WANDERING DOMAINS IN NON-ARCHIMEDEAN POLYNOMIAL DYNAMICS

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ABSTRACT. We extend a recent result on the existence of wandering domains of polynomial functions defined over the p-adic field $\mathbb{C}_p$ to any algebraically closed complete non-archimedean field $\mathbb{C}_K$ with residue characteristic $p > 0$. In fact, we prove polynomials with wandering domains form a dense subset of a certain one-dimensional family of degree $p+1$ polynomials in $\mathbb{C}_K[z]$.

Given a rational function $\phi \in K(z)$ with coefficients in a field K, one may consider the dynamical system given by the action of the iterates ϕ^n on $\mathbb{P}^1(K) = K \cup \{\infty\}$, for $n \geq 0$. Here, ϕ^n denotes the n-fold composition $\phi \circ \cdots \circ \phi$, so that ϕ^0 is the identity function, $\phi^1 = \phi$, $\phi^2 = \phi \circ \phi$, and so on. The case of complex dynamics, when $K = \mathbb{C}$, has been studied intensively for several decades; see [1, 10, 17] for expositions. In particular, one may study the action of $\{\phi^n\}$ on the Fatou set $\mathcal{F} = \mathcal{F}_\phi$, to be defined below. It is well known that the connected components of $\mathcal{F}$ are mapped onto one another by ϕ and that, according to Sullivan's deep No Wandering Domains Theorem [26], every complex Fatou component is preperiodic under application of ϕ.

It is also possible to define Fatou sets for other metric fields. The study of the resulting dynamics has seen growing interest in the past decade or two; see, for example, [2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 15, 16, 19, 20, 21, 25]. In this paper we will study wandering domains over certain non-archimedean fields, and we fix the following notation.

$\mathbb{C}_K$	a complete and algebraically closed non-archimedean field
$\lvert \cdot \rvert$	the absolute value on $\mathbb{C}_K$
$\hat{k}$	the residue field of $\mathbb{C}_K$
p	the residue characteristic char $\hat{k}$
$\mathbb{P}^1(\mathbb{C}_K)$	the projective line $\mathbb{C}_K \cup \{\infty\}$

We will assume throughout that $p > 0$; that is, that $\mathbb{C}_K$ has *positive residue characteristic*. Equivalently, $0 \leq |p| < 1$, when p is viewed as an element of $\mathbb{C}_K$ under the unique nontrivial homomorphism of $\mathbb{Z}$ into $\mathbb{C}_K$. Note that it is possible that char $\mathbb{C}_K = 0$ with char $\hat{k} = p > 0$; but if char $\mathbb{C}_K > 0$, then char $\hat{k} = \mathrm{char}\, \mathbb{C}_K$.

Recall that "non-archimedean" means the ultrametric triangle inequality

$$|x + y| \leq \max\{|x|, |y|\}$$

holds for all $x, y \in \mathbb{C}_K$. If $|x| \neq |y|$, it is immediate that $|x + y| = \max\{|x|, |y|\}$. Note that $|n| \leq 1$ for all $n \in \mathbb{Z}$. Recall also that the residue field $\hat{k}$ is defined to be $\mathcal{O}_K / \mathcal{M}_K$, where $\mathcal{O}_K$ is the ring $\{x \in \mathbb{C}_K : |x| \leq 1\}$ of integers in $\mathbb{C}_K$, and $\mathcal{M}_K$ is the maximal ideal

of $\{x \in K : |x| < 1\}$ of $\mathcal{O}_K$. It is easy to check that $\hat{k}$ is algebraically closed because $\mathbb{C}_K$ is.

The best known example of such field is $\mathbb{C}_p$, constructed as follows. Given a prime integer $p \geq 2$, the (complete non-archimedean) field $\mathbb{Q}_p$ of p-adic rational numbers is formed by completing the usual rational numbers $\mathbb{Q}$ with respect to the p-adic absolute value given by $|p^e r| = p^{-e}$, where r is a rational number with numerator and denominator both prime to p. Its algebraic closure is $\overline{\mathbb{Q}}_p$, and the completion of $\overline{\mathbb{Q}}_p$ is $\mathbb{C}_p$; the absolute value extends uniquely to $\overline{\mathbb{Q}}_p$ and hence to $\mathbb{C}_p$. The residue field is $\hat{k} = \overline{\mathbb{F}}_p$, the algebraic closure of the field $\mathbb{F}_p$ of p elements. Note that $\operatorname{char} \mathbb{C}_p = \operatorname{char} \mathbb{Q}_p = 0$, while $\operatorname{char} \hat{k} = p > 0$.

As another example, if L is any abstract field, then $K = L((T))$, the field of formal Laurent series with coefficients in L, is a complete non-archimedean field. (We may define an absolute value $| \cdot |$ on K by $|f| = 2^{-n}$, where $n \in \mathbb{Z}$ is the smallest integer for which the T^n term of the formal Laurent series f has a nonzero coefficient.) Once again, the absolute value extends uniquely to an algebraic closure of K. If we denote by $\mathbb{C}_K$ the completion of an algebraic closure of K, then $\mathbb{C}_K$ is algebraically closed and complete; its residue field $\hat{k}$ is an algebraic closure of L. In this case, $\operatorname{char} \mathbb{C}_K = \operatorname{char} \hat{k} = \operatorname{char} K = \operatorname{char} L$. We refer the reader to [14, 23, 24] for surveys of non-archimedean fields.

In complex dynamics, given a rational function $\phi \in \mathbb{C}(z)$ with complex coefficients, one considers the action of the iterates $\{\phi^n\}$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. The *Fatou set* $\mathcal{F}$ of ϕ is defined to be the the set of all points $x \in \mathbb{P}^1(\mathbb{C})$ such that the family $\{\phi^n : n \geq 0\}$ is equicontinuous at x, with respect to the spherical metric on $\mathbb{P}^1(\mathbb{C})$; the *Julia set* $\mathcal{J}$ is the complement $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{F}$. (Recall that a family F of functions from a metric space X to a metric space Y is called *equicontinuous* at $x_0 \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ for all $f \in F$ and for all $x \in X$ satisfying $d_X(x, x_0) < \delta$. The key point is that ε is chosen independent of f.) Intuitively, the Fatou set is the region of order, where the iterates $\{\phi^n\}$ are well behaved; the Julia set is the region of chaos, where a small error becomes huge after many iterations of ϕ.

Alternately, by the Arzelà-Ascoli Theorem, one may define the Fatou set of a complex function to be the set of points where $\{\phi^n\}$ forms a normal family. However, because $\mathbb{C}_K$ is not locally compact, and therefore the Arzelà-Ascoli Theorem fails, non-archimedean Fatou sets are usually defined in terms of equicontinuity.

It is easy to verify that the Fatou set is open and that $\phi(\mathcal{F}) = \mathcal{F}$; similarly, the Julia set is closed, and $\phi(\mathcal{J}) = \mathcal{J}$. One may partition the (complex) Fatou set of $\phi \in \mathbb{C}(z)$ into connected components; the function ϕ then maps each component into (and in fact onto) a component. Thus, we may speak of fixed, periodic, and preperiodic components. A component which is not preperiodic (that is, a component with infinite forward orbit) is called a *wandering domain* of ϕ. In 1985, Sullivan [26] prove, using quasiconformal conjugations, that a complex rational function $\phi \in \mathbb{C}(z)$ cannot have wandering domains.

A similar theory of components of the Fatou set exists for non-archimedean rational functions $\phi \in \mathbb{C}_K(z)$; see Section 2 below. However, in spite of the existence of certain non-archimedean versions of Teichmüller spaces [18], there is no apparent analogue of actual quasiconformal maps suitable for use as conjugating functions in $\mathbb{C}_K(z)$. In particular, the author's constructions of wandering domains over any algebraically closed complete non-archimedean field (see [8] for residue characteristic zero, [6] for $\mathbb{C}_p$, and

this paper for general fields of positive residue characteristic) suggest that Sullivan's proof has no analogue for $\mathbb{C}_K$, and that no appropriate versions of quasiconformal maps exist in that context.

Nevertheless, the author showed in [2, 4] that given some weak hypotheses on $\phi \in \mathbb{C}_p(z)$ (including the assumption that the coefficients of ϕ are algebraic over $\mathbb{Q}_p$), the Fatou set has no wandering domains. For more general fields, the arguments in [3] imply that if $\phi \in \mathbb{C}_K(z)$ is defined over a locally compact subfield of $\mathbb{C}_K$, and if the Julia set of ϕ contains no critical points, then ϕ has no wandering domains.

We suspect that the condition concerning critical points is unnecessary. However, the condition that ϕ be defined over a locally compact field is crucial. Indeed, the main result of this paper is the following theorem. It shows that if the residue field $\hat{k}$ has positive characteristic, then there are many polynomials defined over the (not locally compact) field $\mathbb{C}_K$ which have wandering domains in $\mathbb{C}_K[z]$.

Theorem. *Let $\mathbb{C}_K$ be an algebraically closed complete non-archimedean field with residue field $\hat{k}$ and residue characteristic* $\operatorname{char} \hat{k} = p > 0$. *Let $a_0 \in \mathbb{C}_K$ with $|a_0| > 1$, and let $\varepsilon > 0$. There is some $a \in \mathbb{C}_K$ with $|a - a_0| \leq \varepsilon$ such that the Fatou set $\mathcal{F}$ of the function*

$$\phi_a(z) = (1-a)z^{p+1} + az^p$$

has a wandering component.

A weaker version of the above result appeared in [6], with some details of the proof omitted. There, it was assumed that $\mathbb{C}_K = \mathbb{C}_p$, and it was only shown that there is at least one $a \in \mathbb{C}_p$ giving a wandering domain. Indeed, an examination of that proof reveals that the point a constructed has $|a| = |p|^{-(p-1)}$. The Theorem above is much stronger; not only does it apply to a more general class of fields $\mathbb{C}_K$ (including fields of characteristic p, not just residue characteristic p), but it shows that the set of parameters a giving wandering domains is dense in the large open subset $\{a \in \mathbb{C}_K : |a| > 1\}$ of the parameter space. That large subset cannot be extended; there can be no wandering domains for ϕ_a with $|a| \leq 1$, because such ϕ_a are conjugate to maps of the form $\psi(z) = z^{p+1} + cz^p$ with $|c| \leq 1$, which have good reduction and therefore empty Julia set.

Our proof uses some computations specific to this family, but the general method should apply to a larger class of functions. In particular, the main properties of ϕ_a required in the proof are that it maps a disk U_0 (in this case, about 0) onto itself with degree divisible by p, and another disk U_1 (in this case, about 1) onto U_0 with degree prime to p. Rivera-Letelier has observed [22] that an extension of our argument shows that there is an open set in the higher-dimensional parameter space of all degree $(p+1)$ polynomials in $\mathbb{C}_K[z]$ for which a dense subset of parameters exhibit wandering domains. However, the proof of the Theorem is already so heavy with notation that we prefer to avoid the further tedium of proving the higher-dimensional density statement. Still, we hope that the interested reader will be able to glean enough from our proof to deduce how the statement can be extended to the higher-dimensional parameter space.

Some comments are in order regarding the transcendence of the parameters a in the Theorem. As previously mentioned, the results of [3] imply that if a were defined over a locally compact subfield $K \subseteq \mathbb{C}_K$, and if ϕ_a had no critical points in its Julia set (that is, if ϕ_a is *hyperbolic*), then ϕ_a would have no wandering domains. It will become clear in the course of the proof, however, that the parameter a we choose cannot lie in

a discretely valued field; in particular, a cannot lie in any locally compact subfield of $\mathbb{C}_K$. In the case $\mathbb{C}_K = \mathbb{C}_p$, for example, the parameters a we construct must therefore be transcendental over $\mathbb{Q}_p$. In other words, the truth of the Theorem above requires the freedom to choose $a \in \mathbb{C}_K$; a condition like $a \in \overline{\mathbb{Q}}_p$ would be too restrictive.

Another way to see the transcendence of a is to invoke the results of [3] directly. Note that ϕ_a has at most three distinct critical points: at 0, ∞, and $\beta = (pa)/[(p+1)(a-1)]$. The first two are always in the Fatou set; they are superattracting fixed points, implying that each is contained in a disk U such that $\phi_a(U) \subseteq U$, which forces equicontinuity on U. Only β might be in the Julia set. Certainly if $|a| \leq |p|^{-(p-1)}$, then β is contained in the Fatou component at 0; and if $|a| > |p|^{-p}$, then the iterates of β escape to ∞. Furthermore, β turns out to be Fatou even for many parameters a with $|p|^{-(p-1)} < |a| \leq |p|^{-p}$. (Indeed, it appears that hyperbolic functions form an open dense subset of the parameter space.) Thus, most of the functions ϕ_a described in the Theorem (including all with $|a| \leq |p|^{-(p-1)}$ or $|a| > |p|^{-p}$) are hyperbolic. In those cases, the resulting parameters a cannot lie in a locally compact field.

1. DISKS AND MAPPING LEMMAS

We will denote the closed disk of radius $r > 0$ about a point $a \in \mathbb{C}_K$ by $\overline{D}_r(a)$, and the open disk by $D_r(a)$. We recall some basic properties of non-archimedean disks. Every disk is both open and closed as a topological set. Any point in a disk U is a center, but the radius of U is a well-defined real number $\mathrm{rad}(U)$, which is the same as the diameter of U. If two disks in $\mathbb{C}_K$ intersect, then one contains the other.

The following lemmas describe the behavior of disks under polynomials maps. All are easy to prove using standard techniques of non-archimedean analysis, such as Hensel's Lemma and the Weierstrass Preparation Theorem. (Some proofs, which are standard, may be found in [7, Section 2]; see also [23, Chapter 6] for a broad introduction over p-adic fields.)

Lemma 1.1. *Let $U \subseteq \mathbb{C}_K$ be a disk, and let $f \in \mathbb{C}_K[z]$ be a non-constant polynomial. Then $f(U)$ is a disk.*

Lemma 1.2. *Let $U \subseteq \mathbb{C}_K$ be a disk with $\mathrm{rad}(U) = r$, let $f \in \mathbb{C}_K[z]$ be a non-constant polynomial, and let $s = \mathrm{rad}(f(U))$. Then for any $x, y \in U$,*

$$|f(x) - f(y)| \leq \frac{s}{r} \cdot |x - y|.$$

Lemma 1.3. *Let $U \subseteq \mathbb{C}_K$ be a disk with $\mathrm{rad}(U) = r$, let $f \in \mathbb{C}_K[z]$ be a non-constant polynomial, and let $s = \mathrm{rad}(f(U))$. Then $f : U \to f(U)$ is bijective if and only if for every $x, y \in U$,*

$$|f(x) - f(y)| = \frac{s}{r} \cdot |x - y|.$$

2. NON-ARCHIMEDEAN FATOU AND JULIA SETS

The projective line $\mathbb{P}^1(\mathbb{C}_K)$ is equipped with a natural spherical metric

$$\Delta(x, y) = \frac{|x - y|}{(\max\{|x|, 1\})(\max\{|y|, 1\})}$$

analogous to the spherical metric on $\mathbb{P}^1(\mathbb{C})$. It is easy to verify that $\Delta(x, y) = |x - y|$ if $|x|, |y| \leq 1$; that $\Delta(x, y) = |x^{-1} - y^{-1}|$ if $|x|, |y| \geq 1$; and that $\Delta(x, y) = 1$ otherwise.

Let $\phi \in \mathbb{C}_K(z)$ be a rational function, and consider the dynamical system of the iterates $\{\phi^n\}$ acting on $\mathbb{P}^1(\mathbb{C}_K)$. As in complex dynamics, we define the *Fatou set* $\mathcal{F}$ of ϕ to the the set of all points $x \in \mathbb{P}^1(\mathbb{C}_K)$ such that $\{\phi^n : n \geq 0\}$ is equicontinuous at x with respect to Δ; we define the *Julia set* $\mathcal{J}$ to be the complement $\mathbb{P}^1(\mathbb{C}_K) \setminus \mathcal{F}$.

Following any of [2, 3, 5, 8, 20, 21], it is possible to partition the Fatou set of ϕ into "components" using a different definition than the usual topological connected components. By the results of [5], if $\phi \in \mathbb{C}_K[z]$ is a polynomial with Fatou set $\mathcal{F}$, and if $x \in \mathcal{F}$ has bounded forward orbit (that is, there exists $R > 0$ such that $|\phi^n(x)| < R$ for all $n \geq 0$), then the component U of $\mathcal{F}$ containing x is a disk $D_r(x)$ or $\overline{D}_r(x)$, depending on the precise definition of "component" one chooses. Moreover, for such x, the component of $\phi^n(x)$ is precisely $\phi^n(U)$.

Conversely, given a disk $V \subseteq \mathcal{F}$ containing a point x, the component U of x must contain V. In fact, U is the largest disk (or open disk, for some definitions of component) containing x and contained in $\mathcal{F}$. However, it is conceivable that the iterates of the smaller disk V may wander even though U may be preperiodic. (For example, consider the disk $V = D_{1/p}(a)$ for $1/p < |a| < 1$; the iterates of V wander for $\phi(z) = z^2$, even though V is contained in the fixed disk $D_1(0)$.) Still, if the actual component U were preperiodic, then the orbit of V would follow a preperiodic trajectory through a finite set of larger disks. In our main theorem, we will construct a disk which follows a non-preperiodic trajectory (8), implying that the correpsonding Fatou component must wander.

We recall Hsia's criterion [13] for equicontinuity, which is a non-archimedean analogue of the Montel-Carathéodory Theorem.

Lemma 2.1. (Hsia) *Let $F \subseteq \mathbb{C}_K[z]$ be a family of polynomials, and let $U \subseteq \mathbb{C}_K$ be a disk. Suppose that there is a point $y \in \mathbb{C}_K$ such that for all $f \in F$ and $x \in U$, we have $f(x) \neq y$. Then F is an equicontinuous family.*

Hsia stated his result for arbitrary non-archimedean meromorphic functions; in that setting, the criterion is that there are two points in $\mathbb{P}^1(\mathbb{C}_K)$ which are omitted by every $f \in F$. In our simpler rephrasing above, the first point is y, and the second is ∞. Lemma 2.1 can be proven easily with the Lemmas of the preceding section by first making a change of coordinates to move y to 0. Any $f(U)$ is a disk not containing 0 (by Lemma 1.1 and the hypotheses); therefore it either does not intersect $\overline{D}_1(0)$ or is contained in $\overline{D}_1(0)$. If the former, then $f(U)$ has radius at most 1. If the latter, then the image of $f(U)$ under $1/z$ is a disk of radius at most 1. Because $z \mapsto 1/z$ preserves the spherical metric, equicontinuity follows from Lemma 1.2.

3. The family

Recall that the residue field $\hat{k}$ has positive characteristic char $\hat{k} = p > 0$. For any $a \in \mathbb{C}_K$, we define
$$\phi_a(z) = (1 - a)z^{p+1} + az^p$$
as in the Theorem. Note that 0 and 1 are fixed points of ϕ_a. The point at 0 is super-attracting (meaning that $\phi'_a(0) = 0$, and therefore nearby points are strongly attracted

to 0 under iteration), and if $|a| > 1$, then the point at 1 is repelling (meaning that $|\phi_a'(1)| > 1$, and therefore nearby points are pushed away from 1 under iteration).

The following lemma gives a fairly accurate estimate for the expansion or contraction of distances under a single application of ϕ_a.

Lemma 3.1. *Let $a \in \mathbb{C}_K$ with $|a| > 1$, and let $y_1, y_2 \in \mathbb{C}_K$ with $|y_1| \geq |y_2|$. Then*

$$|\phi_a(y_1) - \phi_a(y_2)| \leq |y_1 - y_2| \cdot |a| \cdot |y_1|^{p-1} \cdot \max\left\{ |p|, |y_1|, \left(\frac{|y_1 - y_2|}{|y_1|} \right)^{p-1} \right\}.$$

Furthermore, if $y_1, y_2 \in D_1(1)$, then

$$|\phi_a(y_1) - \phi_a(y_2)| = |a| \cdot |y_1 - y_2|.$$

Proof of Lemma 3.1. We have

$$\phi_a(y_1) - \phi_a(y_2) = (1 - a)(y_1^{p+1} - y_2^{p+1}) + a(y_1^p - y_2^p)$$

$$= a(y_1 - y_2) \cdot \left[\left(\frac{1}{a} - 1 \right) \left(\sum_{j=0}^{p} y_1^j y_2^{p-j} \right) + (y_1 - y_2)^{p-1} + p \left(\sum_{j=0}^{p-1} A_j y_1^j y_2^{p-1-j} \right) \right], \quad (1)$$

with $A_j \in \mathbb{Z}$. Thus,

$$|\phi_a(y_1) - \phi_a(y_2)| \leq |y_1 - y_2| \cdot |a| \cdot \max\left\{ |y_1|^p, |y_1 - y_2|^{p-1}, |p| \cdot |y_1|^{p-1} \right\},$$

as claimed. Finally, if $y_1, y_2 \in D_1(1)$, then

$$\sum_{j=0}^{p} y_1^j y_2^{p-j} \in D_1(1) \quad \text{and} \quad |y_1 - y_2| < 1.$$

In that case, then, (1) has absolute value $|a| \cdot |y_1 - y_2|$. $\qquad \square$

4. Mapping Properties of the Iterates

In this section, we will apply Lemma 3.1 to describe the behavior of iterates ϕ_a^i near the fixed points at $z = 0, 1$, for i relatively small.

Fix $a_0 \in \mathbb{C}_K$ with $|a_0| > 1$ (as chosen in the hypotheses of the Theorem), and to simplify future notation, define

$$\begin{aligned} R &= |a_0|^{-1/(p-1)}, \\ \mu &= \max\{|p|, R\}, \quad \text{and} \\ S &= \mu R^3. \end{aligned} \quad (2)$$

Note that $0 < S < R^{-2}S < R \leq \mu < 1$. The quantity R represents the radius of a particular disk about 0. Specifically, for $a \in \mathbb{C}_K$ with $|a| = |a_0| > 1$,

$$\phi_a(\overline{D}_R(0)) = \overline{D}_R(0);$$

the mapping is p-to-1. The smaller quantity S is important because ϕ_a^i acts in a predictable manner on disks of radius at most S just outside $\overline{D}_R(0)$, as shown in the following lemma.

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Lemma 4.1. *Let $a \in \mathbb{C}_K$ with $|a| = |a_0|$, and let $m \geq 1$. Let $x \in \mathbb{C}_K$ with $R < |x| \leq R^{1-p^{-m}}$. Then for all $0 \leq i \leq m$, we have*

$$|\phi_a^i(x)| = R^{1-p^i}|x|^{p^i} \leq 1,$$

and

$$\phi_a^i(\overline{D}_S(x)) \subseteq \overline{D}_{S_i'}(\phi_a^i(x)),$$

where

$$S_i' = \mu^i R^{-e_i} S < \mu^i R^{-2} S,$$

and

$$e_i = p^{1-m} + p^{2-m} + \cdots + p^{i-m}.$$

Proof of Lemma 4.1. Observe that e_i is a partial sum of a geometric series, so that $e_i < p/(p-1) \leq 2$. In particular, the inequality $S_i' < \mu^i R^{-2} S$ is valid.

For any $y \in D_1(0)$, $|\phi_a(y)| = |a| \cdot |y|^p$, and so the first statement follows from a simple induction.

We now prove the second statement of the lemma by induction on i. If $i = 0$, in which case $e_i = 0$, the statement is clear. Given that the statement holds for some $0 \leq i \leq m-1$, write $y = \phi_a^i(x)$ to simplify notation. Note that $S_i' = \mu^{i+1} R^{3-e_i} < \mu R$, and that

$$\frac{(\mu R)^{p-1}}{|y|^p} < \frac{\mu}{|y|} \leq \max\left\{\frac{|p|}{|y|}, 1\right\}.$$

Therefore, by Lemma 3.1,

$$\phi_a\left(\overline{D}_{S_i'}(y)\right) \subseteq \overline{D}_{\lambda(y) \cdot S_i'}(\phi_a(y)), \tag{3}$$

where

$$\lambda(y) = |a| \cdot |y|^p \max\left\{1, \frac{|p|}{|y|}\right\}.$$

We compute

$$\lambda(y) = \lambda\left(\phi_a^i(x)\right) = |a| \cdot \left|\phi_a^i(x)\right|^p \cdot \max\left\{1, \left|\frac{p}{\phi_a^i(x)}\right|\right\}$$

$$\leq |a| \left(R^{1-p^i}|x|^{p^i}\right)^p \cdot \max\left\{1, \frac{|p|}{R}\right\}$$

$$= |a| R^{p-1-p^{i+1}}|x|^{p^{i+1}}\mu = \mu R^{-p^{i+1}}|x|^{p^{i+1}} \leq \mu R^{-p^{i+1-m}}.$$

Thus, $S_{i+1}' \leq \lambda(y) \cdot S_i'$. By the inductive hypothesis, $\phi_a^i(\overline{D}_S(x)) \subseteq \overline{D}_{S_i'}(y)$; therefore, by (3),

$$\phi_a^{i+1}(\overline{D}_S(x)) \subseteq \overline{D}_{S_{i+1}'}(\phi_a(y)),$$

and the Lemma follows. $\square$

Remark 4.2. *a.* The upper bound of S_i' for the image radius in Lemma 4.1 is far from sharp. The sharp bound is not difficult to compute, but its formula is complicated. Our computations will require only a bound which decreases to 0 as i increases, and therefore the simpler bound given above will suffice.

b. Intuitively, Lemma 4.1 says the following. A point x just outside of $\overline{D}_R(0)$ is pushed further away from 0 by ϕ_a. However, $|\phi_a'(x)| < 1$, so that locally (i.e., on $\overline{D}_S(x)$), ϕ_a

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is contracting. Thus, if $|x|$ is only slightly bigger than R, the iterates of x will stay inside the larger disk $D_1(0)$ for some time; and the corresponding iterates of $\overline{D}_S(x)$ will become very small in radius. In broad terms, ϕ_a is locally contracting on the whole annulus $R < |x| < 1$, but every point in the annulus is pushed away from 0 much as it would be from a repelling fixed point.

c. Ultimately, our main interest in Lemma 4.1 lies in the case that $i = m$. However, the cases $i < m$ are important both for the inductive proof and because we need to have some idea of the orbit traversed by x during those m iterations. An analogous remark applies to the quantity M in Lemma 4.3 below.

Lemma 4.3. *Let $a \in \mathbb{C}_K$ with $|a| = |a_0|$, and let $M \geq 1$. Let $x \in \mathbb{C}_K$ with $|x - 1| \leq |a|^{-M}$. Then for all $0 \leq i \leq M$, we have*
$$|\phi_a^i(x) - 1| = |a|^i \cdot |x - 1|,$$
and for any $0 < r \leq |a|^{-M}$,
$$\phi_a^i(\overline{D}_r(x)) = \overline{D}_{|a|^i r}(\phi_a^i(x)).$$

Proof of Lemma 4.3. The result follows easily by induction on i using the first statement of Lemma 3.1 and the fact that $\phi_a(1) = 1$. $\qquad\square$

5. Varying the parameter

We wish to study the effects of perturbations of a on the dynamics of ϕ_a, and so we introduce the following notation. For any $z \in \mathbb{P}^1(\mathbb{C}_K)$, $a \in \mathbb{C}_K$, and $n \geq 0$, define
$$\Phi_n(a, z) = \phi_a^n(z). \tag{4}$$
For any fixed $x \in \mathbb{C}_K$, $\Phi_n(a, x)$ is a polynomial in the variable a. In particular, for any disk $D \in \mathbb{C}_K$, the set $\Phi_n(D, x)$ is also a disk, by Lemma 1.1.

Lemma 5.1. *Let $a \in \mathbb{C}_K$ with $|a| = |a_0|$, let $M \geq 0$, let $n \geq 1$, and let $x \in \mathbb{C}_K$ with $|\phi_a^n(x) - 1| \leq |a|^{-M}$. Let $\varepsilon \in (0, |a|^{1-M}]$. Suppose that $\Phi_n(\cdot, x)$ maps $\overline{D}_\varepsilon(a)$ bijectively onto $\overline{D}_{\varepsilon/|a|}(\phi_a^n(x))$. Then $\Phi_{n+M}(\cdot, x)$ maps $\overline{D}_\varepsilon(a)$ bijectively onto $\overline{D}_{\varepsilon \cdot |a|^{(M-1)}}(\phi_a^{n+M}(x))$.*

Proof. We proceed by induction on M; the case $M = 0$ is vacuously true. For $M \geq 1$, let $b_1, b_2 \in \overline{D}_\varepsilon(a)$, let $y_1 = \Phi_{n+M-1}(b_1, x)$, and let $y_2 = \Phi_{n+M-1}(b_2, x)$. Note that
$$
\begin{aligned}
|y_1 - 1| &= |a|^{M-1} \cdot |\Phi_n(b_1, x) - 1| \\
&\leq |a|^{M-1} \cdot \max\left\{ |\Phi_n(b_1, x) - \phi_a^n(x)|, |\phi_a^n(x) - 1| \right\} \\
&\leq |a|^{M-1} \cdot \max\left\{ |a|^{-1} \cdot |b_1 - a|, |a|^{-M} \right\} = |a|^{-1},
\end{aligned}
$$
where the first equality is by Lemma 4.3, and the rest is by the hypotheses, ultrametricity, and Lemma 1.3. Therefore,
$$
\begin{aligned}
|\phi_{b_1}(y_1) - \phi_{b_2}(y_1)| &= \left| (1 - b_1)y_1^{p+1} + b_1 y_1^p - (1 - b_2)y_1^{p+1} - b_2 y_1^p \right| \\
&= |y_1|^p \cdot |y_1 - 1| \cdot |b_1 - b_2| \leq |a|^{-1} \cdot |b_1 - b_2|,
\end{aligned}
$$
by the above bound and because $|y_1| = 1$. On the other hand, by Lemma 3.1,
$$|\phi_{b_2}(y_1) - \phi_{b_2}(y_2)| = |a| \cdot |y_1 - y_2| = |a|^{M-1} \cdot |b_1 - b_2|,$$

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where the final equality is by the inductive hypothesis and Lemma 1.3. In particular, if $b_1 \neq b_2$, then

$$|\phi_{b_1}(y_1) - \phi_{b_2}(y_1)| < |\phi_{b_2}(y_1) - \phi_{b_2}(y_2)|,$$

and therefore

$$|\phi_{b_1}(y_1) - \phi_{b_2}(y_2)| = |\phi_{b_2}(y_1) - \phi_{b_2}(y_2)|$$

by ultrametricity. It follows that

$$|\Phi_{n+M}(b_1, x) - \Phi_{n+M}(b_2, x)| = |a|^{M-1} \cdot |b_1 - b_2|. \tag{5}$$

The same equality is clear if $b_1 = b_2$; thus, (5) holds for all $b_1, b_2 \in \overline{D}_\varepsilon(a)$.

By Lemma 1.1, $\Phi_{n+M}(\overline{D}_\varepsilon(a), x)$ is a disk about $\phi_a^{n+M}(x)$ of some radius s. By (5) and Lemma 1.3, $s = \varepsilon \cdot |a|^{(M-1)}$, and the mapping is bijective, as claimed. $\qquad \square$

Lemma 5.2. *Let $a \in \mathbb{C}_K$ with $|a| = |a_0|$, let $m \geq 1$, let $n \geq 0$, and let $x \in \mathbb{C}_K$ with $|\phi_a^n(x)| = R^{1-p^{-m}}$. Let $A > |a|^{-1}$ be a real number, let $\varepsilon \in (0, A^{-1}S]$, and suppose that*

$$\Phi_n(\overline{D}_\varepsilon(a), x) \subseteq \overline{D}_{A\varepsilon}(\phi_a^n(x)) \quad \text{and} \quad \mu^m < A^{-1} R^{p+1}.$$

Then $\Phi_{n+m}(\cdot, x)$ maps $\overline{D}_\varepsilon(a)$ bijectively onto $\overline{D}_{\varepsilon/|a|}(\phi_a^{n+m}(x))$.

Proof. For any $0 \leq i \leq m$, let $e_i = p^{1-m} + p^{2-m} + \cdots + p^{i-m}$ as in Lemma 4.1, and let

$$r_i = \max \left\{ \mu^i R^{-e_i} A, R^{p-p^{-(m-i)}} \right\}.$$

Note that $r_i \leq A$, and therefore $r_i \varepsilon \leq A\varepsilon \leq S$. We claim that for any such i,

$$\Phi_{n+i}\left(\overline{D}_\varepsilon(a), x\right) \subseteq \overline{D}_{r_i\varepsilon}\left(\phi_a^{n+i}(x)\right), \tag{6}$$

with $\Phi_{n+i}(\cdot, x)$ mapping $\overline{D}_\varepsilon(a)$ bijectively onto $\overline{D}_{r_i\varepsilon}(\phi_a^{n+i}(x))$ if

$$i \geq 1 \quad \text{and} \quad \mu^i R^{-e_i} A < R^{p-p^{-(m-i)}}. \tag{7}$$

Condition (7) holds for $i = m$ because $\mu^m < A^{-1} R^{p+1}$ by hypothesis and because $e_m < 2$; moreover, $r_m = R^{p-1} = |a|^{-1}$. Thus, to prove the lemma, it suffices to prove the claim.

We proceed by induction on i. We have $r_0 = A$, and therefore the $i = 0$ case is true by hypothesis. For $1 \leq i \leq m$, pick any two distinct points $b_1, b_2 \in \overline{D}_\varepsilon(a)$, and let $y_1 = \Phi_{n+i-1}(b_1, x)$ and $y_2 = \Phi_{n+i-1}(b_2, x)$. By Lemma 1.2, the inductive hypothesis implies that $|y_1 - y_2| \leq r_{i-1} \cdot |b_1 - b_2|$. Note that $|y_1| = |y_2| = R^{1-p^{-(m-i+1)}}$, by Lemma 4.1. Therefore,

$$\begin{aligned}
|\phi_{b_1}(y_1) - \phi_{b_2}(y_1)| &= \left| (1 - b_1)y_1^{p+1} + b_1 y_1^p - (1 - b_2)y_1^{p+1} - b_2 y_1^p \right| \\
&= |b_1 - b_2| \cdot |y_1 - 1| \cdot |y_1|^p = |b_1 - b_2| \cdot |y_1|^p \\
&= |b_1 - b_2| \cdot R^{p-p^{-(m-i)}} \leq r_i |b_1 - b_2|,
\end{aligned}$$

with equality if condition (7) holds. Meanwhile, $|y_1 - y_2| \leq r_{i-1}\varepsilon \leq S < \mu R$; by Lemma 3.1,

$$|\phi_{b_2}(y_1) - \phi_{b_2}(y_2)| \leq \lambda(y_1) \cdot |y_1 - y_2| \leq r_{i-1} \cdot \lambda(y_1) \cdot |b_1 - b_2|,$$

where $\lambda(y_1) = |a| \cdot |y_1|^p \max\{1, |p|/|y_1|\}$ as in the proof of Lemma 4.1. Adding, we have

$$|\Phi_{n+i}(b_1, x) - \Phi_{n+i}(b_2, x)| = |\phi_{b_1}(y_1) - \phi_{b_2}(y_2)| \leq \max\{r_i, r_{i-1} \cdot \lambda(y_1)\} \cdot |b_1 - b_2|,$$

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with equality if the maximum is uniquely attained. Thus, to prove (6), it suffices to show that

$$\lambda(y_1) \cdot r_{i-1} \leq r_i.$$

By Lemmas 1.1 and 1.3, the full claim would follow if in fact $\lambda(y_1) \cdot r_{i-1} < r_i$ whenever condition (7) holds.

Note that

$$\lambda(y_1) = R^{-p^{-(m-i)}} \cdot \max\left\{R, |p| \cdot R^{p^{-(m-i+1)}}\right\} \leq \mu R^{-p^{-(m-i)}}.$$

Thus, if $r_{i-1} = \mu^{i-1} R^{-e_{i-1}} A$, then

$$\lambda(y_1) \cdot r_{i-1} \leq \mu^i R^{-e_i} A \leq r_i,$$

and this inequality is strict if condition (7) holds. On the other hand, if $r_{i-1} = R^{p-p^{-(m-i+1)}}$, then

$$\lambda(y_1) \cdot r_{i-1} = R^{p-p^{-(m-i)}} \cdot \max\left\{R^{1-p^{-(m-i+1)}}, |p|\right\} < R^{p-p^{-(m-i)}} \leq r_i.$$

The claim is proven, and the lemma follows. $\qquad\square$

Remark 5.3. *a.* The two conditions displayed in the hypotheses of Lemma 5.2 say, first, that A is large enough to measure the size of a certain image disk, and second, that m is large enough that μ^m is smaller than the already small $A^{-1} R^{p+1}$.

b. Lemmas 5.1 and 5.2 both describe the behavior of iterates $\phi_a^{n+i}(x)$ as a varies; the former is for $\phi_a^n(x)$ close to 1, and the latter is for $\phi_a^n(x)$ just outside $\overline{D}_R(0)$. The conclusion of Lemma 5.2 exactly matches the key hypothesis of Lemma 5.1, and the conclusion of Lemma 5.1 implies the key hypothesis of Lemma 5.2. (We shall need the full strength of Lemma 5.1 elsewhere in the main proof.) Thus, we will be able to alternate applications of each Lemma in order to describe the behavior of ϕ_a^n for larger and larger values of n.

6. Proof of the Theorem

To prove the Theorem, we will first choose several auxiliary sequences $\{m_i\}$, $\{M_i\}$, $\{n_i\}$ and $\{N_i\}$ of positive integers, and $\{\varepsilon_i\}$ and $\{r_i\}$ of positive real numbers. The carefully chosen integer sequences will be used to describe the orbit of a certain point x under various maps ϕ_a; the real values ε_i will be radii of disks in which the parameters a lie; and the r_i will be radii of disks in which certain images $\phi_a^n(x)$ lie.

After those choices, we will argue inductively using Lemmas 5.1 and 5.2 to produce a sequence $\{a_i\}$ of parameters with limit a such that ϕ_a has a wandering domain.

6.1. **Auxiliary sequences.** We are given $\varepsilon > 0$ and $a_0 \in \mathbb{C}_K$ with $|a_0| > 1$. Define R, S, and μ as in (2). We assume without loss that $\varepsilon \leq S$.

Choose a strictly increasing sequence of positive integers $\{M_i\}_{i \geq 1}$ so that

$$|a_0|^{1-M_1} \leq \varepsilon, \qquad \text{and for all } i \geq 2, \quad |a_0|^{M_{i-1}-M_i} \leq S.$$

Let $\varepsilon_0 = \varepsilon$, and for all $i \geq 1$, let

$$\varepsilon_i = |a_0|^{1-M_i} S.$$

Then $\{\varepsilon_i\}$ is a strictly decreasing sequence of positive real numbers with limit 0, and $|a_0|^{1-M_i} \leq \varepsilon_{i-1}$ for all $i \geq 1$.

Choose another strictly increasing sequence of positive integers $\{m_i\}_{i\geq 0}$ so that
$$\mu^{m_i} \leq |a_0|^{-M_{i+1}} \cdot R^2 \quad \text{for all } i \geq 0.$$

For all $i \geq 0$, define
$$r_i = R^{1-p^{-m_i}}.$$

Then $\{r_i\}$ is a strictly decreasing sequence of positive real numbers with limit R. Note that the starting index for $\{m_i\}$, $\{\varepsilon_i\}$, and $\{r_i\}$ is $i = 0$, while that for $\{M_i\}$ is $i = 1$.

We will find $x \in D_1(0)$ and $a \in \overline{D}_\varepsilon(a_0)$ such that the orbit $\{\phi_a^j(x)\}_{j\geq 0}$ can be described by
$$\underbrace{0,\ldots,0}_{m_0},\underbrace{1,\ldots,1}_{M_1},\underbrace{0,\ldots,0}_{m_1},\underbrace{1,\ldots,1}_{M_2},\underbrace{0,\ldots,0}_{m_2},\ldots \tag{8}$$
where a 0 in the j-th position in the sequence indicates that $\phi_a^{j-1}(x) \in D_1(0)$, and a 1 indicates that $\phi_a^{j-1}(x) \in D_1(1)$.

To simplify future notation, define
$$n_i = \sum_{j=1}^{i}(m_{j-1} + M_j), \quad \text{and} \quad N_i = n_i + m_i$$

for all $i \geq 0$. That is, n_i is the number of terms in (8) up to but not including the block of m_i 0's, and N_i is the number of terms up to but not including the block of M_{i+1} 1's.

6.2. **Choosing the parameter.** Recall that $r_0 = R^{1-p^{-m_0}}$. Pick $y \in \mathbb{C}_K$ with $|y| = r_0$; by Lemma 4.1, $|\phi_{a_0}^{m_0}(y)| = 1$. (Such a y exists because $r_0 \in \mathbb{C}_K$.) In addition, $\phi_{a_0}^{m_0}(0) = 0$. Therefore, by Lemma 1.1, $\phi_{a_0}^{m_0}\left(\overline{D}_{r_0}(0)\right)$ is a disk containing both 0 and $\phi_{a_0}^{m_0}(y)$, and hence
$$\overline{D}_1(0) \subseteq \phi_{a_0}^{m_0}\left(\overline{D}_{r_0}(0)\right).$$

In particular, there is some $x \in \overline{D}_{r_0}(0)$ with $\phi_{a_0}^{m_0}(x) = 1$. Again by Lemma 4.1, we have $|x| = r_0$.

For every $i \geq 0$, we will find $a_i \in \overline{D}_{\varepsilon_{i-1}}(a_{i-1})$ (or just the original a_0, for $i = 0$) so that for every $a \in \overline{D}_{\varepsilon_i}(a_i)$, the orbit $\{\phi_a^j(x)\}$ follows (8) up to the $j = N_i$ iterate, with $\phi_{a_i}^{N_i}(x) = 1$ and such that
$$\Phi_{N_i}(\cdot, x) : \overline{D}_{\varepsilon_i}(a_i) \to \overline{D}_{\varepsilon_i/|a_i|}(1) \quad \text{is bijective.} \tag{9}$$

Because $\{\varepsilon_i\}$ is decreasing, each a_i will lie in $\overline{D}_{\varepsilon_0}(a_0)$, and therefore $|a_i| = |a_0|$.

We choose the sequence $\{a_i\}$ by induction on i. For $i = 0$, we already have $\phi_{a_0}^{N_0}(x) = 1$, and by Lemma 4.1, the orbit $\{\phi_{a_0}^j(x)\}$ follows (8) up to the $N_0 = m_0$ iterate. By Lemma 5.2 (with $n = n_0 = 0$, $m = m_0$, $a = a_0$, $A = 1$, and $\varepsilon = \varepsilon_0$), condition (9) holds. Also, by Lemma 4.1, the orbit $\{\phi_a^j(x)\}$ is correct up to $j = N_0$ for every $a \in \overline{D}_{\varepsilon_0}(a_0)$. Hence, the $i = 0$ case is already done.

For $i \geq 1$, assume that we are given a_{i-1} with the desired properties. Let $\rho = |a_0|^{1-M_i} \leq \varepsilon_{i-1}$; then by the inductive hyphothesis, for every $a \in \overline{D}_\rho(a_{i-1})$, the orbit $\{\phi_a^j(x)\}$ agrees with (8) up to $j = N_{i-1}$. Furthermore, because $\Phi_{N_{i-1}}(\cdot, x)$ maps $\overline{D}_{\varepsilon_{i-1}}(a_{i-1})$ bijectively onto $\overline{D}_{\varepsilon_{i-1}/|a_{i-1}|}(1)$ with $\Phi_{N_{i-1}}(a_{i-1}, x) = 1$, we see that $\Phi_{N_{i-1}}(\cdot, x)$

also maps $\overline{D}_\rho(a_{i-1})$ bijectively onto $\overline{D}_{\rho/|a_{i-1}|}(1)$, by Lemma 1.3. By Lemma 5.1 (with $a = a_{i-1}$, $M = M_i$, $n = N_{i-1}$, and $\varepsilon = \rho$), $\Phi_{n_i}(\cdot, x)$ maps $\overline{D}_\rho(a_{i-1})$ bijectively onto $\overline{D}_1(1)$. In particular, there exists $b \in \mathbb{C}_K$ such that $\Phi_{n_i}(b, x) = 0$. By Lemma 1.3, we must have $|b - a_{i-1}| = \rho$; and because $r_i = R^{1-p^{-m_i}} < 1$,

$$\Phi_{n_i}(\cdot, x) : \overline{D}_\sigma(b) \to \overline{D}_{r_i}(0) \quad \text{is bijective,} \tag{10}$$

where $\sigma = r_i \cdot |a_0|^{1-M_i} = r_i \cdot \rho \in (0, \rho)$. Moreover, for all $a \in \overline{D}_\sigma(b)$,

$$|\Phi_{N_{i-1}}(a, x) - 1| = |a_0|^{-1}|a - a_{i-1}| = |a_0|^{-1}|b - a_{i-1}| = |a_0|^{-M_i}.$$

By Lemma 4.3, then, the orbit $\{\phi_a^j(x)\}$ is correct up to $j = n_i$ for every $a \in \overline{D}_\sigma(b)$.

Choose $c \in \overline{D}_\sigma(b)$ so that $|\Phi_{n_i}(c, x)| = r_i$. (Such c exists because of (10) and the fact that $r_i \in |\mathbb{C}_K|$.) By Lemma 4.1, $|\Phi_{N_i}(c, x)| = 1$. Furthermore, it is clear that $\Phi_{N_i}(b, x) = 0$. Because the polynomial image of a disk is a disk (Lemma 1.1), it follows that $\Phi_{N_i}(\overline{D}_\sigma(b, x)) \supseteq \overline{D}_1(0)$. We may therefore choose $a_i \in \overline{D}_\sigma(b)$ so that $\Phi_{N_i}(a_i, x) = 1$.

By Lemma 4.1, we must have $|\Phi_{n_i}(a_i, x)| = r_i$. Observe also that $\varepsilon_i \leq \sigma$. By (10) and Lemma 1.3, $\Phi_{n_i}(\overline{D}_{\varepsilon_i}(a_i), x)$ must be a disk of radius $\varepsilon_i \cdot |a_0|^{M_i - 1} = S$. Therefore, by Lemma 5.2 (with $n = n_i$, $m = m_i$, $a = a_i$, $A = |a_i|^{M_i - 1}$, and $\varepsilon = \varepsilon_i$), condition (9) holds on $\overline{D}_{\varepsilon_i}(a_i)$. By Lemma 4.1, the orbit $\{\phi_a^j(x)\}$ is correct up to $j = N_i$ for every $a \in \overline{D}_{\varepsilon_i}(a_i)$. Our construction of a_i is complete.

The sequence $\{a_i\}_{i \geq 0}$ is a Cauchy sequence, because for any $0 \leq i \leq j$, we have $|a_i - a_j| \leq \varepsilon_i$, and $\varepsilon_i \to 0$. Therefore, the sequence has a limit $a \in \mathbb{C}_K$, with $|a - a_0| \leq \varepsilon_0$. We claim that ϕ_a has a wandering domain.

6.3. **The end of the proof.** By construction, $a \in \overline{D}_{\varepsilon_i}(a_i)$ for every $i \geq 0$; hence, the orbit $\{\phi_a^j(x)\}$ follows (8) exactly. In light of Lemmas 4.1 and 4.3, we must have

$$|\phi_a^{n_i}(x)| = r_i = R^{1-p^{-m_i}}, \quad \text{and} \quad |\phi_a^{N_i}(x) - 1| = |a|^{-M_{i+1}}$$

for any $i \geq 0$. Let $U = \overline{D}_S(x)$; we will show that U is contained in a wandering domain of the Fatou set $\mathcal{F}$ of ϕ_a.

Every iterate $U_n = \phi_a^n(U)$ is a disk; we claim that for any $i \geq 0$, the radius of U_{n_i} is at most S, and the radius of U_{N_i} is at most $|a|^{-M_{i+1}}S$. The claim is easily proven by induction, as follows. For $i = 0$, $U_{n_0} = U_0 = U$, which has radius S; and by Lemma 4.1, U_{N_0} has radius at most $\mu^{m_0}R^{-2}S \leq |a|^{-M_1}S$. For $i \geq 1$, we assume the radius of $U_{N_{i-1}}$ is at most $|a|^{-M_i}S$. By Lemma 4.3, the radius of U_{n_i} is at most S; and by Lemma 4.1, the radius of U_{N_i} is at most $\mu^{m_i}R^{-2}S \leq |a|^{-M_{i+1}}S$.

If some U_{n_i} contained the point 1, then by Lemma 4.3, some $U_{n_{i+j}}$ would contain $\overline{D}_1(0)$ and therefore would have radius greater than S. Thus, no U_{n_i} contains the point 1; and because 1 is fixed, it follows that no U_n contains 1. By Hsia's criterion (Lemma 2.1), then, the family $\{\phi_a^n\}$ is equicontinuous on U, and therefore $U \subseteq \mathcal{F}$.

The iterates of U are bounded away from $\infty \in \mathbb{P}^1(\mathbb{C}_K)$, and ϕ_a is a polynomial. Therefore, by any of the definitions of components in [2, 3, 8, 20, 21], the component V of $\mathcal{F}$ containing U must be a disk [5, Theorem 5.4.d]. As before, no iterate of V can contain 1; the symbolic dynamics of V are therefore described by equation (8). Because those dynamics are not preperiodic, it follows that V must be wandering. $\square$

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