Critical points and Julia stability in non-archimedean dynamics

*Rob Benedetto, Amherst College
Laura DeMarco, Harvard University

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Julia Stability in Complex Dynamics

Definition

Let $f \in \mathrm{Rat}_d(\mathbb{C})$ with Julia set $\mathcal{J}_f$. We say f is *J*-**stable** if there is a neighborhood $W \subseteq \mathrm{Rat}_d(\mathbb{C})$ of f such that

1. for every $g \in W$, there is a homeomorphism $\eta : \mathcal{J}_f \to \mathcal{J}_g$ such that $\eta \circ f = g \circ \eta$, and
2. $\mathcal{J}_g$ varies continuously for $g \in W$.

Theorem (Mañé, Sad, Sullivan 1983)

(At least for one-parameter families in $\mathrm{Rat}_d(\mathbb{C})$*),*
If $f \in \mathbb{C}(z)$ *is hyperbolic, then* f *is J-stable.*

(Hyperbolic (over $\mathbb{C}$) means expanding on $\mathcal{J}$. Equivalently, the closure of the post-critical set is disjoint from $\mathcal{J}$.)

The Berkovich Projective Line

$\mathbb{C}_v$: a complete, algebraically closed non-archimedean field with absolute value $|\cdot|$.

[E.g. p-adic field $\mathbb{C}_p$ or Puiseux series field $\widehat{\mathbb{C}((t))}$.]

$f \in \mathbb{C}_v(z)$ acts on $\mathbb{P}^1(\mathbb{C}_v) = \mathbb{C}_v \cup \{\infty\}$,
but even better, f acts on the $\boxed{\text{Berkovich line } \mathbb{P}^1_{\text{an}}}$, which:

- is compact, Hausdorff, and path-connected
- contains $\mathbb{P}^1(\mathbb{C}_v)$ as a subspace ("Type I points")
- contains one point $\zeta(a,r)$ for each closed disk $\overline{D}(a,r) \subseteq \mathbb{C}_v$ ("Type II and Type III points")
- contains one point ζ for each (equivalence class of) sequences of nested disks $D_1 \supsetneq D_2 \supsetneq \cdots$ with empty intersection ("Type IV points")

Rational Functions Acting on $\mathbb{P}^1_{\text{an}}$

For $f \in \mathbb{C}_v(z)$ of degree $d \geq 2$,

- f maps $\mathbb{P}^1_{\text{an}}$ continuously onto itself.
- for $x \in \mathbb{P}^1(\mathbb{C}_v)$ of type I, $f(x)$ is the usual $f(x) \in \mathbb{P}^1(\mathbb{C}_v)$.
- If $f\left(\overline{D}(a,r)\right) = \overline{D}(b,s)$, then $f\left(\zeta(a,r)\right) = \zeta(b,s)$.

  [**Note**: if f is a polynomial, then $f\left(\overline{D}(a,r)\right) = \overline{D}(b,s)$.]

Non-archimedean Dynamics

$f \in \mathbb{C}_v(z)$ has an associated

- (Berkovich) *Fatou set* $\mathcal{F}_{\mathrm{an},f}$, and
- (Berkovich) *Julia set* $\mathcal{J}_{\mathrm{an},f} := \mathbb{P}^1_{\mathrm{an}} \smallsetminus \mathcal{F}_{\mathrm{an},f}$

contained in $\mathbb{P}^1_{\mathrm{an}}$, such that:

- $\mathcal{F}_{\mathrm{an},f}$ is open in $\mathbb{P}^1_{\mathrm{an}}$, and $\mathcal{J}_{\mathrm{an},f}$ is closed (and hence compact).
- $f^{-1}(\mathcal{F}_{\mathrm{an},f}) = \mathcal{F}_{\mathrm{an},f}$ and $f^{-1}(\mathcal{J}_{\mathrm{an},f}) = \mathcal{J}_{\mathrm{an},f}$.
- Both $\mathcal{F}_{\mathrm{an},f}$ and $\mathcal{J}_{\mathrm{an},f}$ are nonempty.
- $\mathcal{J}_{\mathrm{an},f}$ is the smallest nonempty closed subset of $\mathbb{P}^1_{\mathrm{an}}$ that is invariant under f.

Some Previous Non-Archimedean J-Stability Results

- A. Silverman, 2017: Stability in one-parameter families, assuming existence of a type I repelling fixed point and special conditions at non-classical Berkovich points in the moduli space.
- RB–J. Lee, 2022: Stability in Rat_d assuming a uniform expansion condition on $\mathcal{J}_{\mathrm{an},f}$.
- J. Kiwi–H. Nie, preprint 2022: Stability in Poly_d for *tame* polynomials, assuming Böttcher coordinates at ∞ extend nicely to whole attracting basin at ∞.

Recall: $f \in \mathbb{C}_v(z)$ is *tame* if for all $\zeta \in \mathbb{P}^1_{\mathrm{an}}$, the ramification of f at ζ is *not* divisible by the residue characteristic p of $\mathbb{C}_v$.

A new J-Stability Result for Polynomials

Theorem (RB, DeMarco, 2026)

Let $\mathcal{M}$ be a dynamical moduli space of polynomials of degree $d \geq 2$ with marked critical points. Let $f \in \mathcal{M}(\mathbb{C}_v)$, and suppose f has a neighborhood W on which any Julia type I critical points are persistently preperiodic.

Then f is J-stable. *That is, f has a neighborhood $W' \subseteq W$ such that for every $g \in W'$, there is a homeomorphism $\eta : \mathcal{J}_f \to \mathcal{J}_g$ such that $\eta \circ f = g \circ \eta$, and $\mathcal{J}_g$ varies continuously for $g \in W'$.*

$$\begin{array}{ccc} \mathcal{J}_f & \xrightarrow{f} & \mathcal{J}_f \\ \downarrow \eta & & \downarrow \eta \\ \mathcal{J}_g & \xrightarrow{g} & \mathcal{J}_g \end{array}$$

Moreover, η is an isometry w.r.t. the metric on $\mathbb{H} = \mathbb{P}^1_{\text{an}} \setminus \mathbb{P}^1(\mathbb{C}_v)$.

Persistent Periodicity

A critical point $c_f \in \mathbb{P}^1(\mathbb{C}_v)$ of $f \in \mathcal{M}(\mathbb{C}_v)$ is **persistently preperiodic** if there is a neighborhood $W \ni f$ and integers $n > m \geq 0$ such that for all $g \in W$, we have $g^n(c_g) = g^m(c_g)$

Example. Fix $m \geq 2$ integer. For $a \in \mathbb{C}_v$ with $|a| > 1$,

Let $f_a(z) = az^{m+1} - az^m + 1$.

Then $0 \mapsto 1 \mapsto 1$ with $f_a'(1) = a$.

The critical point at 0 is persistently preperiodic.

And this (one-parameter) family is J-stable.

An Unstable Example

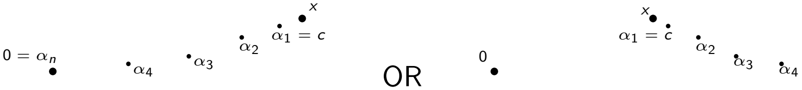
Fix $a \in \mathbb{C}_v$ with $|a| > 1$.

Let $g_c(z) = az^{m+1} - az^m + c$.

For $c = 1$, $0 \mapsto 1 \mapsto 1$, with 1 a repelling fixed point. $(g_1'(1) = a.)$

But for most $c \approx 1$, the critical point at 0 is **not** preperiodic.
Instead, there is a repelling fixed point $x \approx c$, and in fact:

- for some such c, the orbit of 0 can be periodic (and hence is in a bounded cycle of Fatou components).
- for other such c, the orbit of 0 can escape to ∞.



Polynomials with Wandering Domains

Suppose $\mathbb{C}_v$ has residue characteristic $p \geq 2$. Define

$$f_a(z) = (1-a)z^{p+1} + az^p$$

for any $a \in \mathbb{C}_v$ with $|a| > 1$.

- If $\mathbb{C}_v$ is the completion of $K = \overline{\mathbb{Q}}_p$ or of $K = \overline{\mathbb{F}_p(t)}$, and if $a \in K$, then f_a has **no** wandering domains.
- For $|a_0| > 1$ and $\varepsilon > 0$, there are parameters $a \in \mathbb{C}_v$ with $|a - a_0| < \varepsilon$ such that f_a **does** have wandering domains.

But a conjugacy $\eta : \mathcal{J}_{f_a} \to \mathcal{J}_{f_b}$ is still possible, with some points of type IV mapping to points of type II or III of the same diameter.

Nonarchimedean Quasi-analytic Conjugacies

Recall that $\mathbb{H} = \mathbb{P}^1_{\text{an}} \smallsetminus \mathbb{P}^1(\mathbb{C}_v)$, the set of Berkovich points in $\mathbb{P}^1_{\text{an}}$ of types II, III, and IV, has a metric $d_{\mathbb{H}}$.

Definition

We say $\eta : \mathbb{P}^1_{\text{an}} \to \mathbb{P}^1_{\text{an}}$ is a *quasi-analytic homeomorphism* if

1. η is a homeomorphism of $\mathbb{P}^1_{\text{an}}$, and
2. $\eta|_{\mathbb{H}}$ is an isometry with respect to $d_{\mathbb{H}}$.

Example. Over $\mathbb{C}_p$, define $\eta : \mathbb{P}^1_{\text{an}} \to \mathbb{P}^1_{\text{an}}$ by

$$\eta(z) = \begin{cases} z+1 & \text{if } z \in D(n,1) \text{ for some } n \in \{0,1,\dots,p-1\}, \\ z & \text{otherwise.} \end{cases}$$

Then η is a quasi-analytic homeomorphism.

Proof Strategy

To show: Suppose any Julia type I critical points of $f \in \mathcal{M}(\mathbb{C}_v)$ are persistently preperiodic. $\boxed{\text{Then } f \text{ is } J\text{-stable.}}$

- Let U_0 be a disk containing $\mathcal{J}_f$, and let $U_n := f^{-n}(U_0)$.
- For $g \in W'$, let $U'_n := g^{-n}(U_0)$.
- Let $\eta_N : \mathbb{P}^1_{\text{an}} \to \mathbb{P}^1_{\text{an}}$ be the identity map.
- $\boxed{\textbf{Key Step}}$ For each $n \geq N$, adjust η_n on U_n to get:

$$\begin{array}{ccc} U_n \setminus U_{n+1} & \xrightarrow{f} & U_{n-1} \setminus U_n \\ \big\downarrow \eta_{n+1} & & \big\downarrow \eta_n \\ U'_n \setminus U'_{n+1} & \xrightarrow{g} & U'_{n-1} \setminus U'_n \end{array}$$

- Let $\eta := \lim_{n\to\infty} \eta_n$, and restrict to $\mathcal{J}_f$.

Key Lemma

[For purposes of this presentation: assume char $\mathbb{C}_v = 0$.]

Suppose

$$f : U = \{r < |z| < s\} \to V = \{R < |z| < S\}$$

with Weierstrass degree $m \geq 1$, and such that

$$f^{(p^e)} \text{ has no zeros in } U \text{ for } e = 0, 1, \ldots, v_p(m).$$

Let $\eta : \mathbb{P}^1_{\text{an}} \to \mathbb{P}^1_{\text{an}}$ be a quasi-analytic homeomorphism with $\eta(V) = V$, and $g \in \mathbb{C}_v[z]$ with $\|f - g\|_{\zeta(0,s)}$ sufficiently small.
Then there is a quasi-analytic homeomorphism $\tilde{\eta} : \mathbb{P}^1_{\text{an}} \to \mathbb{P}^1_{\text{an}}$ with $\tilde{\eta}(U) = U$ and

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow \tilde{\eta} & & \downarrow \eta \\ U & \xrightarrow{g} & V \end{array}$$

Idea of Proof of Key Lemma

Write $f(z) = \sum_{n \geq 0} a_n z^n$ and $g(z) = \sum_{n \geq 0} b_n z^n$.

For each type I point $c \in U$,

$$f\big(c(1+x)\big) - f(c) = \sum_{j=1}^{\infty} \frac{1}{j!} f^{(j)}(c) c^j x^j,$$

and similarly for g. The hypotheses imply that $\left|f^{(p^e)}(c)\right| = \left|g^{(p^e)}(c)\right|$ depends only on $|c|$.

Thus, the Newton polygons of $f\big(c(1+x)\big) - f(c)$ and $g\big(c(1+x)\big) - g(c)$ coincide and depend only on $|c|$.

So for any disk $D \subseteq V$, the geometric arrangement of the disks of $f^{-1}(D)$ and $g^{-1}(\eta(D))$ are isometric.

This allows us to construct $\tilde{\eta} : U \to U$ locally as $g^{-1} \circ \eta \circ f$.

Main Result, Again

Theorem (RB, DeMarco, 2026)

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Moreover, η is an isometry w.r.t. the metric on $\mathbb{H} = \mathbb{P}^1_{\text{an}} \setminus \mathbb{P}^1(\mathbb{C}_v)$.

Thank you!