A QUOTIENT OF ELLIPTIC CURVES - WEAK NÉRON MODELS FOR LATTÈS MAPS

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1. Introduction

Let R be a Dedekind domain with quotient field F, and let A be an abelian variety defined over F. In studying the canonical height function on A, A. Néron [13] introduced the notion of a *Néron model* $\mathcal{A}$ over R. He showed that A has a Néron model over R and that for each non-Archimedean place v of F, the canonical local height over the local field F_v can be interpreted as an intersection multiplicity on the special fiber of $\mathcal{A} \otimes_R R_v$ over the ring R_v of integers of F_v. Call and Silverman [8] extended Néron's theory by introducing the notion of a weak Néron model $\mathcal{V}$ for a smooth variety V over non-Archimedean local field $k = F_v$ together with a k-morphism $\phi : V \to V$.

Definition 1. (Weak Néron Model) Let V be a smooth variety over k and let $\phi : V \to V$ be a finite morphism over k. A smooth scheme $\mathcal{V}$ over the ring of integers $\mathcal{O}_k$ is called a *weak Néron model* for $(V/k, \phi)$ if the following conditions hold:
(i) $\mathcal{V}_k$ (the generic fiber) $\simeq V$ over k,
(ii) $\mathcal{V}(\mathcal{O}_k) \simeq V(k)$, and
(iii) there exists an $\mathcal{O}_k$-morphism $\Phi : \mathcal{V} \to \mathcal{V}$ such that $\Phi |_{\mathcal{V}_k} = \phi$.

Note that the Néron model of an abelian variety A over k is a weak Néron model for $(A/k, [m])$, where $[m] : A \to A$ is the multiplication-by-m map. In [8], Call and Silverman showed if a weak Néron model $\mathcal{V}$ exists for $(V/k, \phi)$, then the canonical local height associated to ϕ [8, Sect. 2] can also be interpreted as an intersection multiplicity on the special fiber of $\mathcal{V}$. Thus, weak Néron models play an important role in the study of the canonical heights associated to a given morphism ϕ, just as Néron models do in the arithmetic theory of abelian varieties.

It is natural to ask whether a weak Néron model always exists for the pair $(V/k, \phi)$ if V is not an abelian variety. In fact, as shown in [10], if V is not an abelian variety over k, one cannot expect that a weak Néron model always exists An obstruction to the existence of a weak Néron model is closely related to the existence of an "unstable subset" of $V(k^{\mathrm{nr}})$ (where k^{nr} is the maximal unramified extension field of k) under the action of ϕ on $V(k^{\mathrm{nr}})$ via iterations of ϕ. In the case that $V = \mathbb{P}^1$ over k, the unstable subset is called the (non-Archimedean) Julia set $J_\phi(k^{\mathrm{nr}})$ of ϕ, which is the non-Archimedean analogue of the classical Julia set in the theory of complex dynamical systems in one variable. For background on complex dynamics, we refer the reader to any of [1, 9, 12]; papers on non-Archimedean dynamics and Julia sets include [2, 3, 5, 6, 10, 11, 14, 15].

In the case $V = \mathbb{P}^1$ over k, if ϕ has *good reduction*, then the projective line $\mathbb{P}^1_{\mathcal{O}_k}$ over $\mathcal{O}_k$ is a weak Néron model for $(\mathbb{P}^1/k, \phi)$. If ϕ does not have good reduction, then either there is a non-empty Julia set $J_\phi(k^{\mathrm{nr}})$, so that no weak Néron model exists for $(\mathbb{P}^1/k, \phi)$, or else $J_\phi(k^{\rm nr})$ is empty. If $J_\phi(k^{\rm nr})$ is empty, it could happen that $J_\phi(L)$ is non-empty after some (ramified) extension L over $k^{\rm nr}$, and we could conclude that no weak Néron model exists for ϕ over L. Otherwise, $J_\phi(L)$ is empty for any extension L of $k^{\rm nr}$. In many such examples, ϕ turns out to have good reduction over some extension L over $k^{\rm nr}$ (see Example 1 below).

It is an interesting problem to characterize morphisms $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ which do not have good reduction over any extension L of $k^{\rm nr}$ and yet have weak Néron models over $\mathcal{O}_L$. The aim of this note is to exhibit a particular family of such morphisms; our examples are special cases of so-called Lattès maps on $\mathbb{P}^1$. They might not have good reduction, but they have weak Néron models over any discretely valued field that is an algebraic extension of k.

Throughout this note, we fix the notation k to denote a non-Archimedean field that is complete with respect to a discrete valuation v. The ring of integers of k is denoted by $\mathcal{O}_k$, and the maximal ideal of $\mathcal{O}_k$ is denoted by $\mathcal{M}_k$. Also, the residue field of k is denoted by $\widetilde{k} := \mathcal{O}_k/\mathcal{M}_k$. In our discussion of weak Néron models, it is convenient to work with a non-Archimedean field with algebraically closed residue field. We let $k^{\rm nr}$ be the maximal unramified extension of k, so that the maximal ideal of $\mathcal{O}_{k^{\rm nr}}$ is $\mathcal{M}_k^{\rm nr} = \mathcal{M}_k\,\mathcal{O}_{k^{\rm nr}}$, and $\widetilde{k^{\rm nr}}$ is an algebraic closure of $\widetilde{k}$.

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2. Weak Néron models for rational maps on $\mathbb{P}^1$

We consider the case that $V = \mathbb{P}^1$, and that $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map defined over k. Fixing an affine coordinate z on $\mathbb{P}^1$, we have $\phi(z) = f(z)/g(z)$, where f, g are polynomials in z. It is clear that f, g can be chosen to be relatively prime polynomials with coefficients in $\mathcal{O}_k$. We say ϕ has *good reduction* (with respect to the coordinate z) if the resultant $Res(f, g)$ of f and g is a unit in $\mathcal{O}_k$. If ϕ has good reduction, then ϕ extends to an $\mathcal{O}_k$-morphism on the projective line $\mathbb{P}^1_{\mathcal{O}_k}$ over $\mathcal{O}_k$. By the definition of weak Néron model, if ϕ has good reduction, then $\mathbb{P}^1_{\mathcal{O}_k}$ is a weak Néron model for $(\mathbb{P}^1/k, \phi)$.

For general ϕ , we cannot always expect a weak Néron model to exist for $(\mathbb{P}^1/k, \phi)$. A necessary condition for the existence of a weak Néron model is given Proposition 2.1 below (see [10, Theorem 3.1] for a general statement). Before we can state the result, recall that the set of *periodic points* Per_ϕ of ϕ is the set of points P of $\mathbb{P}^1$ such that $\phi^n(P) = P$ for some $n \geq 1$, where $\phi^n = \phi \circ \phi \circ \cdots \circ \phi$ denotes the n-th iterate of ϕ. For any field K/k, let $\mathrm{Per}_\phi(K)$ denote the set of K-rational periodic points of ϕ.

Proposition 2.1. *Suppose that there is a weak Néron model for $(\mathbb{P}^1/k, \phi)$ over $\mathcal{O}_k$. Assume that $\mathrm{Per}_\phi(k^{\rm nr})$ is non-empty. Then for any $P \in \mathrm{Per}_\phi(k^{\rm nr})$ with $\phi^n(P) = P$, we have $|(\phi^n)'(P)| \leq 1$.*

The number $(\phi^n)'(P)$ is called the *multiplier* of periodic point P. We call a periodic point P such that $\phi^n(P) = P$ *repelling* if $|(\phi^n)'(P)| > 1$ and *non-repelling* otherwise.

Proposition 2.1 says that if a weak Néron model exists for the pair $(V/k, \phi)$, then all the k^{nr}-rational periodic points $\text{Per}_\phi(k^{\text{nr}})$ are non-repelling. As there are rational maps ϕ with repelling periodic points, we see from the above proposition that for such rational maps weak Néron models do not exist. It is natural to ask the following question: if all periodic points $P \in \text{Per}_\phi$ are non-repelling, is it true that weak Néron models exist for $(\mathbb{P}^1, \phi)$ over any finite extension L of k?

Example. 1. Suppose that $p \neq 2$ and that k is a finite extension of $\mathbb{Q}_p$. Let $\alpha \in k^*$ with $|\alpha| < 1$, and consider the following rational maps :

$$\begin{aligned} \phi_1(z) &= \frac{\alpha z^2 - z}{\alpha^2 z^2 + 1}, \\ \phi_2(z) &= \alpha z^d + c, \quad d \geq 2, c \in \mathcal{O}_k \\ \psi(z) &= \frac{z^p + c}{\alpha}, \quad c \in \mathcal{O}_k^* \end{aligned}$$

Note that all the maps above have bad reduction (with respect to the coordinate z).

(i) For ϕ_1 : let $z = f(w) = w/\alpha$. Then

$$\phi_1^f(w) := (f^{-1} \circ \phi_1 \circ f)(w) = \frac{w^2 - w}{w^2 + 1},$$

which has good reduction (with respect to the coordinate w).

(ii) For ϕ_2 : choose a $(d-1)$-th root $\delta \in \bar{k}$ of α and make the change of coordinate $z = f(w) = w/\delta$. Then

$$\phi_2^f(w) = w^d + c\,\delta,$$

which has good reduction with respect to the coordinate w. If α is not a $(d-1)$-st power in k, then the above coordinate change is defined over the field $k(\delta)$, which is a finite ramified extension of k. However, there is no coordinate change $z = f(x)$ over k^{nr} such that ϕ_2^f has good reduction. Nevertheless, applying a sequence of blow-ups at closed points on the special fiber [10], it can be shown that there is a weak Néron model for $(\mathbb{P}^1/k, \phi)$.

(iii) For ψ : let $s > 0$ be such that $|\alpha| = |p|^s$, and let $L = k(\delta, \text{Fix}(\psi))$, where $\text{Fix}(\psi) = \{a \in \bar{k} \mid \psi(a) = a\}$, and δ is a $(p-1)$-th root of α.

Case I. $0 < s \leq 1$: Consider the coordinate change $z = f(w) = \lambda + \delta w$, where $\lambda \in \text{Fix}(\psi)$ is any of the fixed points. (Note that $v(\lambda) \geq 0$.) By a direct computation, one can show that $\psi^f(w)$ has good reduction with respect to the coordinate w.

Case II. $s > 1$: For $\lambda \in \text{Fix}(\psi)$, we have $|\psi'(\lambda)| = |p|^{1-s} > 1$. By Proposition 2.1, we see that $(\mathbb{P}^1, \psi)$ does not have a weak Néron model over any field containing $k(\text{Fix}(\psi))$.

Remark. 1. For the map ψ in the above example, if $\text{Fix}(\psi)$ does not contain any k^{nr}-rational points, then it can still be shown that there is a weak Néron model $\mathfrak{X}$ over $\mathcal{O}_k$ for $(\mathbb{P}^1/k, \psi)$.

The above examples have the property that after some finite (ramified) extension of k, the maps in question either have good reduction with respect to some coordinate or have repelling periodic points defined over the extended field; in the latter case, we can conclude that there does not exist a weak Néron model. As mentioned in Section 1, we will seek rational maps ϕ over k that do not have good reduction (with respect to any coordinate on $\mathbb{P}^1$) over any algebraic extension L of k but admit weak Néron models over every $\mathcal{O}_L$. By Proposition 2.1, all periodic points of these rational maps are necessarily non-repelling.

3. Lattès Maps

We start by recalling the definition of Lattès maps.

Definition 2. A morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ over k is called a Lattès map if there exist an elliptic curve E/k, a morphism $\psi : E \to E$, and a finite separable morphism $\alpha : E \to \mathbb{P}^1$ over k so that the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\alpha} \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

For a discussion on properties of Lattès maps, we refer the reader to [18, Chap. 6]. *For simplicity, we'll only consider the case where $\alpha : E \to \mathbb{P}^1$ is given by the quotient $E \to E/\{\pm 1\} \simeq \mathbb{P}^1$. Moreover, we assume that the residue characteristic of $\widetilde{k}$ is odd.*

As an explicit example of a Lattès map, suppose that E is given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

and the quotient map to $\mathbb{P}^1_k$ is simply $\alpha = x : (x, y) \mapsto x$. If $\psi = [2]$, the multiplication-by-2 map, then the map ϕ is given by the formula

$$\phi(x) = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6},$$

where the $\{b_i\}$ are the usual functions of the $\{a_i\}$. (See [16, III.2.3].)

The following proposition [18, Prop. 6.52] gives a description of periodic points of ϕ and computes the multiplier of periodic points of Lattès map ϕ. For the proof (and the observation that part (b) really does cover all possible cases), we refer the reader to [18].

Proposition 3.1. *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map and assume that $\psi(P) = [m]P$.*
(a) *The set of n-periodic points of ϕ (i.e., the set of periodic points P with $\phi^n(P) = P$) is*

$$\mathrm{Per}_n(\phi) = \alpha\left(E[m^n - 1]\right) \cup \alpha\left(E[m^n + 1]\right),$$

where $E[\ell]$ denotes the group ℓ-torsion points of E.
(b) *Let ζ be a periodic point of ϕ of exact period n (i.e. $\phi^n(P) = P$ but $\phi^d(P) \neq P$ for $0 < d < n$). Then*

$$(\phi^n)'(\zeta) = \begin{cases} m^n & \text{if } \zeta \in \alpha(E[m^n - 1]) \text{ and } \zeta \notin \alpha(E[2]), \\ -m^n & \text{if } \zeta \in \alpha(E[m^n + 1]) \text{ and } \zeta \notin \alpha(E[2]), \\ m^{2n} & \text{if } \zeta \in \alpha(E[m^n + 1]) \cap \alpha(E[2]). \end{cases}$$

As the valuation on k is non-Archimedean, we see that all the periodic points of ϕ are non-repelling. In fact, in [4] we can show the following result.

Theorem 3.2. *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map over k. Then, for any discretely valued field extension L of k, there is a weak Néron model over $\mathcal{O}_L$ for $(\mathbb{P}^1/L, \phi)$.*

In the next section, we will sketch the construction of weak Néron models for Lattès maps.

4. A construction of weak Néron models for Lattès maps

Fix a Lattès map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ as defined in Section 3, and let the corresponding elliptic curve E be defined by the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad \text{with } a_i \in \mathcal{O}_k.$$

We'll assume that the Weierstrass model of E is minimal. By this, we mean that the valuation $v(\Delta)$ of the discriminant Δ of E is as small as possible. (Of course, $v(\Delta) \geq 0$, since $a_i \in \mathcal{O}_k$.) Let $\psi : E \to E$ be the endomorphism on E so that the diagram in Definition 2 is commutative (with $\alpha(x, y) = x$), and write $\Gamma = \{\pm 1\} \subset \mathrm{Aut}(E)$.

Let $\mathcal{E}$ be the Néron model of E over $\mathcal{O}_k$. (See [7] for an extensive treatment of Néron models, or [17], Chapter IV for a more accessible exposition for Néron models of elliptic curves.) As Néron models are invariant under faithfully flat base change (see [7, Section 7.2]; see also [17, IV.4.5] for the case of elliptic curves and finite unramified extensions), we'll assume that $k = k^{\mathrm{nr}}$, and thus that the residue field $\widetilde{k}$ is algebraically closed. This assumption will not affect our final results.

It is tempting to take the quotient of $\mathcal{E}$ directly, just as $\mathbb{P}^1$ is a quotient of E. By the Néron mapping property, the map ψ extends to a morphism $\Psi : \mathcal{E} \to \mathcal{E}$ over $\mathcal{O}_k$, and the group Γ extends to a group, which we still denote by Γ, of automorphisms of $\mathcal{E}$ over $\mathcal{O}_k$. Let $\mathcal{Y} := \mathcal{E}/\Gamma$. Note that $\mathcal{Y}$ is a smooth model over $\mathcal{O}_k$ for $\mathbb{P}^1_k$. It's not hard to see that ϕ extends to an $\mathcal{O}_k$-morphism $\Phi : \mathcal{Y} \to \mathcal{Y}$. However, there may be a point $P \in \mathbb{P}^1(k)$ that does not extend to an $\mathcal{O}_k$-valued point on $\mathcal{Y}$. This is possible because $\alpha^{-1}(P) = \{Q, -Q\} \subset E(\ell)$ for some ramified quadratic extension ℓ of k, and if $Q \in E(\ell) \setminus E(k)$, then Q does not extend to $\mathcal{O}_k$-valued point on $\mathcal{E}$. Hence, in general one cannot expect that $\mathcal{Y}$ is a weak Néron model for $(\mathbb{P}^1/k, \phi)$.

Instead of considering $\mathcal{Y}$ over $\mathcal{O}_k$, we consider the quadratic ramified extension of k. Let $\ell = k(\sqrt{\pi_k})$, where π_k is a uniformizer of k. Recall that by assumption, the characteristic of $\widetilde{k}$ is different from 2; hence ℓ is the only ramified quadratic extension over k. Let $E_\ell := E \otimes_k \ell$, and let $\mathcal{E}'$ denote the Néron model over $\mathcal{O}_\ell$ for E_ℓ. Again by the Néron mapping property, ψ and Γ extend to Ψ' and Γ' on $\mathcal{E}'$ over $\mathcal{O}_\ell$. Let $\mathcal{Y}' := \mathcal{E}'/\Gamma'$. Then $\mathcal{Y}'$ is a smooth model over $\mathcal{O}_\ell$ for $\mathbb{P}^1_\ell$. Moreover, every k-rational point on $\mathbb{P}^1$ extends to an $\mathcal{O}_\ell$-valued point on $\mathcal{Y}'$. We claim that a smooth $\mathcal{O}_k$-model $\mathfrak{X}$ for $\mathbb{P}^1_k$ can then be constructed from $\mathcal{Y}'$ so that every k-rational point extends to an $\mathcal{O}_k$-valued point on $\mathfrak{X}$, and ϕ extends to an $\mathcal{O}_k$-morphism on $\mathfrak{X}$.

To prove the claim, we introduce the following notation. Let $\widetilde{\mathcal{Y}}'$ denote the special fiber of $\mathcal{Y}'$, and write

$$\widetilde{\mathcal{Y}}' = \bigcup_i Z_i,$$

where the Z_i's are the irreducible components of $\widetilde{\mathcal{Y}}'$. For each $P \in \mathbb{P}^1(k)$, let $\overline{P} \in \mathcal{Y}'(\mathcal{O}_\ell)$ be the closure of P in $\mathcal{Y}'$, and write $\widetilde{P}$ for the corresponding point on the special fiber. That is, $\overline{P} = \{P, \widetilde{P}\}$. The construction of the desired weak Néron model now comes from following observations about $\widetilde{\mathcal{Y}}'$, proven in [4].

Proposition 4.1. *For every irreducible component Z_i of $\widetilde{\mathcal{Y}}'$,*
(1) Z_i is isomorphic to a Zariski open subset of $\mathbb{P}^1_{\widetilde{k}}$.
(2) $Z_i(\widetilde{k})$ contains either infinitely many $\widetilde{P}$ with $P \in \mathbb{P}^1(k)$ or at most one $\widetilde{P}$ with $P \in \mathbb{P}^1(k)$.
(3) If $Z_i(\widetilde{k})$ contains infinitely many $\widetilde{P}$ with $P \in \mathbb{P}^1(k)$, then Z_i is the special fiber of $X_i \otimes_{\mathcal{O}_k} \mathcal{O}_\ell$, for some $\mathcal{O}_k$-scheme X_i.
(4) If $Z_i(\widetilde{k})$ contains only one $\widetilde{P}$ with $P \in \mathbb{P}^1(k)$, let E_i be the blowing-up of $\widetilde{P}$. Then $E_j(k)$ contains infinitely many $\widetilde{P}$ with $P \in \mathbb{P}^1(k)$. Moreover, E_i is the special fiber of some $X_i \otimes_{\mathcal{O}_k} \mathcal{O}_\ell$, for some $\mathcal{O}_k$-scheme X_i.

Now blow up all components Z_i of $\widetilde{\mathcal{Y}}'$ with exactly one point of the form $\widetilde{P}$ for $P \in \mathbb{P}^1(k)$, and then remove all components Z_i with no such points $\widetilde{P}$. Using Proposition 4.1, it is not difficult to complete the proof that the resulting scheme $\mathfrak{X}$ is a weak Néron model for ϕ over $\mathcal{O}_k$.

We close with an example of the above construction of a weak Néron model in the case that E is a Tate curve.

Example. 2. Assume that the elliptic curve E has multiplicative reduction over k, and that E is defined by a minimal Weierstrass equation. Let $\mathcal{E}^0$ denote that identity component of the Néron model $\mathcal{E}$ of E. The group of automorphisms $\Gamma = \{\pm 1\}$ acts on $\mathcal{E}(\mathcal{O}_k)/\mathcal{E}^0(\mathcal{O}_k) \simeq \widetilde{\mathcal{E}}(\widetilde{k})/\widetilde{\mathcal{E}^0}(\widetilde{k})$. Hence, the action of Γ induces an action on the irreducible components of $\widetilde{\mathcal{E}}$, which form a cyclic group of order $n = v(\Delta)$.

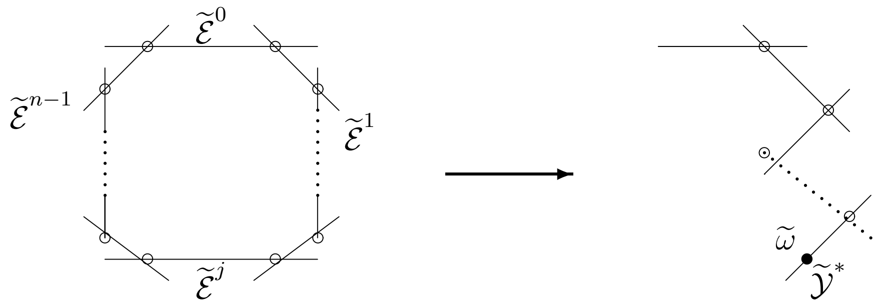
Identify the irreducible components of $\widetilde{\mathcal{E}}$ with $\mathbb{Z}/n\mathbb{Z}$, and denote by $\widetilde{\mathcal{E}}^i$ the component of $\widetilde{\mathcal{E}}$ corresponding to $\overline{i} \in \mathbb{Z}/n\mathbb{Z}$ (fig. 1). Then the action of Γ is just multiplication by ± 1 on the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Note that $\widetilde{\mathcal{E}}^0$ is simply the special fiber of the identity component. If n is even, then there are two components fixed by Γ, namely $\widetilde{\mathcal{Y}}^0$ and $\widetilde{\mathcal{Y}}^{n/2}$; if n is odd, then only the identity component is fixed by Γ. In either case, the special fiber $\widetilde{\mathcal{Y}}$ of $\mathcal{Y}$ can be read off easily from $\widetilde{\mathcal{E}}$ by identifying $\widetilde{\mathcal{E}}^i$ with $\widetilde{\mathcal{E}}^{n-i}$ (see fig. 2). Denote by $\widetilde{\mathcal{Y}}^*$ the component

$$\widetilde{\mathcal{Y}}^* = \begin{cases} \widetilde{\mathcal{E}}^{n/2}/\Gamma & \text{if } n \text{ is even,} \\ \{\widetilde{\mathcal{E}}^{(n-1)/2}, \widetilde{\mathcal{E}}^{(n+1)/2}\}/\Gamma & \text{if } n \text{ is odd,} \end{cases}$$

which is the component that is, in a sense, furthest from $\widetilde{\mathcal{Y}}^0$.

Note that Γ identifies smooth closed points with smooth closed points on $\widetilde{\mathcal{E}}$. It follows that the double points of both $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{Y}}$ are not actually part of the models, as $\mathcal{E}$ and $\mathcal{Y}$ are smooth over $\mathcal{O}_k$. Moreover, if n is even, then all k-rational points reduce to smooth closed points, so that the missing double points don't cause any violations to property (ii) in the definition of a weak Néron model. However, if n is odd, then there is a missing point $\widetilde{\omega}$ on $\widetilde{\mathcal{Y}}^{*}$, corresponding to the double point where $\widetilde{\mathcal{E}}^{(n-1)/2}$ and $\widetilde{\mathcal{E}}^{(n+1)/2}$ intersected. Unfortunately, there *are* points $P \in \mathbb{P}^1(k)$ that reduce to the missing point $\widetilde{\omega}$. This is the phenomenon noted earlier in this section; such points P correspond to points (x, y^+) and (x, y^-) on E with $x \in k$ but $y^{\pm} \notin k$.

fig. 1 $\widetilde{\mathcal{E}}$ fig. 2 $\widetilde{\mathcal{Y}}$



Instead, our algorithm stipulates that we consider the Néron model $\widetilde{\mathcal{E}}'$ over the ramified quadratic extension $\ell = k(\sqrt{\pi_k})$, with component group isomorphic to $\mathbb{Z}/2n\mathbb{Z}$. As before, we take the quotient $\widetilde{\mathcal{Y}}'$, which consists of $n+1$ components: $\widetilde{\mathcal{Y}'}^{0}$ and $\widetilde{\mathcal{Y}'}^{*} = \widetilde{\mathcal{Y}'}^{n}$ are each quotients of a single component $\widetilde{\mathcal{E}'}^{i}$ fixed by Γ, and every other component $\widetilde{\mathcal{Y}'}^{i}$ is the glueing of the pair $\widetilde{\mathcal{E}'}^{i}$ and $\widetilde{\mathcal{E}'}^{2n-i}$, for $1 \le i \le n-1$. We now need to turn $\widetilde{\mathcal{Y}}'$ into a model defined over k.

It can be shown that none of the components $\widetilde{\mathcal{Y}'}^{i}$ for odd $1 \le i \le n-1$ contain any points $\widetilde{P}$ with $P \in \mathbb{P}^1(k)$. If n is even, all the remaining components have infinitely many closed points $\widetilde{P}$ with $P \in \mathbb{P}^1(k)$; by statement (3) of Proposition 4.1, we have our desired weak Néron model $\mathfrak{X}$, up to base change. If n is odd, only the last component, $\widetilde{\mathcal{Y}'}^{n}$, is the exception to this rule: it contains a single point $\widetilde{P}$ with $P \in \mathbb{P}^1(k)$. Following our algorithm, we blow up (over ℓ) at $\widetilde{P}$, which has the effect of replacing $\widetilde{P}$ by a new component $\widetilde{Z}$ that is a copy of $\mathbb{P}^1$. Removing the old $\widetilde{\mathcal{Y}'}^{n}$ (which has now had the point $\widetilde{P}$ removed), and descending back to k, we are left with a weak Néron model $\mathfrak{X}$.

In this special case of multiplicative reduction, we can actually construct the model $\mathfrak{X}$ from the original quotient $\mathcal{Y}$, as follows. If n is even, then the scheme $\mathcal{Y}'$ with the odd-numbered components removed is simply $\mathcal{Y} \otimes_{\mathcal{O}_k} \mathcal{O}_\ell$. If n is odd, then add the missing point $\widetilde{\omega}$ into $\mathcal{Y}$ and blow up (over k) at $\widetilde{\omega}$ to obtain a scheme $\mathcal{Y}^{\circledast}$. Either way, we have found a scheme isomorphic to the Néron model found above.

The observant reader will have noted that in either case, the odd-numbered components removed from $\widetilde{\mathcal{Y}}'$ correspond to the intersection points of adjacent components of $\widetilde{\mathcal{Y}}$. However, the special points $\widetilde{\omega}$ and $\widetilde{P}$ are *not* quite the same thing; $\widetilde{\omega}$ corresponds to an open disk in $\mathbb{P}^1(k)$ of radius $|c|$ for some $c \in |k|$, whereas $\widetilde{P}$ corresponds to a slightly smaller disk, of radius $|c\sqrt{\pi_k}|$. In both the $\mathcal{Y}$ and the $\mathcal{Y}'$ version of the construction, the final blowing-up effectively shrinks this open disk to a *closed* disk of radius $|c\pi_k|$, so that it now appears on the special fiber as a copy of $\mathbb{P}^1$ defined over k.

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