérisque* 2003 (Section 3.2) on the iterative logarithm,

$$F(c) = 0 \text{ iff } z = 0 \text{ is periodic under } f_c,$$
$$\text{i.e., iff } \alpha(c) \text{ is preperiodic under } f_c.$$

If F is identically zero, then for each $c \in D(0,S)$, there are integers $n(c) > m(c) \geq 0$ so that $f_c^{n(c)}(\alpha(c)) = f_c^{m(c)}(\alpha(c))$.

Some such pair $n > m$ occurs uncountably often, so $f_c^n(\alpha(c)) = f_c^m(\alpha(c))$ for **all** $c \in D(0,S)$.

Otherwise, for any $0 < s < S$, there are only finitely many $c \in D(0,s)$ for which $\alpha(c)$ is preperiodic under f_c.

Summary of the Proof so far

For each critical point $\alpha = \alpha(c)$ of f_c:

- After coordinate changes, we have

$$\alpha(c) \overset{f_c^N}{\longmapsto} 0 \in D(0,1) \xrightarrow{f_c^N} D(0,1)$$

- Let $\lambda_0 := \left(f_0^N\right)'(0)$.
  Two cases: $|\lambda_0|_p < 1$ (attracting) or $|\lambda_0|_p = 1$ (indifferent).
- In both cases, we showed either:
  - there are some $n > m \geq 0$ so that $f_c^n(\alpha(c)) = f_c^m(\alpha(c))$ for all $c \in D(0,S)$, or
  - for every $0 < s < S$, there are only finitely many $c \in D(0,s)$ for which $\alpha(c)$ and all critical points in $D(0,1)$ are preperiodic.

Conclusion of the Proof

Applying the preceding arguments to each critical point $\alpha_i(c)$ of $f_c(z)$, then either

1. For every $i = 1, \ldots, 2d-2$, there are integers $n_i > m_i \geq 0$ such that $f_c^{n_i}(\alpha_i(t)) = f_c^{m_i}(\alpha_i(c))$ for all $c \in D(0, S)$, or
2. For every $0 < s < S$, there are only finitely many $c \in D(0, s)$ for which f_c is PCF.

If (1) happens, Thurston Rigidity (Douady and Hubbard, 1993) says that either

- Every f_c is Lattès, or
- f_c is conjugate to $f_{c'}$ for uncountably many distinct c, c', and hence for all $c, c' \in D(0, S)$.

(2) and the two above possibilities for (1) are the three outcomes stated in the Theorem. QED

Stability Condition and Main Theorem, again

Stability Condition: for all $c \in D(b, S)$,

- $f_c(z) \in \mathbb{C}_p(z)$ with $\deg(f_c) = d$,
- the critical points of f_c are $\alpha_1(c), \ldots, \alpha_{2d-2}(c)$, and
- for each $i = 1, \ldots, 2d-2$, there are open disks $U_{i,j}$ with

$$\alpha_i(c) \in U_{i,0} \xrightarrow{f_c} U_{i,1} \xrightarrow{f_c} U_{i,2} \xrightarrow{f_c} \cdots \xrightarrow{f_c} U_{i,N_i}$$

with $U_{i,N_i} \subseteq U_{i,M_i}$, where $N_i > M_i \geq 0$.

Theorem (B-Ih)

Let $f_c(z)$ be a one-parameter family satisfying the stability condition on $D(b, S)$. Then either

1. *f_c is conjugate to f_b for all $c \in D(b, S)$, or*
2. *f_c is flexible Lattès for all $c \in D(b, S)$, or*
3. *for any $0 < s < S$, there are only finitely many $c \in D(b, s)$ for which f_c is PCF.*