WANDERING DOMAINS AND NONTRIVIAL REDUCTION IN NON-ARCHIMEDEAN DYNAMICS

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Abstract. Let K be a non-archimedean field with residue field k, and suppose that k is not an algebraic extension of a finite field. We prove two results concerning wandering domains of rational functions $\phi \in K(z)$ and Rivera-Letelier's notion of nontrivial reduction. First, if ϕ has nontrivial reduction, then assuming some simple hypotheses, we show that the Fatou set of ϕ has wandering components by any of the usual definitions of "components of the Fatou set". Second, we show that if k has characteristic zero and K is discretely valued, then the existence of a wandering domain implies that some iterate has nontrivial reduction in some coordinate.

The theory of complex dynamics in dimension one, founded by Fatou and Julia in the early twentieth century, concerns the action of a rational function $\phi \in \mathbb{C}(z)$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. Any such ϕ induces a natural partition of the sphere into the closed Julia set $\mathcal{J}_\phi$, where small errors become arbitrarily large under iteration, and the open Fatou set $\mathcal{F}_\phi = \mathbb{P}^1(\mathbb{C}) \setminus \mathcal{J}_\phi$. There is also a natural action of ϕ on the connected components of $\mathcal{F}_\phi$, taking a component U to $\phi(U)$, which is also a connected component of the Fatou set. In 1985, using quasiconformal methods, Sullivan [32] proved that $\phi \in \mathbb{C}(z)$ has no wandering domains; that is, for each component U of $\mathcal{F}_\phi$, there are integers $M \geq 0$ and $N \geq 1$ such that $\phi^M(U) = \phi^{M+N}(U)$. We refer the reader to [1, 13, 24] for background on complex dynamics.

Recall that a non-archimedean metric on a space X is a metric d which satisfies the *ultrametric* triangle inequality

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \quad \text{for all } x,y,z \in X.$$

In the past two decades, beginning with a study of linearization at fixed points by Herman and Yoccoz [19], there have been a number of investigations of non-archimedean dynamics; for a small sampling, see [4, 5, 6, 10, 20, 27, 28, 31]. It is natural to ask which properties of complex dynamics extend to the non-archimedean setting and which do not. We fix the following notation throughout this paper.

K	a complete non-archimedean field with absolute value $\lvert\cdot\rvert$
$\hat{K}$	an algebraic closure of K
$\mathbb{C}_K$	the completion of $\hat{K}$
$\mathcal{O}_K$	the ring of integers $\{x \in K : \lvert x\rvert \leq 1\}$ of K
k	the residue field of K
$\mathcal{O}_{\mathbb{C}_K}$	the ring of integers $\{x \in \mathbb{C}_K : \lvert x\rvert \leq 1\}$ of $\mathbb{C}_K$
$\hat{k}$	the residue field of $\mathbb{C}_K$

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$\mathbb{P}^1(\mathbb{C}_K)$ the projective line $\mathbb{C}_K \cup \{\infty\}$

Recall that the absolute value $|\cdot|$ extends in unique fashion to $\hat{K}$ and to $\mathbb{C}_K$. Recall also that the residue field k is defined to be $\mathcal{O}_K/\mathcal{M}_K$, where $\mathcal{M}_K$ is the maximal ideal of $\{x \in K : |x| < 1\}$ of $\mathcal{O}_K$. The residue field $\hat{k}$ is defined similarly. There is a natural inclusion of the residue field k into $\hat{k}$, making $\hat{k}$ an algebraic closure of k.

The best known complete non-archimedean field is $K = \mathbb{Q}_p$, the field of p-adic rational numbers (for any fixed prime number p). Its algebraic closure is $\hat{K} = \overline{\mathbb{Q}}_p$, and the completion $\mathbb{C}_K$ is frequently denoted $\mathbb{C}_p$. The ring of integers is $\mathcal{O}_K = \mathbb{Z}_p$, with residue field $k = \mathbb{F}_p$ (the field of p elements), and $\hat{k} = \overline{\mathbb{F}}_p$. Note that $\operatorname{char} K = 0$, but $\operatorname{char} k = p$. Thus, we say the characteristic of $\mathbb{Q}_p$ is 0, but the *residue characteristic* of $\mathbb{Q}_p$ is p. As another example, if L is any abstract field, then $K = L((T))$, the field of formal Laurent series with coefficients in L, is a complete non-archimedean field with $\mathcal{O}_K = L[[T]]$ (the ring of formal Taylor series) and $k = L$. In this case, $\operatorname{char} K = \operatorname{char} k = \operatorname{char} L$. The absolute value $|\cdot|$ on L may be defined by $|f| = 2^{-n}$, where $n \in \mathbb{Z}$ is the smallest integer for which the T^n term of the formal Laurent series f has a nonzero coefficient. We refer the reader to [17, 22, 29, 30] for surveys of non-archimedean fields.

In the study of non-archimedean dynamics of one-variable rational functions, we consider a rational function $\phi \in K(z)$, which acts on $\mathbb{P}^1(\mathbb{C}_K)$ in the same way that a complex rational function acts on the Riemann sphere. (One could instead "generalize" to $\phi \in \mathbb{C}_K(z)$; however, this setting is not really more general, because we could choose our original field K to be $\mathbb{C}_K$. By studying $\phi \in K(z)$, we may consider phenomena which only occur when the field of definition is *not* algebraically closed, just as functions $\phi \in \mathbb{R}(z)$ share certain dynamical properties which do not extend to all functions in $\mathbb{C}(z)$.)

In [2, 3, 6], the author defined non-archimedean Fatou and Julia sets and proposed two analogues of "connected components" of the Fatou set: D-components and analytic components, both of which we will define in Section 1. Rivera-Letelier proposed an alternate but related definition in [27, 28]. His definition was stated only over the p-adic field $\mathbb{C}_p$; in Section 1, we define an equivalent version of his components, which we call dynamical components, for all non-archimedean fields. In that section, we also will propose a fourth analogue, called dynamical D-components, which will be useful for proving some of our results. In practice, the various different types of components frequently coincide and are always closely related.

In this paper we will study wandering domains in particular. In [2, 3], the author proved a no wandering domains theorem (for D-components and analytic components) for a large class of rational functions $\phi \in \overline{\mathbb{Q}}_p(z)$. However, if the field $\overline{\mathbb{Q}}_p$ is replaced by its completion $\mathbb{C}_p$, the author later showed [7] that there *are* rational (in fact, polynomial) functions in $\mathbb{C}_p(z)$ which have wandering domains. In [9], the result was extended to all complete algebraically closed non-archimedean fields $\mathbb{C}_K$ of *positive* residue characteristic; that is, to fields K such that $K = \mathbb{C}_K$ and $\operatorname{char} k > 0$.

On the other hand, in [6, Example 2], for $K = \mathbb{Q}((T))$, with valuation given by $v(T) = 1$, the rational function $\phi(z) = (z^3 + (1 + T)z^2)/(z + 1) \in K(z)$ was shown to have a wandering domain. The wandering domain U found in that paper, as well as all of its foward images $\phi^n(U)$, are open disks of the form $D_1(b)$, with $|b| \leq 1$. In fact, the map ϕ almost has good reduction, in the sense that for all but finitely many of the disks $D_1(a)$ with $|a| \leq 1$, the image $\phi(D_1(a))$ is just $D_1(a')$, where $\overline{a}' = \overline{\phi}(\overline{a})$, and $\overline{\phi}(z) = z^2 \in k(z)$. In Rivera-Letelier's language [27, 28], there is a fixed point in the hyperbolic space $\mathbb{H}$; equivalently, ϕ has nontrivial reduction. We will make these concepts more precise in Section 2.

The first main result of the current paper will generalize that example to any non-archimedean field K whose residue field k is not an algebraic extension of a finite field. More precisely, Theorem 4.2 and Example 1 imply the following statement.

Theorem A. *Let K be a non-archimedean field with residue field k, where k is not an algebraic extension of a finite field.*

(a). *Let $\phi \in K(z)$ be a rational function of nontrivial reduction $\overline{\phi}$ with* $\deg \overline{\phi} \geq 2$. *Then ϕ has a wandering dynamical component U such that for every integer $n \geq 0$, $\phi^n(U)$ is an open disk of the form $D_1(b_n)$, with $|b_n| \leq 1$. U is also a wandering dynamical D-component; moreover, if the Julia set of ϕ intersects infinitely many residue classes of $\mathbb{P}^1(\mathbb{C}_K)$, then U is also a wandering D-component, and a wandering analytic component.*

(b). *There exist functions $\phi \in K(z)$ satisfying all of the hypotheses of part* (a) *above.*

Theorem 4.2 is an even stronger result, showing that under the hypotheses of Theorem A, there are actually infinitely many different grand orbits of wandering domains of the form $D_1(b)$. In addition, Theorem 4.3 will given sufficient conditions for the Julia set of ϕ to intersect infinitely many residue classes.

Theorem A and Theorem 4.2 are stated for maps with reduction $\overline{\phi}$ of degree at least two. If ϕ has a nontrivial reduction $\overline{\phi}$ of degree one, the situation is a little more complicated. In Examples 2–5, we will see that in some such cases there is a wandering domain of the form $D_1(b)$, and in other cases there is not.

Still, there are many maps with nontrivial reduction of degree at least two, and by Theorem A, all such maps over appropriate fields K have residue classes U which are wandering domains. More generally, if $\phi \in K(z)$ is a rational function and $g \in \mathrm{PGL}(2, \mathbb{C}_K)$ is a linear fractional transformation, suppose that some conjugated iterate $\psi(z) = g \circ \phi^n \circ g^{-1}(z)$ has nontrivial reduction of degree at least two. Even if the original map ϕ has trivial (i.e., constant) reduction, Theorem 4.2 shows that ϕ has a wandering domain, because ψ does.

The existence of rational functions with wandering domains should not come as a surprise for fields K satisfying the hypotheses of Theorem A. Because the residue field k is not finite, nor even algebraic over a finite field, the larger field K fails to be locally compact in spectacular fashion. By comparison, although the p-adic field $\mathbb{C}_p$ is also not locally compact, it does contain a dense subset $\overline{\mathbb{Q}}_p$ which is a countable union of locally compact subfields; however, no such dense subset exists for K. Thus, the impact of Theorem A is not so much the fact that wandering domains exist, but that they may be produced by such simple reduction conditions.

In what is perhaps a more interesting fact than the existence of such wandering domains, our next theorem shows that for certain fields K, the *only* wandering domains possible for rational functions are those described above. That is, any wandering domain for such a field *must* come from a nontrivial reduction.

Theorem B. *Let K be a non-archimedean field with residue field k. Suppose that K is discretely valued and that* $\operatorname{char} k = 0$. *Let $\phi(z) \in K(z)$ be a rational function, and suppose that some $U \subset \mathbb{P}^1(\mathbb{C}_K)$ is a wandering domain (analytic, dynamical, D-component, or dynamical D-component) of ϕ. Then there are integers $M \geq 0$ and $N \geq 1$ and a change of coordinate $g \in \mathrm{PGL}(2, \mathbb{C}_K)$ with the following property:*

Let $\psi(z) = g \circ \phi^N \circ g^{-1}(z) \in \mathbb{C}_K(z)$. Then $D_1(0)$ is the component of the Fatou set of ψ containing $g(\phi^M(U))$, and ψ has nontrivial reduction.

The clause "$D_1(0)$ is the component of the Fatou set of ψ containing $g(\phi^M(U))$" is equivalent to "$g(\phi^M(U)) = D_1(0)$" if we are dealing with analytic or dynamical components. On the other hand, if U is a D-component or dynamical D-component, then it is possible that $\phi^M(U)$ is a proper subset of a (dynamical) D-component. We will prove a slightly stronger version of Theorem B in Theorem 5.1 of Section 5, showing the the function h can be defined over a finite extension of K.

Recall that in [7, 9], it was shown that rational functions, including polynomials, may have wandering domains if $K = \mathbb{C}_K$ and if $\operatorname{char} k > 0$. By contrast, Theorem B shows that rational functions have no wandering domains (besides those arising from a nontrivial reduction, as in Example 1) if the field of definition K is discretely valued and has residue characteristic *zero*. However, an algebraically closed non-archimedean field (such as $\mathbb{C}_K$) cannot be discretely valued. Thus, one is naturally led to ask the following open questions:

(1) If K is locally compact (implying both that K is discretely valued and has residue characteristic $p > 0$), do there exist polynomial or rational functions $\phi \in K(z)$ with wandering domains?
(2) If K is complete and *algebraically closed*, with residue characteristic zero, do there exist functions $\phi \in K(z)$ with wandering domains other than those arising from a nontrivial reduction?

In [3], Theorem 1.2, the author showed that if K is a finite extension of $\mathbb{Q}_p$, and if $\phi \in K(z)$ has no recurrent Julia critical points, then ϕ has no wandering domains. (At the time of this writing, no rational functions $\phi \in K(z)$ with recurrent Julia critical points have been found.) Moreover, the proof of [4, Corollary 3.1] may be applied to any locally compact non-archimedean field K to show that if $\phi \in K(z)$ has no Julia critical points, then ϕ has no wandering domains. (It is a simple matter to produce a map with a Julia critical point, however.) Thus, it appears that the answer to the first question above is probably "no". Meanwhile, we know of no progress on the second question.

We will begin in Section 1 with a review of fundamental definitions and properties of rational functions, non-archimedean analysis, and dynamics, including definitions of the various types of Fatou components. In Section 2, we will introduce Rivera-Letelier's notion of nontrivial reduction and state three important lemmas. In Section 3, we will recall a few facts from the theory of diophantine height functions. Heights will be used only in the proof of Lemma 4.1; the reader unfamiliar with the theory may skip both Section 3 and the proof of Lemma 4.1 without loss of continuity. Finally, in Section 4 we will prove Theorem A, and in Section 5 we will prove Theorem B.

1. Background

1.1. **General properties of rational functions.** We recall some basic facts about rational functions $\phi \in L(z)$, for an abstract field L with algebraic closure $\hat{L}$. A point $x \in L$ is called a *pole* of ϕ if $\phi(x) = \infty$. We may define the derivative $\phi'(z)$ by the usual formal differentiation rules; if L has a metric structure, then the formal definition of ϕ' agrees with the difference quotient definition of ϕ', if ϕ is viewed as a map from $L \setminus \{\text{poles of } \phi\}$ to L.

If $x \in \mathbb{P}^1(\hat{L})$ maps to $\phi(x)$ with multiplicity greater than one (i.e., if $\phi'(x) = 0$), we say x is a *critical point* or *ramification point* of ϕ. After a coordinate change in the domain and range, we may assume that $x = \phi(x) = 0$, and we may expand ϕ locally about 0 as a power series

$$\phi(z) = \sum_{n=1}^{\infty} c_n z^n.$$

We say that x *maps to* $\phi(x)$ *with multiplicity* m if m is the smallest integer such that $c_m \neq 0$. Note that if $\operatorname{char} L = p > 0$, the multiplicity is *not* the same as the number of the first nonzero derivative at x. For example, if $\phi(z) = z^p$ where $\operatorname{char} L = p$, then $\phi'(z) = 0$, but every point x maps to its image with multiplicity p, not infinite multiplicity.

In general, if $\phi'(z)$ is not identically zero, we say ϕ is *separable.* If $\operatorname{char} L = 0$, then all nonconstant functions are separable. If $\operatorname{char} L = p > 0$, then $\phi \in L(z)$ is separable if and only if ϕ cannot be written as $\phi(z) = \psi(z^p)$ for any $\psi \in L(z)$.

A function $\phi \in L(z)$ may be written as $\phi = f/g$, where $f, g \in L[z]$ are relatively prime polynomials. The *degree* $\deg \phi$ is defined to be

$$\deg \phi = \max\{\deg f, \deg g\}.$$

Any point $y \in \mathbb{P}^1(\hat{L})$ has exactly $\deg \phi$ preimages in $\phi^{-1}(y)$, counting multiplicity. If ϕ is a separable function of degree d, then ϕ has exactly $2d - 2$ critical points in $\mathbb{P}^1(\hat{L})$, counting multiplicity.

1.2. **Non-archimedean analysis.** Given $a \in \mathbb{C}_K$ and $r > 0$, we denote by $D_r(a)$ and by $\overline{D}_r(a)$ the open disk and the closed disk, respectively, of radius r centered at a. (We will follow the convention that all disks have *positive* radius by definition, so that singleton sets and the empty set are *not* considered to be disks.) By the non-archimedean triangle inequality, any point of such a disk may be considered a center, and if $U_1, U_2 \subset \mathbb{C}_K$ are two overlapping disks, then either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. Moreover, if $U \subset \mathbb{C}_K$ is an open or closed disk of radius r, then r is also the diameter of U; that is,

$$r = \operatorname{diam}(U) = \sup\{|x - y| : x, y \in U\}.$$

We will write $r = \operatorname{rad}(U)$ for the radius of such a disk.

The set $|K^*| = \{|x| : x \in K \setminus \{0\}\} \subset \mathbb{R}_{>0}$ may be a discrete subset of $\mathbb{R}_{>0}$; if so, we say that K is *discretely valued.* In that case, there is a real number $0 < \varepsilon < 1$ such that $|K^*| = \{\varepsilon^m : m \in \mathbb{Z}\}$.

Meanwhile, the set $|\mathbb{C}_K^*| = \{|x| : x \in \mathbb{C}_K \setminus \{0\}\}$ must be dense in $\mathbb{R}_{>0}$, but it need not contain all positive real numbers. For example, $|\mathbb{C}_p^*| = \{p^q : q \in \mathbb{Q}\}$. Therefore, we say that a disk U is *rational* if $\operatorname{rad}(U) \in |\mathbb{C}_K^*|$, and U is *irrational* otherwise. If $a \in \mathbb{C}_K$ and $r \in |\mathbb{C}_K^*|$, then $D_r(a) \subsetneq \overline{D}_r(a)$; but if $r \in (\mathbb{R}_{>0} \setminus |\mathbb{C}_K^*|)$, then $D_r(a) = \overline{D}_r(a)$. Thus, every disk is exactly one of the following three types: rational open, rational closed, or irrational. The distinctions between the three indicate metric properties, but not topological properties; *all* disks are both open and closed as *topological sets*.

More generally, a set $U \subset \mathbb{P}^1(\mathbb{C}_K)$ is a *rational open disk* if either $U \subset \mathbb{C}_K$ is a rational open disk or $\mathbb{P}^1(\mathbb{C}_K) \setminus U$ is a rational closed disk. Similarly, $U \subset \mathbb{P}^1(\mathbb{C}_K)$ is a *rational closed disk* if either $U \subset \mathbb{C}_K$ is a rational closed disk or $(\mathbb{P}^1(\mathbb{C}_K) \setminus U) \subset \mathbb{C}_K$ is a rational open disk; and $U \subset \mathbb{P}^1(\mathbb{C}_K)$ is an *irrational disk* if either $U \subset \mathbb{C}_K$ is an irrational disk or $(\mathbb{P}^1(\mathbb{C}_K) \setminus U) \subset \mathbb{C}_K$ is an irrational disk. There is a natural spherical metric on $\mathbb{P}^1(\mathbb{C}_K)$ (see, for example, [5, 8, 25]), but not all the disks we have just defined in $\mathbb{P}^1(\mathbb{C}_K)$ are disks with respect to the spherical metric.

If $U_1, U_2 \subset \mathbb{P}^1(\mathbb{C}_K)$ are disks such that $U_1 \cap U_2 \neq \emptyset$ and $U_1 \cup U_2 \neq \mathbb{P}^1(\mathbb{C}_K)$, then either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. In particular, both $U_1 \cap U_2$ and $U_1 \cup U_2$ are also disks; and if U_1 and U_2 are both rational closed (respectively, rational open, irrational), then so are $U_1 \cap U_2$ and $U_1 \cup U_2$.

The group $\mathrm{PGL}(2, \mathbb{C}_K)$ acts on $\mathbb{P}^1(\mathbb{C}_K)$ by linear fractional transformations. Any $g \in \mathrm{PGL}(2, \mathbb{C}_K)$ maps rational open disks to rational open disks, rational closed disks to rational closed disks, and irrational disks to irrational disks.

The following four lemmas concern the mapping properties of non-archimedean rational functions on disks. In fact, all four apply more generally to power series on disks, though we do not need to define the necessary terminology of rigid analyticity to state the lemmas. We omit the proofs, which are easy applications of the Weierstrass Preparation Theorem, Newton polygons, and other fundamentals of non-archimedean analysis. Some proofs may be found in [8]; see any of [11, Chapter 5], [14, Chapter II], [22, Chapter IV], or [29, Chapter 6] for the theory surrounding such results.

Lemma 1.1. *Let $U \subset \mathbb{P}^1(\mathbb{C}_K)$ be a disk, and let $\phi \in \mathbb{C}_K(z)$ be a rational function. Suppose that $\mathbb{P}^1(\mathbb{C}_K) \setminus \phi(U)$ contains at least two points. Then $\phi(U)$ is a disk of the same type (rational closed, rational open, or irrational) as U.*

Lemma 1.2. *Let $a, b \in \mathbb{C}_K$, let $r, s > 0$, and let $\phi \in \mathbb{C}_K(z)$ be a rational function with no poles in $\overline{D}_r(a)$, such that $\phi(D_r(a)) = D_s(b)$. Then $\phi(\overline{D}_r(a)) = \overline{D}_s(b)$.*

Lemma 1.3. *Let $U \subset \mathbb{C}_K$ be a disk, let $a \in U$, and let $\phi \in \mathbb{C}_K(z)$ be a rational function with no poles in U. Then the following two statements are equivalent.*

a. *ϕ is one-to-one on U.*
b. *For all $x, y \in U$, $|\phi(x) - \phi(y)| = |\phi'(a)| \cdot |x - y|$.*

Lemma 1.4. *Let K be a non-archimedean field with residue field k, let $U \subset \mathbb{C}_K$ be a disk, and let $\phi \in \mathbb{C}_K(z)$ be a rational function. Suppose that* $\operatorname{char} k = 0$. *Then the following two statements are equivalent.*

a. *ϕ is one-to-one on U.*
b. *ϕ has no critical points in U.*

We will need some basic facts and definitions from the non-archimedean theory of rigid analysis. We refer the reader to [11, Part C] or [15] for detailed background, or to [16] for a broader (but still technical) overview of the subject; however, the discussion that follows is mostly self-contained.

A *connected affinoid* is a set $W \subset \mathbb{P}^1(\mathbb{C}_K)$ of the form

$$W = \mathbb{P}^1(\mathbb{C}_K) \setminus (U_1 \cup U_2 \cup \cdots \cup U_N),$$

where $N \geq 0$, and where the $\{U_i\}$ are pairwise disjoint disks. If each U_i is rational open, we say W is a *connected rational closed affinoid*; if each U_i is rational closed, we say W is a *connected rational open affinoid*; and if each U_i is irrational, we say W is a *connected irrational affinoid*.

If W_1 and W_2 are connected affinoids, and if $W_1 \cap W_2 \neq \emptyset$, then $W_1 \cap W_2$ and $W_1 \cup W_2$ are also connected affinoids. In that case, if W_1 and W_2 are both rational closed (respectively, rational open, irrational), then so are $W_1 \cap W_2$ and $W_1 \cup W_2$.

In general, an *affinoid* is a finite union of connected affinoids. However, we will not need that notion in this paper. Note that our definition allows the full set $\mathbb{P}^1(\mathbb{C}_K)$ and the empty set $\emptyset$ to be considered connected affinoids, while traditional rigid analysis does not. Also note that we consider $\mathbb{P}^1(\mathbb{C}_K)$ to be a connected affinoid of all three types. Every other connected affinoid is at most one of the three types; or, it may be none of them, if, for example, U_1 is a rational open disk and U_2 is a rational closed disk.

Intuitively, connected affinoids are supposed to behave like connected sets, even though *topologically*, all subsets of $\mathbb{P}^1(\mathbb{C}_K)$ are totally disconnected. In particular, it is well known (as can be shown using standard rigid analysis techniques) that if $\phi \in \mathbb{C}_K(z)$ is a rational function of degree d, and if W is a connected affinoid, then:

- $\phi(W)$ is also a connected affinoid. Moreover, if W is rational closed (respectively, rational open, irrational), then so is $\phi(W)$.
- $\phi^{-1}(W)$ is a disjoint union of connected affinoids $V_1, \ldots, V_N$, with $1 \leq N \leq d$. For every $i = 1, \ldots, N$, ϕ maps V_i onto W. Moreover, if W is rational closed (respectively, rational open, irrational), then so are $V_1, \ldots, V_N$.

The following lemma shows that for any given rational function ϕ, most disks $U \subset \mathbb{P}^1(\mathbb{C}_K)$ have preimage $\phi^{-1}(U)$ consisting simply of a finite union of disks. It appeared as [2, Lemma 3.1.4], but we include a partial proof here for the convenience of the reader.

Lemma 1.5. *Let $U_1, \ldots, U_n \subset \mathbb{P}^1(\mathbb{C}_K)$ be disjoint disks, and let $\phi \in K(z)$ be a rational function of degree $d \geq 1$. Suppose that for each $i = 1, \ldots, n$, the inverse image $\phi^{-1}(U_i)$ is not a finite union of disks. Then $n \leq d - 1$.*

Sketch of proof. If U_i is an open disk, then it can be written as a nested union $\bigcup_{m \geq 1} V_m$ of rational closed disks V_m, with $V_m \subset V_{m+1}$. If each $\phi^{-1}(V_m)$ is a union of at most d disks, then the same is true of $\phi^{-1}(U_i)$. Thus, we may assume that each U_i is a rational closed disk.

Let $W = \mathbb{P}^1(\mathbb{C}_K) \setminus (U_1 \cup \cdots \cup U_n)$. Then W is a rational open connected affinoid. By the discussion above, the inverse image $\phi^{-1}(W)$ is a disjoint union of at most d rational open connected affinoids. Thus, as we leave to the reader to verify, the complement $\mathbb{P}^1(\mathbb{C}_K) \setminus \phi^{-1}(W)$ is a union of some rational closed disks and at most $d - 1$ connected affinoids which are not disks. (Note, for example, that if V is a closed affinoid that is neither a disk nor all of $\mathbb{P}^1(\mathbb{C}_K)$, then the complement of V consists of at least two connected components. Thus, if $\mathbb{P}^1(\mathbb{C}_K) \setminus \phi^{-1}(W)$ consisted of d or more non-disk connected components, then $\phi^{-1}(W)$ would consist of at least $d+1$ connected components.) However, the complement of $\phi^{-1}(W)$ is precisely the disjoint union $\bigcup_{i=1}^{n} \phi^{-1}(U_i)$. It follows that $n \leq d-1$. □

1.3. **Dynamics.** Let X be a set, and let $f : X \to X$ be a function. For any $n \geq 1$, we write $f^1 = f$, $f^2 = f \circ f$, and in general, $f^{n+1} = f \circ f^n$; we also define f^0 to be the identity function on X. Let $x \in X$. We say that x is *fixed* if $f(x) = x$; that x is *periodic of period* $n \geq 1$ if $f^n(x) = x$; that x is *preperiodic* if $f^m(x)$ is periodic for some $m \geq 0$; or that x is *wandering* if x is not preperiodic. Note that all fixed points are periodic, and all periodic points are preperiodic. We define the *forward orbit* of x to be the set $\{f^n(x) : n \geq 0\}$; the *backwards orbit* of x to be $\bigcup_{n\geq 0} f^{-n}(x)$; and the *grand orbit* of x to be

$$\{y \in X : \exists\, m, n \geq 0 \text{ such that } f^m(x) = f^n(y)\}.$$

Equivalently, the grand orbit of x is the union of the backwards orbits of all points in the forward orbit of x. We say a grand orbit S is *preperiodic* if it contains a preperiodic point, or S is *wandering* otherwise. Note that S is preperiodic (respectively, wandering) if and only if every point in S is preperiodic (respectively, wandering).

Suppose X is a metric space. Recall that the family of functions $\{f^n : n \geq 0\}$ is *equicontinuous* at $x \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d(f^n(x), f^n(y) < \varepsilon$ for all $n \geq 0$ and for all $y \in X$ satisfying $d(x, y) < \delta$. (The key point is that ε is chosen independent of n.)

Now consider $X = \mathbb{P}^1(\mathbb{C}_K)$ and $f = \phi \in \mathbb{C}_K(z)$. The *Fatou set* of ϕ is the set $\mathcal{F} = \mathcal{F}_\phi$ consisting of all points $x \in \mathbb{P}^1(\mathbb{C}_K)$ for which $\{\phi^n : n \geq 0\}$ is equicontinuous on some neighborhood of x, with respect to the spherical metric on $\mathbb{P}^1(\mathbb{C}_K)$. The *Julia set* $\mathcal{J} = \mathcal{J}_\phi$ of ϕ is the complement $\mathcal{J} = \mathbb{P}^1(\mathbb{C}_K) \setminus \mathcal{F}$. Clearly the Fatou set is open, and the Julia set is closed. It is easy to show that $\phi(\mathcal{F}) = \phi^{-1}(\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}_{\phi^n} = \mathcal{F}_\phi$ for all $n \geq 1$, and similarly for the Julia set.

Intuitively speaking, the Fatou set is the region where small errors stay small under iteration, while the Julia set is the region of chaos. Note that because $\mathbb{P}^1(\mathbb{C}_K)$ is not locally compact, the Arzelà-Ascoli theorem fails, which is why non-archimedean Fatou and Julia sets are defined in terms of equicontinuity instead of normality.

It is easy to verify (using, for example, [8, Lemma 2.7] or other well known lemmas on non-archimedean power series) that if $U \subset \mathbb{P}^1(K)$ is a disk and if $R > 0$ such that for all $n \geq 0$, $\phi^n(U)$ is a disk in $\mathbb{C}_K$ of radius at most R, then $U \subset \mathcal{F}_\phi$. It follows that if $V \subset \mathbb{P}^1(K)$ is any open set and if $R > 0$ such that for all $n \geq 0$, $\phi^n(V) \subset \mathbb{C}_K$ and $\operatorname{diam}(\phi^n(V) \leq R$, then $V \subset \mathcal{F}_\phi$. Another criterion, due to Hsia [20] (see also [8, Theorem 3.7]) states that if $U \subset \mathbb{P}^1(\mathbb{C}_K)$ is a disk such that $\bigcup_{n\geq 0} \phi^n(U)$ omits at least two points of $\mathbb{P}^1(\mathbb{C}_K)$, then $U \subset \mathcal{F}_\phi$. Clearly Hsia's criterion also extends to arbitrary open sets V in place of U.

At last, we are prepared to define components of non-archimedean Fatou sets.

Definition 1.6. *Let $\phi \in \mathbb{C}_K(z)$ be a rational function with Fatou set $\mathcal{F}$, and let $x \in \mathcal{F}$.*

a. *The* analytic component *of $\mathcal{F}$ containing x is the union of all connected affinoids $W \subseteq \mathbb{P}^1(\mathbb{C}_K)$ such that $x \in W \subset \mathcal{F}$.*
b. *The* D-component *of $\mathcal{F}$ containing x is the union of all disks $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ such that $x \in U \subset \mathcal{F}$.*

c. *The* dynamical component *of $\mathcal{F}$ containing x is the union of all rational open connected affinoids $W \subseteq \mathbb{P}^1(\mathbb{C}_K)$ such that $x \in W$ and the set*

$$\mathbb{P}^1(\mathbb{C}_K) \setminus \left(\bigcup_{n \geq 0} \phi^n(W) \right)$$

is infinite.

d. *The* dynamical D-component *of $\mathcal{F}$ containing x is the union of all rational open disks $U \subseteq \mathbb{P}^1(\mathbb{C}_K)$ such that $x \in U$ and the set*

$$\mathbb{P}^1(\mathbb{C}_K) \setminus \left(\bigcup_{n \geq 0} \phi^n(U) \right)$$

is infinite.

Clearly all of these components are open sets. Because unions of overlapping connected affinoids or disks are again connected affionids or disks (or all of $\mathbb{P}^1(\mathbb{C}_K)$), the relation "$y$ is in the component of $\mathcal{F}$ containing x" is an equivalence relation between x and y, for each of the four types of components. Note that by Hsia's criterion, any dynamical component or dynamical D-component must in fact be contained in the Fatou set, so the terminology "component of $\mathcal{F}$" is not misleading. D-components must be either disks, all of $\mathbb{P}^1(\mathbb{C}_K)$, or all but one point of $\mathbb{P}^1(\mathbb{C}_K)$. Dynamical D-components must be either open disks, all of $\mathbb{P}^1(\mathbb{C}_K)$, or all but one point of $\mathbb{P}^1(\mathbb{C}_K)$. Analytic and dynamical components may be more complicated geometrically.

Analytic components and D-components were first defined in [2, 3]; dynamical components were defined by Rivera-Letelier in [27, 28], though he called them simply "components", and he used a different, but equivalent, definition. Dynamical D-components have not been mentioned in publications until now.

For any of the four types of components, if $x \in \mathcal{F}_\phi$ and if W is the component containing x, then $\phi(W)$ is contained in the component containing $\phi(x)$, by of the mapping properties discussed in Section 1.2. Thus, ϕ induces an action Φ_D on the set of D-components by

$$\text{for all D-components } U, \quad \Phi_D(U) = \text{the D-component containing } \phi(U).$$

Similarly, ϕ induces actions Φ_{an} on the set of analytic components, Φ_{dyn} on the set of dynamical components, and Φ_{dD} on the set of dynamical D-components.

For analytic and dynamical components, it can be shown [3, 27] that $\Phi_{an}(W) = \phi(W)$ and $\Phi_{dyn}(W) = \phi(W)$. For D-components and dynamical D-components, the corresponding equalities usually hold; but occasionally, the containment may be proper. Fortunately, by Lemma 1.5, for any given $\phi \in \mathbb{C}_K(z)$ of degree d, there are at most $d-1$ D-components U for which there exists a D-component V with $\Phi_D(V) = U$ but $\phi(V) \subsetneq U$. The analogous statement also holds for dynamical D-components.

Observe that for any of the four component types, if we fix $\phi \in \mathbb{C}_K(z)$ and let X be the set of components of $\mathcal{F}_\phi$ of that type, then the appropriate map Φ (Φ_{an}, Φ_D, etc.) maps X to X. Thus, we can discuss fixed components, wandering components, grand orbits of components, etc., as defined before.

2. Nontrivial reduction

As is well known, the natural projection $\mathcal{O}_{\mathbb{C}_K} \to \mathcal{O}_{\mathbb{C}_K}/\mathcal{M}_{\mathbb{C}_K} = \hat{k}$ induces a reduction map red : $\mathbb{P}^1(\mathbb{C}_K) \to \mathbb{P}^1(\hat{k})$. Given $\overline{a} \in \mathbb{P}^1(\hat{k})$, the associated *residue class*, which we shall denote $W_{\overline{a}} \subset \mathbb{P}^1(\mathbb{C}_K)$, is the preimage

$$W_{\overline{a}} = \mathrm{red}^{-1}(\overline{a}).$$

Any such class is either an open disk $W_{\overline{a}} = D_1(a)$ with $a \in \mathbb{C}_K$ and $|a| \leq 1$, or else it is the disk at infinity, $W_{\overline{\infty}} = \mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}_1(0)$.

Given a rational function $\phi \in \mathbb{C}_K(z)$ and a residue class $W_{\overline{a}}$, it will be useful to know whether or not $\phi(W_{\overline{a}})$ is again a residue class. To do so, we recall the following definition of Rivera-Letelier [28], which generalizes the notion of good reduction first stated by Morton and Silverman [25].

Definition 2.1. *Let $\phi \in \mathbb{C}_K(z)$ be a nonconstant rational function. Write ϕ as f/g, with $f, g \in \mathcal{O}_{\mathbb{C}_K}[z]$, such at least one coefficient of f or g has absolute value* 1*. Denote by $\overline{f}$ and $\overline{g}$ the reductions of f and g in $\hat{k}[z]$. Let $\overline{h} = \gcd(\overline{f}, \overline{g}) \in \hat{k}[z]$, let $\overline{f}_0 = \overline{f}/\overline{h}$, and let $\overline{g}_0 = \overline{g}/\overline{h}$. We say that ϕ has* nontrivial reduction *if $\overline{f}_0$ and $\overline{g}_0$ are not both constant. In that case, we define $\overline{\phi} = \overline{f}_0/\overline{g}_0 \in \hat{k}(z)$. If the degree of $\overline{\phi}$ equals that of ϕ, we say ϕ has* good reduction.

If ϕ and ψ have nontrivial reductions $\overline{\phi}$ and $\overline{\psi}$, then $\phi \circ \psi$ has nontrivial reduction $\overline{\phi} \circ \overline{\psi}$. Rivera-Letelier showed that the above definition of good reduction is equivalent to Morton and Silverman's original definition. His analysis is summarized in the following two lemmas. The proofs, stated for the field $\mathbb{C}_p$ but which apply to arbitrary $\mathbb{C}_K$, may be found in [27, Proposition 2.4].

Lemma 2.2. *Let $\phi \in \mathbb{C}_K(z)$ be a rational function. The following are equivalent.*

a. *ϕ has nontrivial reduction.*

b. *There are (not necessarily distinct) points $\overline{a}, \overline{b} \in \mathbb{P}^1(\hat{k})$ such that $\phi(W_{\overline{a}}) = W_{\overline{b}}$.*

Lemma 2.3. *Let $\phi \in \mathbb{C}_K(z)$ be a rational function of nontrivial reduction $\overline{\phi} \in \hat{k}(z)$. Then there is a finite set $T \subset \mathbb{P}^1(\hat{k})$ such that*

$$\phi(W_{\overline{a}}) = W_{\overline{\phi}(\overline{a})} \quad \textit{for all} \quad \overline{a} \in \mathbb{P}^1(\hat{k}) \setminus T,$$

and

$$\phi(W_{\overline{a}}) = \mathbb{P}^1(\mathbb{C}_K) \quad \textit{for all} \quad \overline{a} \in T.$$

Moreover, ϕ has good reduction if and only if $T = \emptyset$.

Given $\phi \in \mathbb{C}_K(z)$ of nontrivial reduction and the corresponding set T from Lemma 2.3, we will call classes $W_{\overline{a}}$ of elements $\overline{a} \in T$ the *bad classes*, and we will call the remaining classes the *good classes*. The bad classes are precisely those classes that contain *both* a zero and a pole of ϕ; that is, they are the classes $W_{\overline{a}}$ corresponding to linear factors $(z - \overline{a})$ of $\overline{h} = \gcd(\overline{f}, \overline{g})$ in Definition 2.1.

We remark that Rivera-Letelier considers the action of ϕ on a space he calls $\mathbb{H}$, which is essentially the Bruhat-Tits tree for $\mathrm{PGL}(2, \mathbb{C}_K)$. Roughly speaking, the "rational" points $\mathbb{H}^{\mathbb{Q}}$ of $\mathbb{H}$ are cosets of the subgroup $\mathrm{PGL}(2, \mathcal{O}_{\mathbb{C}_K})$ in the larger group $\mathrm{PGL}(2, \mathbb{C}_K)$ of linear fractional changes of coordinate on $\mathbb{P}^1(\mathbb{C}_K)$; he calls the identity coset the "canonical" point. The full set $\mathbb{H}$ is then a certain completion of $\mathbb{H}^{\mathbb{Q}}$; any rational function $\phi \in \mathbb{C}_K(z)$ acts on $\mathbb{H}$ in a natural way. (An element $g \in \mathrm{PGL}(2, \mathcal{O}_{\mathbb{C}_K})$ is a rational functions of degree one and of good reduction, so that a composition $\phi \circ g$ has nontrivial reduction if and only if ϕ does. The cosets of this subgroup are then the appropriate objects to study in order to understand the reduction of ϕ in different coordinates.) In Rivera-Letelier's lexicon, then, a map ϕ has nontrivial reduction if and only if it fixes the "canonical" point of $\mathbb{H}$.

The following technical result will be needed to prove Theorem 4.2. We provide a sketch of the proof, using methods similar to those used by Rivera-Letelier.

Lemma 2.4. *Let $\phi \in K(z)$ be a rational function of nontrivial reduction. Let $\overline{a} \in \mathbb{P}^1(\hat{k})$ be a point of ramification of $\overline{\phi}$ which is also fixed by $\overline{\phi}$. Let $0 < r < 1$, and let $a \in \mathbb{P}^1(\mathbb{C}_K)$ be a point in the residue class $W_{\overline{a}}$. If $\overline{a} \neq \infty$, let U be the annulus $D_1(a) \setminus \overline{D}_r(a)$; if $\overline{a} = \infty$, let U be the image of $D_1(1/a) \setminus \overline{D}_r(1/a)$ under the map $z \mapsto 1/z$. Then the set*

$$W_{\overline{a}} \setminus \left(\bigcup_{n \geq 0} \phi^n(U) \right)$$

contains at most one point.

Sketch of proof. After a $\mathrm{PGL}(2, \mathcal{O}_{\hat{K}})$-change of coordinates, we may assume that $a = 0$. If $\overline{0}$ is a good class, then the hypotheses imply that for $z \in D_1(0)$, $\phi(z)$ is given by a power series

$$\phi(z) = \sum_{i=0}^{\infty} c_i z^i$$

with all $|c_i| \leq 1$, with $|c_0|, |c_1| < 1$, and with $|c_m| = 1$ for some $m \geq 2$. (The conditions on c_1 and c_m come from the ramification hypothesis; they imply that the reduction $\overline{\phi}$ looks like $\overline{c}_m z^m + \overline{c}_{m+1} z^{m+1} + \ldots$.) Solving $\phi(z) = z$, it follows easily that $D_1(0)$ contains a fixed point b; without loss, $b = 0$, so that $c_0 = 0$. Then for any $0 < s < 1$, solving the power series equations $\phi(z) = x$ for $x \in D_1(0) \setminus \overline{D}_{s^m}(0)$ shows that

$$D_1(0) \setminus \overline{D}_{s^m}(0) \subseteq \phi\left(D_1(0) \setminus \overline{D}_s(0)\right).$$

Thus, for any nonzero $x \in D_1(0)$, the must be an integer $n \geq 0$ such that $x \in \phi^n(U)$. Hence, $D_1(0) \setminus \bigcup \phi^n(U) \subseteq \{0\}$. (In dynamical language, 0 is an attracting fixed point with basin containing $D_1(0)$.)

If $\overline{0}$ is a bad class, then $D_1(0)$ contains finitely many poles, so that for $z \in D_1(0)$, $\phi(z)$ may be written as

$$\phi(z) = \left(\sum_{i=0}^{\infty} c_i z^i \right) + \sum_{j=1}^{M} \frac{A_j}{(z - \alpha_j)^{e_j}},$$

with the same conditions as before on $\{c_i\}$, and with $|A_j|, |\alpha_j| < 1$. As before, we may change coordinates so that $c_0 = 0$, although this time, 0 itself may not be a fixed point. Let $R = \max\{|\alpha_j|\} < 1$. Then for any $s \in [R, 1)$, again we have

$$D_1(0) \setminus \overline{D}_{s^m}(0) \subseteq \phi\left(D_1(0) \setminus \overline{D}_s(0)\right).$$

Thus, $\phi^n(U)$ contains a pole for some $n \geq 0$; further computations show that $\phi^{n+1}(U) = \mathbb{P}^1(\mathbb{C}_K)$. □

3. Canonical heights

To prove our existence result (Theorem 4.2), we will need a few facts from the theory of diophantine height functions. We present the required statements without proof; instead, we refer the reader to [23, Chapters 2–4] for more details. The results of this section will be used only in the technical proof of Lemma 4.1. The statement of that Lemma, however, requires no knowledge of heights. The reader may therefore wish to skip ahead to the application of Lemma 4.1 in the proof of Theorem 4.2.

We consider a field k_0 which is either a finite extension of $\mathbb{Q}$ or else the field $L(T)$ of rational functions in one variable defined over an arbitrary field L. The standard *height function* h_0 on k_0 is the function $h_0 : k_0 \to \mathbb{R}_{\geq 0}$ given by

$$h_0\left(\frac{f}{g}\right) = \max\{\deg f, \deg g\} \quad \text{for } f, g \in L[T] \text{ relatively prime polynomials}$$

if $k_0 = L(T)$, or

$$h_0\left(\frac{m}{n}\right) = \max\{\log|m|, \log|n|\} \quad \text{for } m, n \in \mathbb{Z} \text{ relatively prime integers}$$

if $k_0 = \mathbb{Q}$. Intuitively speaking, h_0 measures the amount of ink required to write down a given field element in lowest terms. The following lemma states that h_0 can be extended to an algebraic closure of k_0, and that the resulting height function satisfies a useful mapping property.

Lemma 3.1. *Let k_0 be either the function field $L(T)$, for some field L, or the rational field $\mathbb{Q}$, and let $h_0 : k_0 \to \mathbb{R}_{\geq 0}$ be the height function defined above.*

Let k be a field isomorphic to a finite extension of k_0, and let $\hat{k}$ be an algebraic closure of k. Then there is a function $h : \mathbb{P}^1(\hat{k}) \to \mathbb{R}_{\geq 0}$, called the height function *for k, such that both of the following properties hold.*

a. *For all $x \in k_0$, $h(\iota(x)) = h_0(x)$, where ι is the inclusion $k_0 \hookrightarrow k \hookrightarrow \mathbb{P}^1(\hat{k})$.*
b. *For any rational function $\overline{\phi} \in k(z)$ of degree $d \geq 1$, there is a real constant $C = C_{\overline{\phi}} \geq 0$ such that*

$$\text{for all } x \in \mathbb{P}^1(\hat{k}), \quad \left|h\left(\overline{\phi}(x)\right) - dh(x)\right| \leq C. \tag{1}$$

For those readers familiar with the terminology, we mention that Lemma 3.1 can be proven by a lengthy but standard argument, using the fact that k_0 has a proper set of absolute values satisfying a product formula. However, there is no general guarantee that h is non-degenerate, unless $k_0 = \mathbb{Q}$ or $k_0 = L(T)$ with L finite. Fortunately, we will not need any non-degeneracy results in this paper.

The idea of the height function is that "most" elements of $\mathbb{P}^1(k)$ have large height. Thus, for any given $\overline{\phi}$, "most" points $x \in \mathbb{P}^1(k)$ satisfy $h(\overline{\phi}(x))/(dh(x)) \approx 1$. This approximate equality tends to be easy to work around, but it is a constant annoyance. Fortunately, Néron [26] introduced a related *canonical* height function with equality, rather than approximate equality, in its functional equation:

Lemma 3.2. *Let k be a field satisfying the hypotheses of Lemma 3.1, and let $\overline{\phi} \in k(z)$ be a rational function of degree $d \geq 2$. Then there is a function*

$$\hat{h} = \hat{h}_{\overline{\phi}} : \mathbb{P}^1(\hat{k}) \to \mathbb{R}_{\geq 0}$$

and a real constant $C' = C'_{\overline{\phi}} \geq 0$ such that for all $x \in \mathbb{P}^1(\hat{k})$,

$$\hat{h}(\overline{\phi}(x)) = d\hat{h}(x) \quad \text{and} \quad \left|\hat{h}(x) - h(x)\right| \leq C', \tag{2}$$

where h is the height function of Lemma 3.1.

Note that by (2), a preperiodic point x of $\overline{\phi}$ must have canonical height $\hat{h}_{\overline{\phi}}(x) = 0$. (If h were a non-degenerate height, then the converse would be true as well.) Tate [33] simplified Néron's construction, and Call and Silverman [12] extended the proof to our setting, as follows. The function $\hat{h}$ may be defined for $x \in \mathbb{P}^1(\hat{k})$ as

$$\hat{h}(x) = \lim_{n\to 0} d^{-n} h(\overline{\phi}^n(x)).$$

We leave to the reader the straightforward proof that the limit converges and that both conditions of the Lemma are satisfied.

Finally, the following lemma is not directly concerned with heights, but it applies to fields of the type we have been considering in this section. It can be proven using the fact that a field k of the type considered in Lemma 3.1 contains a Dedekind ring of integers $\mathcal{O}_k$ with infinitely many prime ideals.

Lemma 3.3. *Let k be a field satisfying the hypotheses of Lemma 3.1, let $c \in k^*$ such that $c^n \neq 1$ for all $n \geq 1$, and define $\overline{\phi}(z) = cz$. Then there exists an infinite sequence $\{x_i : i \in \mathbb{Z}\}$ of wandering points in $\mathbb{P}^1(k)$ such that for any distinct $i, j \in \mathbb{Z}$, x_i and x_j lie in different grand orbits of $\overline{\phi}$.*

4. Existence of wandering domains

Our strategy for constructing wandering domains of $\phi(z) \in K(z)$ begins with finding wandering *points* in $\mathbb{P}^1(\hat{k})$ of the reduction $\overline{\phi}(z) \in k(z)$. The following lemma shows that outside of trivial counterexamples, such points always exist. As mentioned in the previous section, the reader may wish to skip the proof of the Lemma to see its application in the proof of Theorem 4.2.

Lemma 4.1. *Let k be a field, and let $\overline{\phi}(z) \in k(z)$ be a nonconstant rational function. Suppose that for every $n \geq 1$, $\overline{\phi}^n$ is not the identity function. Then the following five statements are equivalent:*

a. *k is an algebraic extension of a finite field.*
b. *There are only finitely many wandering grand orbits of $\overline{\phi}$ intersecting $\mathbb{P}^1(k)$.*
c. *There are only finitely many wandering grand orbits of $\overline{\phi}$ in $\mathbb{P}^1(\hat{k})$.*
d. *There are no wandering grand orbits of $\overline{\phi}$ intersecting $\mathbb{P}^1(k)$. That is, all points in $\mathbb{P}^1(k)$ are preperiodic under $\overline{\phi}$.*
e. *There are no wandering grand orbits of $\overline{\phi}$ in $\mathbb{P}^1(\hat{k})$. That is, all points in $\mathbb{P}^1(\hat{k})$ are preperiodic under $\overline{\phi}$.*

Proof. (*i*). Clearly (e) implies both (d) and (c); similarly, (d) implies (b), and (c) implies (b).

To show (a) implies (e), suppose that k is an algebraic extension of a finite field. Then we may assume that $\hat{k} \cong \overline{\mathbb{F}}_p$, an algebraic closure of the field $\mathbb{F}_p$ of p elements, for some prime number p. Therefore $\overline{\phi}(z) \in \overline{\mathbb{F}}_p(z)$ is in fact defined over a finite field $\mathbb{F}_q$, where q is some power of p. Given $x \in \mathbb{P}^1(\hat{k})$, there is some $r \geq 1$ such that $x \in \mathbb{P}^1(\mathbb{F}_{q^r})$, which is a finite set and which is mapped into itself by $\overline{\phi}$. Thus, x must be preperiodic under $\overline{\phi}$, proving the implication.

The remaining (and substantive) part of the proof is to show that (b) implies (a). Suppose that k is not an algebraic extension of a finite field; we wish to show that $\mathbb{P}^1(k)$ intersects infinitely many wandering grand orbits of $\overline{\phi}$.

(*ii*). We will now reduce to the case that k is a finite extension either of $\mathbb{Q}$ or of the function field $L(T)$, for some field L.

Clearly, k is a field extension of L_0, where $L_0 = \mathbb{Q}$ if $\operatorname{char} k = 0$, or $L_0 = \mathbb{F}_p$ if $\operatorname{char} k = p > 0$. If k/L_0 is an algebraic extension, then by hypothesis, k must be an algebraic extension of $\mathbb{Q}$.

On the other hand, if k/L_0 is a transcendental extension, then there is a nonempty transcendence basis $B \subset k$ such that k is an algebraic extension of $L_0(B)$ (See, for example, [21, Theorem 8.35].) Pick $T \in B$, let $B' = B \setminus \{T\}$, and let $L = L_0(B')$, so that $L(T) \cong L_0(B)$, and T is transcendental over L. In that case, then, k is an algebraic extension of the function field $L(T)$.

We may now assume that k is a *finite* extension of either $\mathbb{Q}$ or $L(T)$. After all, the finitely many coefficients of $\overline{\phi}$ are each algebraic over $\mathbb{Q}$ or $L(T)$, so there is a single finite extension that contains all of them.

Write $k_0 = \mathbb{Q}$ or $k_0 = L(T)$ as appropriate, so that k is a finite extension of k_0. Let $d = \deg \overline{\phi}$. We consider two cases: that $d = 1$, or that $d \geq 2$.

(*iii*). If $d = 1$, then by a change of coordinates, we may assume that either $\overline{\phi}(z) = z+1$ or $\overline{\phi}(z) = cz$, for some $c \in k^*$. (If there are two distinct fixed points, move one to 0 and one to ∞, to get $\overline{\phi}(z) = cz$. If there is only one, move it to ∞ and then scale to get $\overline{\phi}(z) = z + 1$.) If $\overline{\phi}(z) = z + 1$ and $\operatorname{char} k = p > 0$, then $\phi^p(z) = z$, contradicting the hypotheses. If $\phi(z) = z + 1$ and $\operatorname{char} k = 0$, then $\mathbb{Q} \subset k$, so that there are clearly infinitely many wandering grand orbits; for example, there is one such orbit for each element of $\mathbb{Q} \cap [0, 1)$. On the other hand, if $\overline{\phi}(z) = cz$, then by hypothesis, $c^n \neq 1$ for all $n \geq 1$. Therefore, we have infinitely many wandering grand orbits by Lemma 3.3.

(*iv*). For the remainder of the proof, suppose $d \geq 2$. Define the canonical height function $\hat{h} = \hat{h}_{\overline{\phi}}$ as in Lemma 3.2, and let $C' = C'_{\overline{\phi}} \geq 0$ be the constant defined in that lemma. Let $M = 1 + 2C'$.

We claim that for any real number $r \geq 0$, there exists $x \in \mathbb{P}^1(k)$ satisfying

$$r < \hat{h}(x) \leq r + M.$$

To see this, note that by the definition of the original height function h in Lemma 3.1, there is an element $x \in k_0$ such that

$$r + C' < h(x) \leq r + C' + 1.$$

Because $|\hat{h}(x) - h(x)| \leq C'$, it follows that $\hat{h}(x) \in (r, r + M]$, proving the claim.

(*v*). Let $N \geq 1$ be any positive integer; we will show that $\overline{\phi}$ has at least N distinct wandering grand orbits which intersect $\mathbb{P}^1(k)$.

Let I be the real interval $I = (MN, 2MN]$. By (*iv*), there are at least N different points $x \in \mathbb{P}^1(k)$ such that $\hat{h}(x) \in I$. Recall that the preperiodic points all have canonical height zero; so if $\hat{h}(x) \in I$, then x must be wandering. Thus, it suffices to show that if $x, y \in \mathbb{P}^1(k)$ are two points with $\hat{h}(x), \hat{h}(y) \in I$ but $\hat{h}(x) \neq \hat{h}(y)$, then x and y must lie in different grand orbits.

Suppose not. Then there exist points $x, y \in \mathbb{P}^1(k)$ with $\hat{h}(x), \hat{h}(y) \in I$ but $\hat{h}(x) \neq \hat{h}(y)$, and integers $n \geq m \geq 0$ such that $\overline{\phi}^m(x) = \overline{\phi}^n(y)$. Thus, we have $d^m \hat{h}(x) = d^n \hat{h}(y)$, and therefore $\hat{h}(x) = d^{n-m}\hat{h}(y)$. Since $\hat{h}(x) \neq \hat{h}(y)$, we must have $m < n$. Hence,

$$2MN < 2\hat{h}(y) \leq d^{n-m}\hat{h}(y) = \hat{h}(x) \leq 2MN,$$

because $MN < \hat{h}(y), \hat{h}(x) \leq 2MN$. This contradiction completes the proof. □

We are now prepared to state and prove our existence theorem, which immediately implies part (a) of Theorem A.

Theorem 4.2. *Let K be a non-archimedean field with residue field k, where k is not an algebraic extension of a finite field. Let $\phi \in K(z)$ be a rational function of nontrivial reduction $\overline{\phi}$, and suppose that $\deg \overline{\phi} \geq 2$. Then there is an infinite set $\{\overline{b}_i : i \in \mathbb{Z}\} \subset \mathbb{P}^1(\hat{k})$ such that $\phi^n(W_{\overline{b}_i}) = W_{\overline{\phi}^n(\overline{b}_i)}$ for every $n \geq 0$ and $i \in \mathbb{Z}$, and such that all iterates $\overline{\phi}^n(\overline{b}_i)$ are distinct. Furthermore,*

- a. *Each $W_{\overline{b}_i}$ is a wandering dynamical component and a wandering dynamical D-component for ϕ, and each $W_{\overline{b}_i}$ lies in a different grand orbit of such components.*
- b. *If the Julia set $\mathcal{J}$ of ϕ intersects at least two different residue classes $W_{\overline{a}_1}, W_{\overline{a}_2}$, then each $W_{\overline{b}_i}$ is also a wandering D-component, and each $W_{\overline{b}_i}$ lies in a different grand orbit of such components.*
- c. *If $\mathcal{J}$ has nonempty intersection with infinitely many different residue classes, then each $W_{\overline{b}_i}$ is a wandering analytic component for ϕ, and each $W_{\overline{b}_i}$ lies in a different grand orbit of such components.*

Proof. (*i*). Let $\{\overline{c}_1, \ldots, \overline{c}_m\} \subset \mathbb{P}^1(\hat{k})$ represent the finitely many bad residue classes for $\overline{\phi}$. We claim that there is an infinite set $\{\overline{b}_i : i \in \mathbb{Z}\} \subset \mathbb{P}^1(\hat{k})$ such that no $\overline{b}_i$ is preperiodic under $\overline{\phi}$, such that $\overline{\phi}^n(\overline{b})$ avoids the $\overline{c}_i$'s, and such that for any distinct $i, j \in \mathbb{Z}$, the grand orbits of $\overline{b}_i$ and $\overline{b}_j$ under $\overline{\phi}$ are distinct.

To prove the claim, first note that by Lemma 4.1, there are points $\{\overline{b}_i' : i \in \mathbb{Z}\} \subset \mathbb{P}^1(\hat{k})$, each with infinite forward orbit under $\overline{\phi}$, and no two in the same grand orbit. For each $i \in \mathbb{Z}$, let N_i be the largest nonnegative integer n such that $\overline{\phi}^n(\overline{b}_i')$ equals some $\overline{c}_j$, or else $N_i = -1$ if no such n exists. Then we may choose $\overline{b}_i = \overline{\phi}^{N_i+1}(\overline{b}_i')$, which clearly satisfies the claim.

It follows immediately from Lemma 2.3 that for all $i \in \mathbb{Z}$ and all $n \geq 0$, $\phi^n(W_{\overline{b}_i}) = W_{\overline{\phi}^n(\overline{b}_i)}$. Thus, each $W_{\overline{b}_i}$ wanders and lies in the Fatou set of ϕ. Moreover, $W_{\overline{b}}$ is a rational open disk, and therefore it must be contained in a single component of the Fatou set, by any of the four definitions of components. Thus, it suffices only to show that each $W_{\overline{b}_i}$ is the full Fatou component, for each of the four types.

(*ii*). Fix $\overline{b} = \overline{b}_i$ for some $i \in \mathbb{Z}$. Let V_{dD} be the dynamical D-component containing $W_{\overline{b}}$, V_{dyn} the dynamical component, V_D the D-component, and V_{an} the analytic component.

If $V_{dyn} \supsetneq W_{\overline{b}}$, then V_{dyn} contains a connected open affinoid U such that $U \supsetneq W_{\overline{b}}$. Because it *properly* contains a residue class, U must contain all but finitely many residue classes. Define the finite (and possibly empty) sets

$$T_1 = \left\{ \overline{a} \in \mathbb{P}^1(\hat{k}) : W_{\overline{a}} \not\subset U \right\}$$

and

$$T_2 = \left\{ \overline{a} \in T_1 : \overline{\phi}^{-n}(\overline{a}) \subseteq T_1 \text{ for all } n \geq 0 \right\}.$$

That is, T_2 is the set of all points $\overline{a} \in \mathbb{P}^1(\hat{k})$ none of whose preimages $\overline{c}$ under any $\overline{\phi}^n$ have class $W_{\overline{c}}$ contained in U. Because T_2 is finite and $\overline{\phi}^{-1}(T_2) \subset T_2$, every element of T_2 must be periodic under $\overline{\phi}$. Thus, there is some integer $m \geq 1$ such that $\overline{\phi}^m$ fixes every element of T_2; it follows that for every $\overline{a} \in T_2$, $\overline{\phi}^{-m}(\overline{a}) = \{\overline{a}\}$. But $\overline{\phi}^m$ has degree larger than 1, and therefore every element of T_2 is a fixed ramification point of $\overline{\phi}^m$.

For any $\overline{a} \notin T_2$, there is some $\ell \geq 0$ such that $\phi^\ell(U) \supset W_{\overline{a}}$; hence $V_{dyn} \supset W_{\overline{a}}$. On the other hand, for $a \in T_2$, the intersection $U \cap W_{\overline{a}}$ contains an annulus of the sort described in Lemma 2.4. By that lemma, then, V_{dyn} must contain all but at most one point of $W_{\overline{a}}$, because V_{dyn} contains $\phi^n(U)$ for all $n \geq 0$. Thus, V_{dyn} contains all but finitely many points of $\mathbb{P}^1(\mathbb{C}_K)$, which contradicts the definition of a dynamical component. Therefore $V_{dyn} = W_{\overline{b}}$.

From the definitions, we have $W_{\overline{b}} \subseteq V_{dD} \subseteq V_{dyn}$. It follows that $V_{dD} = W_{\overline{b}}$ also.

(iii). Next, under the assumption that $\mathcal{J}$ intersects at least two different residue classes, suppose that $V_D \supsetneq W_{\overline{b}}$. Then V_D contains a disk $U \supsetneq W_{\overline{b}}$. Such a disk must contain all but one residue class, and therefore U must intersect the Julia set, which is impossible. Therefore $V_D = W_{\overline{b}}$.

Similarly, if $\mathcal{J}$ intersects infinitely many residue classes $W_{\overline{a}}$, and if $V_{an} \supsetneq W_{\overline{b}}$, then V_{an} contains a connected affinoid $U \supsetneq W_{\overline{b}}$. U must contain all but finitely many residue classes, which is impossible because then U would intersect $\mathcal{J}$. Hence $V_{an} = W_{\overline{b}}$. □

The following theorem shows that the condition that the Julia set intersects infinitely many different residue classes holds frequently.

Theorem 4.3. *Let K be a non-archimedean field with residue field k, let $p = \operatorname{char} k \geq 0$, let $\phi \in K(z)$ be a rational function of nontrivial reduction $\overline{\phi}$, and let $\mathcal{J} \subset \mathbb{P}^1(\mathbb{C}_K)$ be the Julia set of ϕ. Suppose either that $\overline{\phi}$ is separable and of degree at least two, or that there is a separable map $\overline{\psi} \in k(z)$ of degree at least two and an integer $r \geq 1$ such that $\overline{\phi}(z) = \overline{\psi}(z^{p^r})$. If $\mathcal{J}$ intersects at least three different residue classes of $\mathbb{P}^1(\mathbb{C}_K)$, then $\mathcal{J}$ intersects infinitely many different residue classes in $\mathbb{P}^1(\mathbb{C}_K)$.*

Proof. Because $\overline{\psi}$ is separable and of degree at least two, then by the Riemann-Hurwitz formula (see [18, Corollary 2.4]), for example) at most two points of $\mathbb{P}^1(\hat{k})$ have only one preimage each under $\overline{\psi}$. Given any $N \geq 3$, then, and any set $S_N \subset \mathbb{P}^1(\hat{k})$ of N distinct points, the number of points in $\overline{\psi}^{-1}(S_N)$ must be strictly greater than N.

Applying this fact inductively to $\overline{\phi}(z) = \overline{\psi}(z^{p^r})$, we see that, given any three distinct points $\overline{c}_1, \overline{c}_2, \overline{c}_3 \in \mathbb{P}^1(\hat{k})$, there are infinitely many points $\overline{a} \in \mathbb{P}^1(\hat{k})$ which eventually map to some $\overline{c}_i$ under some $\overline{\phi}^n$.

Meanwhile, by Lemma 2.3, for any residue class $W_{\overline{a}}$, we have $\phi(W_{\overline{a}}) \supseteq W_{\overline{\phi}(\overline{a})}$. Thus, if $\mathcal{J}$ intersects at least three residue classes, it must intersect all preimages of those three classes, and hence $\mathcal{J}$ intersects infinitely many residue classes. □

To show that wandering domains coming from nontrivial reduction actually exist, we present the following example, which is just a generalization of [6, Example 2]. Our example proves part (b) of Theorem A.

Example 1. Let K be a non-archimedean field with residue field k such that k is not an algebraic extension of a finite field. Then there is an element $b \in K$ that reduces to $\overline{b} \in k$, with the property that $\{\overline{b}^n\}_{n\in\mathbb{Z}}$ is an infinite subset of k. If $m \geq 2$ is an integer not divisible by $\operatorname{char} k$, let $\Psi_m(z)$ denote the m-th cyclotomic polynomial. For example, if $\operatorname{char} k \neq 2$, we may choose $m = 2$ and hence $\Psi_m(z) = z + 1$; if $\operatorname{char} k = 2$, we may choose $m = 3$ and hence $\Psi_m(z) = z^2 + z + 1$. In either case, $\overline{\Psi}_m$ has distinct roots in $\hat{k}$, and if $\zeta \in \mathbb{C}_K$ is any root, then $\overline{\zeta} \neq 1$ but $\zeta^m = 1$.

If $T \in K$ is any element satisfying $0 < |T| < 1$, define the rational function

$$\phi(z) = z^m + \frac{T}{\Psi_m(z)} = \frac{z^m \Psi_m(z) + T}{\Psi_m(z)}.$$

Then ϕ has nontrivial reduction $\overline{\phi}(z) = z^m \in \hat{k}[z]$, which is separable and of degree $m \geq 2$. The only bad residue classes are the roots of $\overline{\Psi}_m$ in $\hat{k}$. Hence, given $\overline{a} \in \mathbb{P}^1(\hat{k})$ which is not a root of $\overline{\Psi}_m$, we have $\phi(W_{\overline{a}}) = W_{\overline{a}^m}$.

Moreover, we claim that the Julia set of ϕ intersects infinitely many distinct residue classes. To show this, let $\zeta \in \mathbb{C}_K$ be a root of Ψ_m. First, we can easily check that ϕ has a fixed point $\alpha \in \mathbb{C}_K$ with $|\alpha - \zeta| = |T|$. Indeed, after substituting $w = z - \zeta$ in the equation $\phi(z) = z$ and using the fact that $|\zeta - 1| = |\Psi'_m(\zeta)| = 1$, the existence of such a fixed point α is clear from the Newton polygon of the resulting polynomial. Second, we compute $|\phi'(\alpha)| = |T|^{-1} > 1$, so that α is a repelling fixed point and hence lies in the Julia set. Furthermore, because $\overline{\phi}(z) = z^m$ is separable, with no ramification points in $\mathbb{P}^1(\hat{k})$ besides 0 and ∞, the set $\{\overline{\zeta}\} \cup \overline{\phi}^{-1}(\overline{\zeta})$ consists of at least three points. Finally, the corresponding residue classes each contain preimages of α, and hence they intersect the Julia set. By Theorem 4.3, our claim is valid.

It follows by Theorem 4.2 that ϕ has infinitely many grand orbits of wandering components (dynamical, analytic, D-components, and dynamical D-components). More precisely, for any $b \in K$ such that $\{\overline{b}^n\}_{n\in\mathbb{Z}}$ is an infinite subset of k, the class $W_{\overline{b}}$ is a wandering domain. After all, no iterate $\overline{\phi}(\overline{b})$ is ever one of the bad classes $\overline{\zeta}$, (otherwise, all future iterates of $\overline{b}$ would be $\overline{1}$), and those iterates are all distinct.

If one is only interested in dynamical components or dynamical D-components, as opposed to analytic or D-components, then it is even easier to produce examples of wandering domains by Theorem 4.2, provided one is working over a field satisfying that theorem's hypotheses. Indeed, any rational function $\phi(z)$ of degree at least two and of good reduction satisfies the hypotheses for statement (a.) of the theorem. Thus, for any $n \geq 2$, $\phi(z) = z^n$ will have wandering domains over such a field K.

As mentioned in the introduction, sufficient conditions for residue classes of $\mathbb{P}^1(\mathbb{C}_K)$ to be wandering domains are more complicated if the map has a nontrivial reduction of degree one. The remaining examples of this section are of functions of reduction degree one, all defined over the field $K = \mathbb{Q}((T))$, whose residue field $\mathbb{Q}$ is not an algebraic extension of a finite field.

Example 2. Let $\phi(z) = z + 1 + T/z \in \mathbb{Q}((T))$. Then ϕ has nontrivial reduction $\overline{\phi}(z) = z + 1$ of degree one. The disk $D_1(0)$ contains the repelling fixed point $-T$; it follows that $D_1(-m)$ intersects the Julia set for every integer $m \geq 0$. On the other hand, the disk $U = D_1(1)$ satisfies $\phi^n(U) = D_1(n+1)$ for every integer $n \geq 0$, so that U lies in the Fatou set. Moreover, any strictly larger affinoid containing U must contain one of the disks $D_1(-m)$ and hence must intersect the Julia set. So U is a wandering analytic component, wandering D-component, wandering dynamical component, and wandering dynamical D-component.

Example 3. Let $\phi(z) = Tz^2 + z + 1 \in \mathbb{Q}((T))$. Then ϕ has nontrivial reduction $\overline{\phi}(z) = z + 1$ of degree one, and as in the previous example, the disk $U = D_1(1)$ lies in the Fatou set and is wandering; in fact, the same is true of every disk $D_1(b)$ for $|b| \leq 1$. However, all these disks are contained in the single disk $D_{1/|T|}(0)$, which is fixed. Thus, although the smaller disks are wandering, none of them is large enough to be a full component of the Fatou set.

In fact, $\phi = h \circ \psi \circ h^{-1}$, where $h(z) = Tz$ and $\psi(z) = z^2 + z + T$, which is a map of good reduction, having reduction $\overline{\psi}(z) = z^2 + z$ of degree two. By Theorem 4.2, ψ does have wandering dynamical components and wandering dynamical D-components, but they are *not* the residue classes $D_1(b)$. Moreover, the whole of $\mathbb{P}^1(\mathbb{C}_K)$ forms a single D-component and a single analytic component; hence, there are no wandering analytic or D-components.

Example 4. Let $\phi(z) = Tz^2 + 2z + T/z \in \mathbb{Q}((T))$. Then ϕ has nontrivial reduction $\overline{\phi}(z) = 2z$ of degree one. The Julia set intersects both W_0 and W_∞, while all of the other residue classes $D_1(b)$ (for $|b| = 1$) are contained in the Fatou set. In fact, every such residue class $D_1(b)$ is a wandering D-component and a wandering dynamical D-component. On the other hand, the open affinoid $V = D_{|T|^{-1}}(0) \setminus \overline{D}_{|T|^{1/2}}(0)$ satisfies $\phi(V) = V$; therefore, the corresponding analytic component and dynamical component are fixed, and they strictly contain all the disks $D_1(b)$.

Example 5. Let $\phi(z) = Tz^2 + z + T/z \in \mathbb{Q}((T))$. Then ϕ has nontrivial reduction $\overline{\phi}(z) = z$ of degree one. As in the previous example, the Julia set intersects both W_0 and W_∞, while all of the other residue classes $D_1(b)$ (for $|b| = 1$) are contained in the Fatou set and are both D-components and dynamical D-components. This time, however, all those disks are fixed by $\overline{\phi}$, so none of them is wandering. As before, the open affinoid $V = D_{|T|^{-1}}(0) \setminus \overline{D}_{|T|^{1/2}}(0)$ satisfies $\phi(V) = V$; therefore, the corresponding analytic component and dynamical component are also fixed, and they strictly contain all the disks $D_1(b)$.

5. Residue characteristic zero

We now prove Theorem B; the following theorem is a slightly stronger result, showing that the desired conjugacy is defined over a certain finite extension of K.

Theorem 5.1. *Let K be a discretely valued non-archimedean field with residue field k and residue characteristic $\operatorname{char} k = 0$. Let $\phi \in K(z)$ be a rational function, and suppose that U is a wandering analytic component, wandering D-component, wandering dynamical D-component, or wandering dynamical component of ϕ. Let $L \subset \mathbb{C}_K$ be any* *finite extension of K such that U contains a point of $\mathbb{P}^1(L)$. Then there is a change of coordinates $g \in \mathrm{PGL}(2, L)$ and there are integers $M \geq 0$ and $N \geq 1$ such that $\psi(z) = g \circ \phi^N \circ g^{-1}(z)$ has nontrivial reduction, $D_1(0)$ is a wandering component (of the same type as U) of ψ, and $g(\phi^M(U)) \subset D_1(0)$.*

Note that a field L satisfying the required properties always exists. Indeed, the algebraic closure $\hat{K}$ of K is dense in $\mathbb{P}^1(\mathbb{C}_K)$, so that the open set U must contain some $a \in \hat{K}$. Then $L = K(a)$ is a finite extension of K.

Proof. We devote the bulk of the proof to the case that U is a wandering dynamical D-component.

(*i*). Let $a \in U \cap \mathbb{P}^1(L)$. Write $U_n = \Phi^n_{dD}(U)$ to simplify notation; recall that $\Phi^n_{dD}(U)$ is the dynamical D-component containing $\phi^n(U)$.

We may assume without loss that $U_n \subset \overline{D}_1(0)$ for every $n \geq 0$. To do so, make a $\mathrm{PGL}(2, L)$-change of coordinates to move a to ∞ and U to a set containing $\mathbb{P}^1 \setminus \overline{D}_1(0)$. Then $U_n \subset \overline{D}_1(0)$ for every $n \geq 1$. Finally, replace U by U_1, and we have the desired scenario.

Let L' be a finite extension of L such that $\mathbb{P}^1(L')$ contains all critical points and all poles of ϕ in $\mathbb{P}^1(\mathbb{C}_K)$. L' is discretely valued, because it is only a finite extension of the original field K. Thus, there is a real number $0 < \varepsilon < 1$ such that $|(L')^*| = \{\varepsilon^m : m \in \mathbb{Z}\}$.

Furthermore, there are only finitely many $n \geq 0$ such that U_n contains a critical point, and by Lemma 1.5, only finitely many $n \geq 0$ such that $\Phi_{dD}(U_n) \neq \phi(U_n)$. Thus, by replacing U by $U_{M'}$ for some $M' \geq 0$, we may assume for all $n \geq 0$ that $U_n = \phi^n(U)$, and that U_n contains no critical points and no poles.

Write $r_n = \mathrm{rad}(U_n) > 0$, so that $U_n = D_{r_n}(\phi^n(a))$, for each $n \geq 0$. Because each U_n contains no critical points or poles, it follows from Lemmas 1.3 and 1.4 that there are integers $\ell_n \in \mathbb{Z}$ such that $r_n = \varepsilon^{\ell_n} r_0$.

(*ii*). We now claim that $r_0 \in |(L')^*|$. To prove the claim, suppose $r_0 \notin |(L')^*|$. Because L' is discretely valued, there exists a real number $s_0 > r_0$ such that no $x \in L'$ satisfies $r_0 \leq |x| < s_0$. For every $n \geq 0$, let $s_n = r_n \cdot s_0/r_0$. By the fact that $r_n = \varepsilon^{\ell_n} r_0$ and $|(L')^*| = \{\varepsilon^m\}$, it follows that no $x \in L'$ satisfies $r_n \leq |x| < s_n$.

Let $V_n = \phi^n(D_{s_0}(a))$, for all $n \geq 0$. We will now show, by induction on n, that V_n is an open disk of radius s_n which contains no critical points or poles. For $n = 0$, V_0 is an open disk of radius s_0 by definition, and it contains no critical points or poles because $V_0 \cap L' = U_0 \cap L'$ by our choice of s_0. Assuming the claim is true for $n \geq 0$, then by Lemmas 1.3 and 1.4, $\mathrm{diam}(V_{n+1})/\mathrm{rad}(V_n) = \mathrm{rad}(U_{n+1})/\mathrm{rad}(U_n)$, since V_n contains no critical points or poles. Thus, V_{n+1} is a set of diameter s_{n+1}, and by Lemma 1.1, it is an open disk. Because no $x \in L'$ satisfies $r_{n+1} \leq |x| < s_{n+1}$, we have $V_{n+1} \cap L' = U_{n+1} \cap L'$, and therefore V_{n+1} contains no critical points or poles, completing the induction.

Recall that each U_n is contained in $\overline{D}_1(0)$; for every $n \geq 0$, then, $s_n \leq s_0/r_0$, and therefore,

$$\phi^n(V_0) = V_n \subseteq \overline{D}_{s_0/r_0}(0).$$

Because $U = U_0 \subsetneq V_0$, we have contradicted the assumption that U is a dynamical D-component.

Thus, $r_0 \in |(L')^*|$, as claimed. It follows that $r_n \in |(L')^*|$ for all $n \geq 0$.

(*iii*). For all $n \geq 0$, let $\overline{U}_n = \overline{D}_{r_n}(\phi^n(a))$, so that $U_n \subsetneq \overline{U}_n \subset \overline{D}_1(0)$.

We claim that for infinitely many $n \geq 0$, $\overline{U}_n$ contains a pole or a critical point of ϕ. To prove the claim, suppose only finitely many of the $\overline{U}_n$ contained poles or critical points, and replace U by $U_{M'}$ (for some appropriate $M' \geq 0$) so that no $\overline{U}_n$ contains a pole or critical point. Because $|(L')^*| = \{\varepsilon^m\}$ is discrete and $r_n \in |(L')^*|$, the larger disk $D_{r_n/\varepsilon}(\phi^n(a))$ also contains no poles or critical points for any $n \geq 0$.

For all $n \geq 0$, define $V'_n = \phi^n(D_{r_0/\varepsilon}(a))$. By an induction argument similar to that in part (*ii*) above, $V'_n = D_{r_n/\varepsilon}(\phi^n(a))$. Since V'_0 is an open disk that properly contains U, we have contradicted the assumption that U is a dynamical D-component, thus proving the claim.

(*iv*). Next, we claim that either $\overline{U}_n$ contains a pole for infinitely many $n \geq 0$, or else there exist $M \geq 0$ and $N \geq 1$ such that $\overline{U}_M = \overline{U}_{M+N}$.

If only finitely many of the $\overline{U}_n$ contain a poles, then there are integers $M \geq 0$ and $N \geq 1$ such that $\overline{U}_M \cap \overline{U}_{M+N}$ is nonempty, and such that for all $n \geq M$, $\overline{U}_n$ contains no poles. By replacing U by U_M and ϕ by ϕ^N, we may assume that $M = 0$ and $N = 1$. Because $\overline{U}_0$ and $\overline{U}_1$ are disks in $\mathbb{C}_K$, either $\overline{U}_0 \supsetneq \overline{U}_1$ or $\overline{U}_0 \subseteq \overline{U}_1$.

If $\overline{U}_0 \supsetneq \overline{U}_1$, then because $|(L')^*| = \{\varepsilon^m\}$, we must have $r_1 \leq \varepsilon r_0 < r_0$. Then $V'' = D_{r_0}(\phi(a))$ is an open disk that properly contains U_1, and $\phi^n(V'') \subset \overline{U}_0$ for all $n \geq 0$, contradicting the supposition that $U_1 = \Phi_{dD}(U)$ is a dynamical D-component.

If $\overline{U}_0 \subseteq \overline{U}_1$, note that by Lemma 1.2, $\phi(\overline{U}_n) = \overline{U}_{n+1}$ for every $n \geq 0$, because ϕ has no poles in any $\overline{U}_n$. Therefore,

$$\overline{U}_0 \subseteq \overline{U}_1 \subseteq \overline{U}_2 \subseteq \cdots .$$

If all the inclusions are proper, then

$$r_0 < r_1 < r_2 < \cdots .$$

Because $|(L')^*| = \{\varepsilon^m\}$, we must have $r_n > 1$ for some $n \geq 0$, contradicting the assumption that every U_n is contained in $\overline{D}_1(0)$. Thus, for some $n \geq 0$, we have $\overline{U}_n = \overline{U}_{n+1}$, and the claim is proven. (In fact, we would have $\overline{U}_0 = \overline{U}_1$, but we do not need that result here.)

(*v*). Consider the case that $\overline{U}_n$ contains a pole for infinitely many $n \geq 0$. Since there are only finitely many poles, there must be some pole $y \in \mathbb{P}^1(L')$ and an infinite set I of nonnegative integers such that $y \in \overline{U}_n = \overline{D}_{r_n}(\phi^n(a))$ for all $n \in I$. Pick $s > 0$ so that $\phi(D_s(y)) \subset \mathbb{P}^1(\mathbb{C}_K) \setminus \overline{D}_1(0)$. By our initial assumptions, no U_n can intersect $D_s(y)$, or else U_{n+1} would not be contained in $\overline{D}_1(0)$. Thus, $s < r_n \leq 1$ for all $n \in I$.

However, we also know that $r_n = \varepsilon^{\ell_n} r_0 \in |(L')^*|$ for all $n \geq 0$. As n ranges over I, then, there are only finitely many possible values that r_n can attain. At least one must be attained infinitely often. In particular, there are integers $M \geq 0$ and $N \geq 1$ such that $M, M+N \in I$ and $r_M = r_{M+N}$. Since y lies in both U_M and U_{M+N}, we have $\overline{U}_M = \overline{U}_{M+N}$.

(*vi*). By (*iv*) and (*v*), then, there exist integers $M \geq 0$ and $N \geq 1$ such that $\overline{U}_M = \overline{U}_{M+N}$. Thus, $r_M = r_{M+N}$, and $|\phi^M(a) - \phi^{M+N}(a)| \leq r_M$. Because $U_M \cap U_{M+N} = \emptyset$, we must in fact have $|\phi^M(a) - \phi^{M+N}(a)| = r_M$. Therefore $r_M \in |L^*|$, since $a \in \mathbb{P}^1(L)$ and $\phi \in K(z) \subseteq L(z)$.

Let $g \in \mathrm{PGL}(2, L)$ be the unique linear fractional transformation satisfying $g(\infty) = \infty$, $g(\phi^M(a)) = 0$, and $g(\phi^{M+N}(a)) = 1$. Thus, $g(U_M) = D_1(0)$ and $g(U_{M+N}) =$

$D_1(1)$. Let $\psi = g \circ \phi^N \circ g^{-1}$. By Lemma 2.2, ψ has nontrivial reduction, and the remaining conclusions of the theorem follow as well, at least for the case of dynamical D-components.

(*vii*). Finally, suppose that U is a wandering D-component, wandering analytic component, or wandering dynamical component containing a point $a \in \mathbb{P}^1(L)$. Let U' be the dynamical D-component containing a Then $U' \subseteq U$, and therefore U' is wandering.

For any integer $n \geq 0$, define $U_n = \Phi^n(U)$ (where Φ is Φ_D, Φ_{an}, or Φ_{dyn}, as appropriate), and define $U'_n = \Phi_{dD}(U')$. Choose g, M, and N for ϕ as in the theorem applied to U'. It suffices to show that $U_M = U'_M$.

Suppose not; then $U'_M \subsetneq U_M$, and therefore $g(U'_M) \subsetneq g(U_M)$. Thus, $g(U_M)$ contains an affinoid strictly containing the residue class $g(U'_M)$; hence, $g(U_M)$ contains all but finitely many of the residue classes $D_1(b)$. However, for every $n \geq 0$, $g(U'_{M+nN})$ is a residue class. In particular, $g(U_M)$ contains $g(U'_{M+nN})$ for some $n \geq 1$. Thus, $U_M \cap U_{M+nN}$ is nonempty, contradicting the wandering assumption and proving the theorem. □

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