Fatou Components in p-adic Dynamics

by

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A.B., Harvard University, 1993
Sc.M., Brown University, 1996

Thesis

Submitted in partial fulfillment of the requirements for
the Degree of Doctor of Philosophy
in the Department of Mathematics at Brown University

May 1998

This dissertation by Robert L. Benedetto is accepted in its present form by the Department of Mathematics as satisfying the dissertation requirement for the degree of Doctor of Philosophy

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Acknowledgments

No thesis was ever written in a vacuum, and I owe much gratitude to the many people who supported and aided me in the writing of this one.

First and foremost, I would like to thank my advisor, Joseph Silverman, who first steered me in dynamical and p-adic directions. His good humor and clear explanations have been as valuable to me as his wide mathematical knowledge and keen insight. I've been able to count on his patience and encouragement in times of few ideas, as well as his energetic enthusiasm when ideas did come. Joe, it's been an honor, and it's been fun.

I'm grateful to my readers, Jonathan Lubin and Michael Rosen, who, in addition to slogging through the pages that follow, were there so many times in the past few years for discussions and help. Thanks to all the other professors who have stopped to give me a hand, especially Siman Wong, Curt McMullen, Fernando Gouvêa, and Ted Gamelin. A special thanks to Natalie Johnson, Carol Oliveira, and Phyllis Hudek for all their support and assisantance throughout grad school.

Thank you to all my friends who kept me going. To Dan, Lawrence, Julie, Steve, Matt, Marshall, Josh, Ottavio, Steve, and all of the math grad students at Brown, thanks for the help, the advice, the wings, the cakes, and the parties. To John and Cathy, Dave and Kristy, Vicki and Jon, John and Kate, Joe, the choir, and everyone else at Manning, thanks for your prayers, friendship, and community.

Finally, thanks to my parents and my brothers for all your love and support throughout my life, and especially in graduate school. Most of all, thanks to my wonderful wife, Danielle, for your constant patience, your confidence and cheering, and your friendship and love.

Contents

Chapter 1

Introduction

The modern theory of discrete complex dynamical systems began in the early part of this century with the work of P. Fatou and G. Julia, concerning the action of a rational function $\phi(z) \in \mathbb{C}(z)$ on the Riemann sphere. Their goal was to understand the "dynamics" of such a map; that is, to study the behaviors which can occur when ϕ is composed with itself repeatedly. The field has attracted considerable interest and is now very well-developed. The purpose of this thesis is to investigate the parallel but relatively undeveloped theory of p-adic dynamical systems, using the theory of complex dynamical systems as a model.

1.1 Complex dynamics

A *discrete dynamical system* is a set X with a map $\phi : X \to X$; by definition, any of the complex dynamical systems of Fatou and Julia are discrete dynamical systems. The first objects to arise in the study of such systems are the so-called fixed points, which are, as the name implies, points of X which are mapped to themselves by ϕ. In complex dynamics, fixed points can be classified by their derivatives into attracting, repelling, and neutral points; these names are indicative of whether nearby points move closer to or further from the fixed point under application of ϕ. Besides fixed points, ϕ may have periodic points, which are points that map back to themselves after several iterations of ϕ, and preperiodic points, which are points that eventually map to periodic points.

Deeper analysis leads naturally to the definitions of the Fatou and Julia sets. The Fatou set, roughly speaking, consists of all areas of the sphere where small errors stay small under iteration; in other words, if two points are close to each other in such an area, then all of their iterates under application of ϕ are also close together. The Julia set is the complement of the Fatou set, and it is the locus of chaos; small errors may become arbitrarily large after many iterations.

Julia sets are almost always complicated fractal sets, and they split the Fatou set into many connected components. The original function ϕ maps any one such component onto another; it therefore induces a map Φ from the set of components to the set of components. The resulting object is another discrete dynamical system, and it therefore makes sense to discuss fixed, periodic, and preperiodic components of the Fatou set. A long-standing conjecture for much of the century states that all Fatou components are preperiodic. D. Sullivan ([32]) finally proved this statement, the celebrated No Wandering Domains Theorem, in 1985, by a deep method involving quasiconformal functions. (There had been proofs in various special cases, but Sullivan's was the first to hold in full generality.) The No Wandering Domains Theorem, combined with the classification of possible dynamics on fixed Fatou components, allows for a complete understanding of dynamics on the Fatou set of a complex rational function.

For a much broader and more detailed study of complex dynamics, we refer the reader to the expositions in [6] and [23].

1.2 The p-adics

The basic definitions and objects of complex dynamics do not depend on any special properties of the complex numbers; all that is needed is an algebraically closed field which is complete with respect to an absolute value. There are, of course, fields besides $\mathbb{C}$ which satisfy these requirements; our interest will be fields of this type which come from number theory.

The main fields of interest in number theory are the field $\mathbb{Q}$ of rational numbers and its finite extensions. The field $\mathbb{C}$ arises naturally in this setting, because it is the algebraic closure of the real line, $\mathbb{R}$, which is the completion of $\mathbb{Q}$ with respect to the standard absolute value.

However, there are other natural absolute values on $\mathbb{Q}$: for any prime p of $\mathbb{Z}$, there is an associated p-adic absolute value. The completion of $\mathbb{Q}$ with respect to the p-adic absolute value is normally denoted $\mathbb{Q}_p$, the field of p-adic rationals. Its algebraic closure, $\overline{\mathbb{Q}}_p$, is unfortunately not complete; however, the completion $\mathbb{C}_p$ (or Ω_p) of $\overline{\mathbb{Q}}_p$ is both algebraically closed and complete. $\mathbb{C}_p$ is therefore a natural analogue of $\mathbb{C}$; our goal in this thesis is to study the dynamics of maps defined over $\mathbb{C}_p$ rather than over $\mathbb{C}$. We will see many similarities and parallels, but also several striking differences, between the complex and p-adic theories.

There has recently been increasing interest in non-Archimedean spaces among theoretical physicists. In 1986, Rammal, Toulouse, and Virasoro ([31]) observed p-adic structures in spin glasses; in 1987, Volovich ([36]) suggested p-adic strings to explain some phenomena in superstring theory. Given such a p-adic physical structure, there have been some attempts to study any resulting dynamical systems; see, for example, [34, 35].

There have been a few other scattered studies of p-adic dynamics in various forms; see, for example, [1, 13, 14, 17, 18, 19, 37]. In this thesis, we will build on the results of such investigations; however, our main focus will be on building a theory analogous to the study of connected components of the complex Fatou set. In particular, our main theorem will be a partial analogue of the No Wandering Domains Theorem.

For details on the construction of $\mathbb{C}_p$ and its properties, see, for example, [16].

1.3 Local and global dynamics

Let K be a number field. Consider a discrete dynamical system where the set X is an algebraic variety and the map ϕ is a morphism, both defined over K. The simplest such example is that X is the projective line, and $\phi \in K(z)$ is a rational function (i.e., a morphism from the projective line to itself). Northcott ([28]) proved that only finitely many preperiodic points of ϕ can be defined over K. Since then, various bounds have been found for the number of K-rational periodic points for certain classes of maps; see, for example, [24, 25, 26, 29, 30, 37]. Morton and Silverman have conjectured that if X is projective n-space, then the number of K-rational preperiodic points is bounded solely in terms of the degree of the map ϕ, the dimension n, and the field extension degree $[K : \mathbb{Q}]$ (see [24]).

This boundedness conjecture is analogous to a similar statement for elliptic curves. Given any elliptic curve E, the multiplication-by-n map, denoted $[n]$, is a morphism from E to E and may therefore be considered a dynamical system. The set of preperiodic points of $[n]$ is precisely the torsion subgroup of E. Merel ([22]), building on the work of Mazur ([20]) and Kamienny ([15]), has proven that number of K-rational torsion points is bounded solely in terms of the degree $[K : \mathbb{Q}]$. Thus, the boundedness conjecture is true for elliptic curves.

Dynamical systems defined over global fields also arise in the study of canonical height functions; see, for example, [5]. In fact, if ϕ is a morphism from a variety X to itself, defined over a number field K, then the set of preperiodic points of ϕ is precisely the zero set of the associated canonical height.

The canonical height can also be written as a sum of local canonical heights, one for each place of the field K. Most of these local heights can be computed without much difficulty; however, the places of bad reduction may cause trouble. As Call and Silverman showed in [5], the local canonical heights for such places can be computed using the corresponding local dynamical system. For an archimedean place, $\mathbb{C}$ is the natural setting for this local dynamical system; for a prime, $\mathbb{C}_p$ is the natural setting. It seems reasonable to hope that a clearer understanding of p-adic dynamical systems, together with the theory of complex dynamical systems, may aid in the study of global dynamics.

1.4 Summary of results

In this thesis, we will develop the theory of p-adic dynamics as follows. In Chapter 2, we will present the necessary background, including definitions of all dynamical terminology and some basic dynamical results. We include a section on rigid analysis, which will be used for the definition of “analytic components” and for a few minor results.

In Chapter 3, we will define two analogues of the connected components from the complex theory. We will also give a detailed analysis (using these components) of the dynamics of quadratic rational maps over the p-adics. For clarity of presentation, the proofs of many of the facts in that analysis will not appear until Appendix A.

In Chapter 4, we will define hyperbolic maps, analogous to the definition in the complex theory. After giving several equivalent characterizations, we will prove a version of our main theorem and related results for hyperbolic maps.

In Chapter 5, we will state and prove our main theorem, which is an analogue of Sullivan's No Wandering Domains Theorem for a very large class of p-adic rational maps. This theorem and related results will be proven using a series of technical lemmas.

In the remaining chapters, we will present some examples and results on related topics. Chapter 6 concerns dynamics on a fixed component, and Chapter 7 is a study of some of the dynamical phenomena which occur for polynomial maps. Finally, in Chapter 8, we will prove some results relating to reduction, a tool which is specific to non-Archimedean fields.

Chapter 2

Fundamentals

2.1 Notation

Throughout this thesis, we will be using the following notation:

p	a prime number
$\mathbb{Z}_p$	the p-adic integers
$\mathbb{Q}_p$	the p-adic rationals
$\overline{\mathbb{Q}}_p$	a (fixed) algebraic closure of $\mathbb{Q}_p$
$\mathbb{C}_p$	the completion of $\overline{\mathbb{Q}}_p$
$v_p(\cdot)$ or $v(\cdot)$	the p-adic valuation on $\mathbb{C}_p$
$\lvert\cdot\rvert_p$ or $\lvert\cdot\rvert$	the p-adic absolute value on $\mathbb{C}_p$
$\mathcal{O}$	the integers of $\mathbb{C}_p$; that is, $\{z \in \mathbb{C}_p : \lvert z\rvert_p \leq 1\}$
$\mathbb{P}^1(\mathbb{C}_p)$	the projective line (closed points only) over $\mathbb{C}_p$

Our valuation and absolute value will be normalized so that $v_p(p) = 1$ and $|p|_p = p^{-1}$. In particular, $v_p(\mathbb{Q}_p^*) = \mathbb{Z}$, $|\mathbb{Q}_p^*|_p = p^{\mathbb{Z}}$, $v_p(\overline{\mathbb{Q}}_p^*) = v_p(\mathbb{C}_p^*) = \mathbb{Q}$, and $|\overline{\mathbb{Q}}_p^*|_p = |\mathbb{C}_p^*|_p = p^{\mathbb{Q}}$. For details on the construction of $\mathbb{Q}_p$, $\overline{\mathbb{Q}}_p$, and $\mathbb{C}_p$, see [16].

We will usually not consider $\mathbb{P}^1(\mathbb{C}_p)$ as a scheme, but as analogous to the Riemann sphere. In other words, we will view the set $\mathbb{P}^1(\mathbb{C}_p)$ as $\mathbb{C}_p \cup \{\infty\}$; we have the inclusions

$$\mathbb{Z}_p \subset \mathbb{Q}_p \subset \overline{\mathbb{Q}}_p \subset \mathbb{C}_p \subset \mathbb{P}^1(\mathbb{C}_p).$$

We will consider $\mathbb{C}_p$ and all its subsets to be metric spaces under the metric induced by $|\cdot|_p$, with the corresponding metric topology. It should be noted that $\mathbb{Q}_p$ and its finite extensions are complete and locally compact, but not algebraically closed. $\overline{\mathbb{Q}}_p$, on the other hand, is algebraically closed but neither complete nor locally compact; $\mathbb{C}_p$ is both algebraically closed and complete, but not locally compact. In Section 2.3, we will define a metric on $\mathbb{P}^1(\mathbb{C}_p)$ which, while not restricting to the same metric on $\mathbb{C}_p$, induces the same topology on $\mathbb{C}_p$.

For $a \in \mathbb{C}_p$ and $r > 0$, we define $D_r(a)$ to be the open disk of radius r centered at a; in other words,

$$D_r(a) = \{z \in \mathbb{C}_p : |z - a| < r\}.$$

Similarly, $\overline{D}_r(a)$ is the closed disk of radius r centered at a:

$$\overline{D}_r(a) = \{z \in \mathbb{C}_p : |z - a| \le r\}.$$

Note that if $r \notin p^{\mathbb{Q}}$, then $D_r(a) = \overline{D}_r(a)$. We will call such disks *irrational.* For $r \in p^{\mathbb{Q}}$, we will say that $D_r(a)$ and $\overline{D}_r(a)$ are *rational* open and closed disks, respectively.

We will see in Proposition 2.2.2 that all disks are both open and closed as topological sets. It should therefore be emphasized that the terms "open" and "closed", when applied to a disk, refer only to whether or not the disk contains points of exact distance r from the center.

For any two subsets S_1 and S_2 of a metric space with distance d, we define

$$\mathrm{dist}(S_1, S_2) = \inf\{d(x, y) : x \in S_1, y \in S_2\}$$

to be the distance between S_1 and S_2. If one or both of S_1 and S_2 is a singleton $\{a\}$ or $\{b\}$, we will often abuse notation and write $\mathrm{dist}(a, S_2)$, $\mathrm{dist}(S_1, b)$, or $\mathrm{dist}(a, b)$. If D is a disk (open or closed), we define the radius of D to be

$$\mathrm{rad}(D) = \sup\{d(x, y) : x, y \in D\}.$$

This value is what is usually referred to as the diameter. However, for disks in a non-Archimedean metric space, the notions of diameter and radius coincide. In other words, if $D \subset \mathbb{C}_p$ is an open disk, $a \in D$, and $r = \mathrm{rad}(D)$, then $D = D_r(a)$; if D is a closed disk, $a \in D$, and $r = \mathrm{rad}(D)$, then $D = \overline{D}_r(a)$.

Given a rational function $\phi \in \mathbb{C}_p(z)$, then ϕ acts on $\mathbb{P}^1(\mathbb{C}_p)$ as a morphism

$$\phi : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p).$$

We will denote by $\phi^n(z)$ the n-fold compostion

$$\phi^n = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}.$$

In this thesis, we will study the dynamics of ϕ as it acts on $\mathbb{P}^1(\mathbb{C}_p)$.

2.2 Basic p-adic properties

The absolute value $|\cdot|_p$ on $\mathbb{C}_p$ is non-Archimedean; that is to say, it satisfies the ultrametric triangle inequality

$$|x + y| \le \max\{|x|, |y|\} \quad \text{for any } x, y \in \mathbb{C}_p$$

It follows that if $|x| \ne |y|$, then in fact $|x+y| = \max\{|x|, |y|\}$. Intuitively, this means that no matter how many "small" numbers you add, the sum can never get "big". As a result, a series converges if and only if its terms approach zero.

We also have the following useful standard theorem on roots of p-adic power series. It is a generalization of Hensel's Lemma; the key tool in the proof is Newton's method. However, we omit the proof, and instead refer the interested reader to [16, pages 97–108].

Theorem 2.2.1. *(Roots of p-adic Power Series.) Let*

$$f(z) = \sum_{i=0}^{\infty} c_i z^i, \quad c_i \in \mathbb{C}_p$$

be a power series in $\mathbb{C}_p[[z]]$*. Fix* $r > 0$ *such that*

$$\lim_{i \to \infty} |c_i| r^i = 0.$$

Then $f = 0$ *has a root* $\alpha \in \mathbb{C}_p$ *with* $|\alpha| = r$ *if and only if there exist nonnegative integers* $m < n$ *with*

$$|c_m| r^m = |c_n| r^n = \sup_{i \geq 0} \{|c_i| r^i\}. \tag{2.1}$$

Furthermore, in this case, if m *is the smallest integer achieving the supremum, and* n *is the largest, then* $n - m$ *is the number of roots of absolute value* r*, counting multiplicity.*

The condition $\lim_{i\to\infty} |c_i| r^i$ merely states that f converges on $\overline{D}_r(0)$, provided $r \in p^{\mathbb{Q}}$. If $r \notin p^{\mathbb{Q}}$, then f can have no roots of absolute value r, since $r \notin |\mathbb{C}_p|$. Condition (2.1) is often written in terms of the p-adic valuation as follows. If $b \in \mathbb{C}_p$ with $|b| = r$, then f has a root $\alpha \in \mathbb{C}_p$ with $v(\alpha) = v(b)$ if and only if there exist integers $n > m \geq 0$ such that

$$v(c_m) + mv(b) = v(c_n) + nv(b) = \inf_{i \geq 0} \{v(c_i) + iv(b)\}.$$

The non-Archimedean property also has interesting consequences for the geometry of $\mathbb{C}_p$, as the following elementary proposition shows.

Proposition 2.2.2.

1. *All disks of positive radius in* $\mathbb{C}_p$ *are both open and closed as topological sets.*
2. *Any point of a disk in* $\mathbb{C}_p$ *is a center. More precisely, if* $b \in D_r(a)$ *(resp.,* $\overline{D}_r(a)$*), then*
$$D_r(a) = D_r(b) \quad \text{(resp., } \overline{D}_r(a) = \overline{D}_r(b)\text{)}.$$
3. *If* D_1 *and* D_2 *are two disjoint disks in* $\mathbb{C}_p$*, and if* $a \in D_1$ *and* $b \in D_2$*, then*
$$\operatorname{dist}(D_1, D_2) = |a - b|.$$
4. *If two disks in* $\mathbb{C}_p$ *intersect, then one contains the other.*

The first three statements are direct consequences of the definitions and the ultrametric triangle inequality. The fourth follows from the second; if two disks both contain a given point, then that point is a center for both, and hence the disk of larger radius contains the other.

We will often consider the restriction of a rational function to a disk in $\mathbb{C}_p$; the following lemma will come in handy.

Lemma 2.2.3. *Let $\phi(z) \in \mathbb{C}_p(z)$ and let $D \subset \mathbb{C}_p$ be a disk (open or closed). Suppose that ϕ has no poles in D, i.e., that $\phi^{-1}(\{\infty\}) \cap D = \emptyset$. Then for any $a \in D$, there is a power series $f(z) \in \mathbb{C}_p[[z-a]]$ convergent on D such that for all $z \in D$, $\phi(z) = f(z)$.*

Proof. We will assume $D = \overline{D}_r(a)$ is a rational closed disk; the proof for open and irrational disks is similar. By a change of coordinates, we may assume that $a = 0$. (Specifically, we can consider the function $\phi(z + a)$ instead of $\phi(z)$.) Write $\phi(z) = g(z)/h(z)$, with $g, h \in \mathbb{C}_p[z]$, in lowest terms. Write $g(z) = b_0 + b_1 z + \cdots b_m z^m$ and $h(z) = c_0 + c_1 z + \cdots + c_n z^n$. By assumption, $h(z) \neq 0$ for all $z \in D$. Thus, the constant term c_0 of h is nonzero, and by Theorem 2.2.1, $|c_i|r^i < |c_0|$ for all $i \geq 1$. It follows that for all $z \in D$, $|c_1 z + c_2 z^2 + \cdots + c_n z^n| < |c_0|$, and hence the power series

$$f_0(z) = \frac{1}{c_0} \sum_{i=0}^{\infty} \left(-\frac{c_1}{c_0} z - \frac{c_2}{c_0} z^2 - \cdots - \frac{c_n}{c_0} z^n \right)^i$$

converges on D and equals $1/h(z)$ for all $z \in D$. Thus, $f(z) = g(z) f_0(z)$ is the desired power series. □

2.3 The projective line and the chordal metric

We have thus far studied properties of the metric on $\mathbb{C}_p$ induced by the p-adic absolute value. This metric is invariant under translation and multiplication by units (i.e., elements of $\mathcal{O}^*$). We would like to have a metric on $\mathbb{P}^1(\mathbb{C}_p)$ based on the metric from $\mathbb{C}_p$ but invariant under $\mathrm{PGL}(2, \mathcal{O})$.

We begin by recalling the action of $\mathrm{PGL}(2, \mathbb{C}_p)$ on $\mathbb{P}^1(\mathbb{C}_p)$. Elements of $\mathbb{P}^1(\mathbb{C}_p)$ may be written as $[x, y]$ with $x, y \in \mathbb{C}_p$ not both zero, with the equivalence $[x, y] = [cx, cy]$ for $c \in \mathbb{C}_p^*$. The inclusion of $\mathbb{C}_p$ in $\mathbb{P}^1(\mathbb{C}_p)$ takes $z \in \mathbb{C}_p$ to $[z, 1]$; $[1, 0]$ is the point at infinity. An element

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\mathrm{PGL}(2, \mathbb{C}_p)$ takes $[x, y]$ to $[ax + by, cx + dy]$. We will often consider $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ and abuse notation by writing

$$f(z) = \frac{az + b}{cz + d}.$$

In fact, the group $\mathrm{PGL}(2, \mathbb{C}_p)$ is the full group of automorphisms of the variety $\mathbb{P}^1(\mathbb{C}_p)$.

Note that for any domain R, $\mathrm{PGL}(2, R)$ is generated by the matrices

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (2.2)$$

where $b \in R$ and $c \in R^*$. Note that these matrices correspond to the functions $z \mapsto z + b$, $z \mapsto cz$, and $z \mapsto 1/z$.

The chordal (or spherical) metric we will define on $\mathbb{P}^1(\mathbb{C}_p)$ is analogous to the standard chordal metric on the Riemann sphere, where the metric is inherited from the standard embedding of the Riemann sphere in $\mathbb{R}^3$. As in [13], if $[x_1, y_1]$ and $[x_2, y_2]$ are two points in $\mathbb{P}^1(\mathbb{C}_p)$, we define

$$d([x_1, y_1], [x_2, y_2]) = \frac{|x_1 y_2 - x_2 y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}$$

or, viewing $\mathbb{P}^1(\mathbb{C}_p)$ as $\mathbb{C}_p \cup \{\infty\}$,

$$d(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}}.$$

It is clear from the above definition that for $z_1, z_2 \in \overline{D}_1(0) = \mathcal{O}$, their distance is $d(z_1, z_2) = |z_1 - z_2|_p$; thus, our new metric on $\mathbb{P}^1(\mathbb{C}_p)$ agrees with the old one on the unit disk. In addition, it is easy to verify that the metric is invariant under the matrices of equation (2.2) for $b \in \mathcal{O}$ and $c \in \mathcal{O}^*$. Thus, the metric is invariant under $\mathrm{PGL}(2, \mathcal{O})$, as desired.

We will be using the chordal metric to define the notion of equicontinuity of a family of maps on $\mathbb{P}^1(\mathbb{C}_p)$. However, because it is the same as the standard metric on $\overline{D}_1(0)$, in practice we will often change coordinates to ensure that all the points of interest are in the unit disk.

2.4 Dynamics

In its greatest generality, a discrete dynamical system is a set X with a map $\phi : X \to X$. In this thesis, we will usually let $X = \mathbb{P}^1(\mathbb{C}_p)$ and let $\phi \in K(z)$ (for K a finite extension of $\mathbb{Q}_p$) be a rational map; in other words, ϕ is any algebraic morphism, defined over $\overline{\mathbb{Q}}_p$, of $\mathbb{P}^1(\mathbb{C}_p)$ to itself. We wish to study the iterates $\{\phi^n\}$ obtained by composition. Our study will parallel that of complex dynamics, where $X = \mathbb{P}^1(\mathbb{C})$ and $\phi \in \mathbb{C}(z)$; we will borrow both terminology and philosophy from that field. We refer the reader to [6] or [23] for a more detailed background on complex dynamics; however, all the terms and concepts needed for this thesis will be defined in this section.

2.4.1 Periodic points

To study the iteration of a map $\phi : X \to X$, we often consider an element $x \in X$ and its *forward orbit*, $\{\phi^n(x)\}_{n \geq 0}$. A point $x \in X$ is said to be *fixed* if $\phi(x) = x$, that is, if its forward orbit consists of one point. Similarly, a point x is said to be *periodic* (of period n) if $\phi^n(x) = x$ for some $n \geq 1$; the forward orbit of x is then called a *(periodic) cycle*. Note that for x to have period n under the map ϕ is the same as to be fixed by the map ϕ^n. If n is the smallest positive integer such that $\phi^n(x) = x$, we say that x has *exact period* n. Finally, if there is $m \geq 0$ such that $\phi^m(x)$ is periodic, we say x is *preperiodic*. Note that x is preperiodic if and only if it has finite forward orbit.

Without more structure on the set X, not much else can be said. However, as in complex dynamics, the metrics on $\mathbb{C}_p$ and $\mathbb{P}^1(\mathbb{C}_p)$ give us much more information. In particular, we can take derivatives. If $x \in \mathbb{C}_p \subset \mathbb{P}^1(\mathbb{C}_p)$ is a fixed point of $\phi(z) \in \mathbb{C}_p(z)$, then we define the *multiplier* of x to be $\phi'(x) \in \mathbb{C}_p$. Similarly, if x is periodic of exact period n, we consider it to be a fixed point of ϕ^n and define its multiplier to be $(\phi^n)'(x)$. If the forward orbit of x is contained in $\mathbb{C}_p$, then by the chain rule, the multiplier of x is the product of the derivatives at all its iterates, i.e.,

$$(\phi^n)'(x) = \phi'(x) \cdot \phi'(\phi(x)) \cdot \phi'(\phi^2(x)) \cdot \cdots \cdot \phi'(\phi^{n-1}(x)).$$

If x is a periodic point with multiplier λ, we say x is

attracting	if $\lvert\lambda\rvert < 1$.
superattracting	if $\lambda = 0$.
repelling	if $\lvert\lambda\rvert > 1$.
neutral	if $\lvert\lambda\rvert = 1$.
rationally neutral	if λ is a root of unity.
irrationally neutral	if $\lvert\lambda\rvert = 1$ and λ is not a root of unity.

If x is a finite fixed point of $\phi(z) \in \mathbb{C}_p(z)$, then ϕ can be expanded as a power series in a neighborhood of x:

$$\phi(z) = \sum_{i=1}^{\infty} c_i (z - x)^i.$$

There is no constant term, since $\phi(x) = x$. Note that c_1 is the multiplier of x. If x is attracting, we can pick a small enough neighborhood U of x such that the linear term dominates, and so all points in U are moved closed to x under iteration. Similarly, if x is repelling, we can choose U so that all points in U are moved further from x under iteration, whence the names "attracting" and "repelling". We will discuss these phenomena more rigorously in Section 2.4.3.

2.4.2 Change of coordinates

If $f \in \mathrm{PGL}(2, \mathbb{C}_p)$ is an automorphism of $\mathbb{P}^1(\mathbb{C}_p)$, and if $\phi \in \mathbb{C}_p(z)$, then the conjugated function $\psi = f^{-1} \circ \phi \circ f$ is also an element of $\mathbb{C}_p(z)$. Furthermore, $\psi^n = f^{-1} \circ \phi^n \circ f$. Thus, $x \in \mathbb{P}^1(\mathbb{C}_p)$ is periodic under ϕ if and only if $f^{-1}(x)$ is periodic under ψ. In fact, ψ can be viewed as the same morphism on $\mathbb{P}^1(\mathbb{C}_p)$, but in a different coordinate system, with change of coordinates given by f. While change of coordinates does not in general preserve values of derivatives, it does preserve multipliers. Change of coordinates also preserves critical points; if $\phi'(x) = 0$, then $\psi'(f^{-1}(x)) = 0$.

If we write $\phi(z) = g(z)/h(z)$ as a quotient of relatively prime polynomials, then the geometric degree d of ϕ (i.e., the number of inverse images of a point, generically) is equal to the maximum of the degrees of g and h. Counting multiplicity, ϕ will have $d+1$ fixed points, and for $n \geq 1$, $d^n + 1$ periodic points of period n (that is, of exact period dividing n); it will also have $2d-2$ critical points. All these points, as well as all preperiodic points, have coordinates which are roots of algebraic equations with coefficients determined algebraically by the coefficients of ϕ. Thus, if ϕ is defined over $\overline{\mathbb{Q}}_p$, then so are all its preperiodic and critical points.

2.4.3 The Fatou and Julia sets

Two of the most fundamental objects in the study of complex dynamics are the Fatou and Julia sets associated to a rational map. The motivating idea is that, given a map ϕ of the sphere, there are some areas where two nearby points stay nearby no matter how many times we apply ϕ, while there are other areas where two nearby points can be moved far apart. Areas of the first sort make up the Fatou set, and those of the second make up the Julia set.

There are essentially two ways to define the Fatou set for complex rational functions. The first uses the notion of normality, and the second uses the notion of equicontinuity; they can be shown to be equivalent via the Arzela-Ascoli theorem. It is possible to define the Fatou set for p-adic functions using normality, but the fact that $\mathbb{C}_p$ is not locally compact would be a continual annoyance. Instead, we turn to equicontinuity, and we begin by recalling its definition.

Definition 2.4.1. *Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let F be a family of functions $\{f : X \to Y\}$. We say that F is an* equicontinuous family *if there exists a constant $C > 0$ such that for all $x_1, x_2 \in X$ and for all $f \in F$,*

$$d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2).$$

Thus, if F is equicontinuous, then every function in F is continuous and, in fact, uniformly continuous. Moreover, and more importantly, they are all uniformly continuous with *the same* constant of uniformity. In other words, if two points in X are close to each other, no element of F can take them very far apart.

In [14], Hsia proved the following useful theorem for p-adic analytic families.

Theorem 2.4.1. *(Hsia) Let D be a disk centered at a, and let F be a family of power series in $\mathbb{C}_p[[z-a]]$ which converge on D. Suppose that there is some value $c \in \mathbb{C}_p$ such that for all $f \in F$ and $x \in D$, $f(x) \neq c$. Then F is an equicontinuous family with respect to the chordal metric on $\mathbb{P}^1(\mathbb{C}_p)$.*

By changing coordinates if needed, Hsia's theorem says that if a family of p-adic power series on a disk omits any two points of the projective line, then it is an equicontinuous family. Note that this p-adic result is stronger than the otherwise analogous theorem of Montel, which, in the complex case, requires the omission of three points of the Riemann sphere to guarantee equicontinuity. This strengthening is the first evidence we will see of a theme which will arise often: equicontinuity is more common in the p-adic universe than in the complex universe.

We are now ready to define p-adic Fatou and Julia sets, following [13].

Definition 2.4.2. *Let $\phi \in \mathbb{C}_p(z)$ be a rational function, and let $F = \{\phi^n\}_{n\geq 1}$ be the family of iterates of ϕ. For any subset S of $\mathbb{P}^1(\mathbb{C}_p)$, let $F_S = \{f|_S : f \in F\}$. We define the* Fatou set *of ϕ to be*

$$\mathcal{F} = \mathcal{F}_\phi = \left\{x \in \mathbb{P}^1(\mathbb{C}_p) : \exists\, U \subset \mathbb{P}^1(\mathbb{C}_p) \text{ open, with } x \in U \text{ and } F_U \text{ equicontinuous}\right\}.$$

We define the Julia set *of ϕ to be*

$$\mathcal{J} = \mathcal{J}_\phi = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathcal{F}_\phi.$$

Once again, equicontinuity in the above definition is with respect to the chordal metric.

2.4.4 First properties of the Fatou and Julia sets

It is immediate from the definitions that the Fatou set is open and the Julia set is closed. It is only slightly less immediate that $\mathcal{F}_\phi$ and $\mathcal{J}_\phi$ are invariant under the application of ϕ. That is, $\phi(\mathcal{F}_\phi) = \phi^{-1}(\mathcal{F}_\phi) = \mathcal{F}_\phi$, and $\phi(\mathcal{J}_\phi) = \phi^{-1}(\mathcal{J}_\phi) = \mathcal{J}_\phi$. A little more work shows that for any $n \geq 1$, $\mathcal{F}_{\phi^n} = \mathcal{F}_\phi$; see [13] for details. In addition, for any $f \in \mathrm{PGL}(2, \mathbb{C}_p)$, if we let $\psi = f^{-1} \circ \phi \circ f$, then $\mathcal{F}_\psi = f^{-1}(\mathcal{F}_\phi)$, and $\mathcal{J}_\psi = f^{-1}(\mathcal{J}_\phi)$.

The idea, mentioned at the end of Section 2.4.1, that attracting and repelling periodic points live up to their names indicates that attracting points should be in the Fatou set, and repelling points should be in the Julia set. In fact, as the following proposition shows, the absolute value of the multiplier of any periodic point determines whether it is Julia or Fatou.

Proposition 2.4.2. *Let $\phi \in \mathbb{C}_p(z)$ be a rational function, and let $\mathcal{F}$ and $\mathcal{J}$ be its Fatou and Julia sets. Let x be a periodic point of ϕ with multiplier λ. If $|\lambda| \leq 1$, then $x \in \mathcal{F}$; if $|\lambda| > 1$, then $x \in \mathcal{J}$.*

Proof. Let n be the exact period of x. Since $\mathcal{J}_{\phi^n} = \mathcal{J}_\phi$, we may consider ϕ^n in place of ϕ. Thus, we may assume without loss that x is a fixed point of ϕ, with $\phi'(x) = \lambda$. Furthermore, by a change of coordinates, we may assume that $x = 0$.

If $|\lambda| > 1$ (i.e., 0 is repelling), suppose that $0 \in \mathcal{F}$. Then there is a neighborhood U on which $\{\phi^n\}$ is equicontinuous. U contains a disk $\overline{D}_r(0)$; by decreasing r if necessary, we can assume that $\overline{D}_r(0)$ contains no poles, and that $r \in p^{\mathbb{Q}}$. By Lemma 2.2.3, we may write $\phi|_{\overline{D}_r(0)}$ as

$$\phi(z) = \sum_{i=1}^{\infty} c_i z^i, \tag{2.3}$$

with $|c_i| r^i \to 0$. Note that $c_1 = \lambda$. Decreasing r again, we may assume that for all $i \geq 2$, $|c_1| r > |c_i| r^i$.

Now let $C > 0$ be the constant of equicontinuity on U; in particular, $|\phi^m(y)| \leq C|y|$, for all $y \in \overline{D}_r(0)$ and $m \geq 0$. Pick a positive integer m such that $|c_1|^m > C$, and pick $y \in \overline{D}_r(0)$ such that $0 < |c_1^m y| < r$. Then

$$|\phi^m(y)| = |c_1^m y| > C|y|,$$

and we have a contradiction; hence, $0 \in \mathcal{J}$.

If $|\lambda| \leq 1$ (i.e., 0 is attracting or neutral), then pick $r \in p^{\mathbb{Q}}$ such that ϕ has no poles on $\overline{D}_r(0)$. Expand ϕ as a power series on $\overline{D}_r(0)$ as in equation (2.3), with $|c_i|r^i \to 0$. Decrease r so that for all $i \geq 1$, $|c_i|r^{i-1} \leq 1$; this is possible because $|c_1| \leq 1$. We will show that $\{\phi^n\}$ is equicontinuous on $\overline{D}_r(0)$ with constant $C = 1$.

Pick any $x, y \in \overline{D}_r(0)$. Then

$$|\phi(x) - \phi(y)| \leq \max_{i\geq 1} \left\{ \left| c_i(x^i - y^i) \right| \right\} = |x - y| \max_{i\geq 1} \left\{ \left| c_i \sum_{j=0}^{i-1} x^j y^{i-j-1} \right| \right\}$$

$$\leq |x - y| \max_{i\geq 1} \{ |c_i| r^{i-1} \} \leq |x - y|.$$

Thus, the distance between any two points in $\overline{D}_r(0)$ cannot increase under application of ϕ; in addition, by taking one of the points to be 0, we see that any point in $\overline{D}_r(0)$ must remain in $\overline{D}_r(0)$ under application of ϕ. So applying any iterate ϕ^n, two points in the disk cannot move further apart. Hence, the family is equicontinuous, and $0 \in \mathcal{F}$, as desired. □

For non-neutral periodic points, the above proof is essentially the same as the proof in the complex case. However, the complex case is very different for neutral points; some complex neutral points (including all rationally neutral points) are in the Julia set. Again, equicontinuity is more common in the non-Archimedean world.

Proposition 2.4.3. *Let $\phi \in \mathbb{C}_p(z)$ be a rational function of degree d. Suppose that none of its fixed points have multiplier* 1. *If we let $\lambda_0, \ldots, \lambda_d$ denote the multipliers of the $d + 1$ fixed points of ϕ in $\mathbb{P}^1(\mathbb{C}_p)$, then*

$$\sum_{i=0}^{d} \frac{1}{1 - \lambda_i} = 1.$$

Proof. We may assume, by changing coordinates if necessary, that ∞ is not a fixed point. Now our desired equality is known to hold in the complex case, by the use of contour integration; see, for example, [23]. To extend the result to $\mathbb{C}_p$, fix the degree of ϕ and note that $f = \sum(1 - \lambda)^{-1}$ is a rational function, symmetric in the λ_i. However, any rational function symmetric in the roots of a polynomial is equal to some rational function of the coefficients of the polynomial. Since the λ_i are the roots of a polynomial whose coefficients are determined by those of ϕ, we see that f is a rational function in the coefficients of ϕ. However, we know from the complex case that if the coefficients of ϕ are in $\mathbb{Q}$, then $f = 1$; thus, f must be identically equal to 1, regardless of the field of definition of ϕ. □

Corollary 2.4.4. *Let $\phi \in \mathbb{C}_p(z)$. Then ϕ has a non-repelling fixed point.*

Proof. If any fixed point multiplier is equal to 1, then we have a neutral fixed point. If not, we apply Proposition 2.4.3 to get $\sum(1 - \lambda)^{-1} = 1$. Suppose that every fixed point were repelling; then all the $|\lambda|$ would be bigger than 1, and so every term of the sum would have absolute value less than 1. But then, by ultrametricity, the terms could not add up to 1. Thus, at least one fixed point must be non-repelling. □

Corollary 2.4.5. *Let $\phi \in \mathbb{C}_p(z)$. Then $\mathcal{F}_\phi \neq \emptyset$.*

Proof. Immediate from Proposition 2.4.2 and Corollary 2.4.4. □

While a p-adic Fatou set must be nonempty, the same is not true for the Julia set. In fact, Morton and Silverman proved in [25] that if $\phi(z) \in \mathbb{C}_p(z)$ has good reduction, then $\mathcal{J}_\phi = \emptyset$. We refer the reader to their paper, and to Chapter 8 of this thesis, for the definition of good reduction and further details.

Thus, in the p-adic case, the Julia set can be empty, while the Fatou set must be nonempty. This is in sharp contrast with the complex case, where the opposite is true: the Fatou set can be empty, while the Julia set (for maps of degree at least two) must be nonempty. Ultrametricity once again allows more equicontinuity. In addition, $\mathbb{C}_p$ is not locally compact; this is a key difference, allowing a map of high degree to be nowhere expanding. It is precisely the local compactness of $\mathbb{C}$ which forces a high degree complex map to have a Julia set.

On the other hand, the failure of local compactness can also cause some difficulties in p-adic dynamics. For example, in the complex case, the Julia set is always compact, because it is a closed subset of a compact space. However, the Julia set of a p-adic map need not be compact; in fact, most nonempty p-adic Julia sets are not compact. Here is a simple example.

Example. Let p be any prime, and let

$$\phi(z) = \frac{z^3 + pz}{z + p^2}.$$

We will prove that ϕ has a non-compact p-adic Julia set. Note that $\phi(0) = 0$, and

$$\phi'(z) = \frac{2z^3 + 3p^2z^2 + p^3}{(z + p^2)^2},$$

so $\phi'(0) = p^{-1}$, and $|p^{-1}| = p > 1$, so 0 is repelling; hence, $0 \in \mathcal{J}$.

Claim. For any $n \geq 1$, there exists $a_n \in \mathbb{C}_p$ with $v_p(a_n) = 2^{-n}$ and $\phi^n(a_n) = 0$.

Let us assume the claim for the moment. Because of the invariance of $\mathcal{J}$ under ϕ, it follows that $a_n \in \mathcal{J}$ for all $n \geq 1$. Furthermore, the sequence $\{a_n\}$ has no accumulation points in $\mathbb{C}_p$; for any positive integers $n \neq m$,

$$|a_n - a_m| = \max\{|a_n|, |a_m|\} = \max\{p^{-2^{-n}}, p^{-2^{-m}}\} \geq p^{-\frac{1}{2}}.$$

Thus, the claim implies that $\mathcal{J}$ is non-compact.

We will prove the claim by induction on n. For $n = 1$, note that $\phi(\sqrt{-p}) = 0$, so we can choose $a_1 = \sqrt{-p}$. To prove the inductive step, it will suffice to show that for any $x \in \mathbb{C}_p$ with $v_p(x) = 2^{-n}$, there is $y \in \mathbb{C}_p$ with $\phi(y) = x$ and $v_p(y) = 2^{-n-1}$.

Solving $\phi(z) = x$ for $v_p(x) = 2^{-n}$, we see that the pre-images of x are the roots of

$$z^3 + (p - x)z - p^2x = 0.$$

Viewing this polynomial as a power series with coefficients $c_0 = -p^2x$, $c_1 = (p - x)$, $c_3 = 1$, and all other coefficients zero, we see that $v(c_0) = 2 + v(x)$, $v(c_1) = v(x)$, and $v(c_3) = 1$; therefore, by Theorem 2.2.1, one root has valuation 2 and the other two have valuation $v(x)/2 = 2^{-n-1}$. We can pick y to be either of the latter two roots, and we are done.

2.5 Rigid Analysis

The theory of rigid analysis, originally proposed by Tate ([33]), was developed for the purpose of analytic continuation in the non-Archimedean setting. In contrast to complex analysis, it is not useful to define non-Archimedean "analytic functions" to be those functions which locally can be written as power series, because too many functions would qualify. For instance, the function

$$f(z) = \begin{cases} 1 & \text{if } |z| \geq 1 \\ 0 & \text{if } |z| < 1 \end{cases}$$

would be analytic on $\mathbb{C}_p$, by this local definition. In such a situation, analytic continuation would be meaningless. The starting point of rigid analysis is to limit the set of allowed functions to the so-called "restricted power series" or "rigid analytic functions"; on $\overline{D}_1(0)$, these are all functions which can be expressed as a single power series centered at the origin and converging on the whole disk.

In this section we will present a brief overview of the basics of rigid analysis. We will state some definitions and fundamental theorems, but we will omit most of the proofs; we refer the interested reader to Tate's original lectures ([33]), as well as to the expositions in [3, 9, 11]. The main facts we will need are those concerning affinoid domains. The reader may feel free to skip even the definition of affinoid domains, and instead begin with their basic properties and their characterization in Proposition 2.5.3.

We will always be working over the field $\mathbb{C}_p$; however, most of what we say is true over any complete non-Archimedean field (though some modifications would have to be made for a field which is not algebraically closed).

2.5.1 Tate algebras and maximal ideal sets

Even though we will be restricting our attention to dimension one, we must use higher-dimensional spaces to define affinoid domains. We therefore begin by defining the *n-dimensional polydisk* to be

$$D^n = \left(\overline{D}_1(0)\right)^n = \left\{(z_1, \ldots, z_n) \in \mathbb{C}_p^n : |z_i| \leq 1\right\}.$$

We will often abbreviate notation by writing ν in place of $(i_1, \ldots, i_n)$; z^ν will denote $z_1^{i_1} \cdots z_n^{i_n}$. Similarly, $\nu \geq 0$ will mean that each index i_j is nonnegative, and $\|\nu\|$ will denote $\max\{i_1, \ldots, i_n\}$. The *ring of restricted power series* $T_n = \mathbb{C}_p\langle z_1, \ldots, z_n\rangle$ on D^n is

$$T_n = \left\{\sum_{\nu \geq 0} c_\nu z^\nu \mid c_\nu \in \mathbb{C}_p \text{ and } c_\nu \to 0 \text{ as } \|\nu\| \to \infty\right\}.$$

In other words, T_n is the set of all power series about 0 which converge on D^n. We define a norm on T_n as follows: for $f(z) = \sum c_\nu z^\nu$,

$$\|f\| = \max_\nu |c_\nu|.$$

Under this norm, T_n is a Banach $\mathbb{C}_p$-algebra. Note that the points of D^n are in one-to-one correspondence with the set $\operatorname{Max} T_n$ of maximal ideals of T_n.

A Banach $\mathbb{C}_p$-algebra A is said to be a *Tate algebra* (or *affinoid algebra*) if there is a continuous surjective $\mathbb{C}_p$-algebra homomorphism $T_n \to A$. In other words, a Tate algebra is a Banach $\mathbb{C}_p$-algebra which is isomorphic to T_n/I for some ideal I; the norm on T_n/I is given by

$$\|f\| = \inf\{\|g\|_{T_n} : f - g \in I\}.$$

Tate showed that all ideals of T_n are closed; hence, any ideal I produces a Tate algebra T_n/I. Tate algebras are also said to be *topologically of finite type.*

Because $\mathbb{C}_p$ is algebraically closed, it can be shown that any maximal ideal x of A has residue field $\mathbb{C}_p$; thus, any $f \in A$ defines a map from $\operatorname{Max} A$ to $\mathbb{C}_p$, where $\operatorname{Max} A$ denotes the set of maximal ideals of A. If $X = \operatorname{Max} A$, we will often refer to X as an *affinoid domain.* We can consider X to be a closed subspace of D^n (where $T_n \to A$ surjectively); it is therefore a topological space, with topology inherited from D^n. However, we will often wish to consider A to be a subset of a lower-dimensional space, as the following example illustrates.

Example. Let $A = \mathbb{C}_p\langle z_1, z_2\rangle/(z_1 z_2 - 1)$. Then $\operatorname{Max} A$ is a subset of D^2. However, A is the ring of two-sided power series $\mathbb{C}_p\langle z, \frac{1}{z}\rangle$ in one variable, convergent for $|z| \le 1$ and $\left|\frac{1}{z}\right| \le 1$; thus, $\operatorname{Max} A$ may be viewed as the annulus $\{z \in \mathbb{C}_p : |z| = 1\}$.

To make this identification more rigorous, we need to define morphisms. We define the category $\mathcal{A}$ of Tate algebras by letting morphisms be continuous $\mathbb{C}_p$-algebra homomorphisms. Tate proved that every maximal ideal of a Tate algebra has finite codimension; it follows that a morphism $\phi : A \to B$ of Tate algebras induces a map $\tilde{\phi} : \operatorname{Max} B \to \operatorname{Max} A$ by

$$\tilde{\phi}(m) = \phi^{-1}(m).$$

In fact, $\tilde{\phi}$ is a continuous map of topological spaces.

2.5.2 Affinoid subdomains

Following Tate, we now define subdomains of affinoid domains functorially.

Definition 2.5.1. *Let A be a Tate algebra, $X = \operatorname{Max} A$, and $Y \subseteq X$ a subset. We say that Y is an* affinoid subdomain *(or* affine subset*) of X if the functor*

$$F : \mathcal{A} \to \{Sets\}$$

given by

$$F(C) = \{\phi \in \operatorname{Hom}(A, C) : \tilde{\phi}(\operatorname{Max} C) \subset Y\}$$

is representable.

In other words, Y is an affinoid subdomain of X if there is a Tate algebra B_Y and a morphism $\psi : A \to B_Y$ such that for any Tate algebra C, the map

$$\operatorname{Hom}(B_Y, C) \to \{\phi \in \operatorname{Hom}(A, C) : \tilde{\phi}(\operatorname{Max} C) \subset Y\}$$

given by

$$\phi' \mapsto \phi' \circ \psi$$

is a natural bijection. In more colloquial language, B_Y behaves as if its maximal ideal space were Y; it turns out that in fact there is a natural bijection $\operatorname{Max} B_Y \cong Y$. This bijection comes from the map $\psi : A \to B_Y$ induced by the identity morphism in $\operatorname{Hom}(B_Y, B_Y)$.

We will soon state a much less abstract characterization of affinoids. However, the functorial definition is useful for proving many fundamental properties. For instance, we have the following proposition.

Proposition 2.5.1.

1. *Let A and B be Tate algebras, and let $X = \operatorname{Max} A$ and $Y = \operatorname{Max} B$. Let $\phi : A \to B$, and let U be an affinoid subdomain of X. Then $\tilde{\phi}^{-1}(U)$ is an affinoid subdomain of Y.*

2. *Let A be a Tate algebra, and let U and V be affinoid subdomains of $X = \operatorname{Max} A$. Then $U \cap V$ is an affinoid subdomain of X. If U and V are disjoint, then $U \cup V$ is an affinoid subdomain of X.*

3. *Let U and V be any affinoid subdomains of D^1. Then $U \cup V$ is an affinoid subdomain of D^1.*

Because of the "rigidity" of affinoids, we have a notion of connectedness, in spite of the fact that $\mathbb{C}_p$ is totally disconnected as a topological space.

Definition 2.5.2. *Let A be a Tate algebra, and $X = \operatorname{Max} A$. If A cannot be written as a direct sum $A_1 \oplus A_2$ of nonzero Tate algebras, then we say X is a* connected *affinoid domain.*

Any affinoid domain X is a finite disjoint union of connected affinoid subdomains; we will refer to these connected subdomains as the *connected components* of X. Note that the disjoint union of two nonempty affinoids cannot be connected.

We still have very few examples of affinoid domains. The following definition will help to remedy this situation.

Definition 2.5.3. *Let A be a Tate algebra and $X = \operatorname{Max} A$. Let $g, f_1, \ldots, f_n \in A$ generate the unit ideal (i.e., there are no zeros common to all). A* rational subdomain *of X is a set of the form*

$$X\left(\frac{f}{g}\right) = \{x \in X : |f_i(x)| \le |g(x)|, i = 1, \ldots, n\}.$$

Any rational subdomain of X is in fact an affinoid subdomain; the representing Tate algebra is

$$A\left(\frac{f}{g}\right) = \left\{\sum_{\nu \ge 0} c_\nu \left(\frac{f}{g}\right)^\nu : c_\nu \to 0 \text{ as } \|\nu\| \to \infty\right\},$$

where $\nu = (i_1, \ldots, i_n)$, $c_\nu \in \mathbb{C}_p$, and

$$\left(\frac{f}{g}\right)^\nu = \frac{f_1^{i_1} \cdots f_n^{i_n}}{g^{i_1+\cdots+i_n}}.$$

In fact, all affinoid subdomains can be constructed from rational subdomains, as the following important theorem shows.

Theorem 2.5.2. *(Gerritzen, Grauert) Let A be a Tate algebra, $X = \operatorname{Max} A$, and $Y \subset X$ an affinoid subdomain. Then Y is a finite union of rational subdomains of X.*

2.5.3 Rigid analysis on the projective line

Our real goal is to define affinoid subdomains of $\mathbb{P}^1(\mathbb{C}_p)$. We can consider $\mathbb{P}^1(\mathbb{C}_p)$ to be a "rigid analytic variety" obtained by gluing two copies of D^1 together along the subdomains $\{z : |z| = 1\}$. We refer the interested reader to [3] for definitions and discussion of rigid analytic varieties. However, we can, for our purposes, loosely define an affinoid subdomain of $\mathbb{P}^1(\mathbb{C}_p)$ to be any proper subset which is in some sense "naturally isomorphic" to an affinoid domain. As before, inverse images, finite unions (provided they do not cover all of $\mathbb{P}^1(\mathbb{C}_p)$), and finite intersections of affinoid subdomains of $\mathbb{P}^1(\mathbb{C}_p)$ are again affinoid subdomains. In addition, an even stronger version of Theorem 2.5.2 holds: *any* affinoid subdomain of $\mathbb{P}^1(\mathbb{C}_p)$ is a rational subdomain.

The following proposition gives a much more down-to-earth characterization of affinoid subdomains of $\mathbb{P}^1(\mathbb{C}_p)$. It follows from Theorem 2.5.2 by analyzing the possible forms of rational subdomains.

Proposition 2.5.3. *Let $X \subset \mathbb{P}^1(\mathbb{C}_p)$ be an connected affinoid subdomain. Then X is of the form*

$$\mathbb{P}^1(\mathbb{C}_p) \setminus (D_1 \cup \cdots \cup D_n),$$

where $n \geq 1$, and each D_i is a rational open $\mathbb{P}^1(\mathbb{C}_p)$-disk. Conversely, any set of this form is a connected affinoid subdomain of $\mathbb{P}^1(\mathbb{C}_p)$. An affinoid subdomain of $\mathbb{P}^1(\mathbb{C}_p)$ is any finite disjoint union of connected affinoid subdomains.

In particular, any rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disk is a connected affinoid subdomain of $\mathbb{P}^1(\mathbb{C}_p)$. In addition, the union of two intersecting connected affinoid subdomains of $\mathbb{P}^1(\mathbb{C}_p)$ is again a connected affinoid subdomain.

We have seen that inverse images of affinoids under Tate algebra morphisms are again affinoids. It is also true that, given an affinoid $X \subset \mathbb{P}^1(\mathbb{C}_p)$ with $X = \operatorname{Max} A$ and a non-surjective, non-constant map $\phi : X \to \mathbb{P}^1(\mathbb{C}_p)$ whose coordinate functions are elements of A, the image $\phi(X)$ is affinoid. For a proof of this statement, see [7], Lemma A5.6. If X is connected, then so is $\phi(X)$. In particular, the image of a (connected) affinoid subdomain of $\mathbb{P}^1(\mathbb{C}_p)$ under a non-constant rational function is either all of $\mathbb{P}^1(\mathbb{C}_p)$ or a (connected) affinoid.

In fact, the discussion in [7] following Lemma A5.6 implies a more general result. Let $\phi : V \to U$ be a quasi-finite morphism of affinoids, where U is an affinoid subdomain of $\mathbb{P}^1(\mathbb{C}_p)$. For any $x \in U$, define $\deg \phi^{-1}(x)$ to be the number of points in the inverse image of x, counting multiplicity. For any $n \geq 0$, let W_n be the subset of U given by

$$W_n = \left\{x \in U : \deg \phi^{-1}(x) \geq n\right\}.$$

Then W_n is an affinoid subdomain of U. We will make use of this result below, in the case that V is an affinoid subdomain of $\mathbb{P}^1(\mathbb{C}_p)$, and ϕ is a non-constant rational function.

We close this section with a lemma which will prove very useful in future sections.

Lemma 2.5.4. *Let $U \subset \mathbb{P}^1(\mathbb{C}_p)$ be a connected affinoid, and let $\phi(z) \in \mathbb{C}_p(z)$ be a rational function of degree $d \geq 1$. Then $\phi^{-1}(U)$ is a disjoint union of at most d nonempty connected affinoids,*

$$\phi^{-1}(U) = V_1 \cup V_2 \cup \cdots \cup V_m$$

(with $1 \leq m \leq d$). In addition, for all $1 \leq i \leq m$, $\phi(V_i) = U$.

Proof. We know that $\phi^{-1}(U)$ is an affinoid; hence, it is a finite disjoint union of nonempty connected affinoids. Let m be the number of such connected affinoids; we must show that $m \leq d$. If V_i is any one of these connected affinoids, and if $n \geq 0$ is a nonnegative integer, define the set $W_{i,n}$ by

$$W_{i,n} = \left\{x \in U : \deg \left(\phi|_{V_i}\right)^{-1}(x) \geq n\right\}.$$

In other words, $W_{i,n}$ is the set of points with at least n inverse images, counting multiplicity, in V_i. By the above comments, $W_{i,n}$ is an affinoid subdomain of U.

Let I denote the set of all m-tuples $\eta = (n_1, \ldots, n_m)$ of nonnegative integers with sum d. For any $\eta \in I$, define

$$X_\eta = W_{1,n_1} \cap W_{2,n_2} \cap \cdots \cap W_{m,n_m}.$$

Then X_{eta} is an affinoid subdomain of $\mathbb{P}^1(\mathbb{C}_p)$. Note that if $\eta \neq \eta'$, then there is some i such that $n_i > n_i'$ and j such that $n_j < n_j'$; hence

$$X_\eta \cap X_{\eta'} = \emptyset.$$

Furthermore,

$$U = \bigcup_{\eta \in I} X_\eta,$$

because each point of U has exactly d inverse images under ϕ. Thus, $\{X_\eta\}$ is a cover of U by disjoint affinoids; since U is connected, all but one X_η must be empty. In other words, there are m nonnegative integers $\{n_i\}$ summing to d such that for each i and for any point $x \in U$, x has exactly n_i inverse images (counting multiplicity) in V_i.

If some n_i were zero, then no point of U would have inverse images in V_i, contradicting the definition of V_i. Thus, $n_i \geq 1$ for any $i = 1, \ldots, m$. Since

$$n_1 + \cdots + n_m = d,$$

it follows that $m \leq d$. Furthermore, every point of U is the image of at least one point from each V_i; thus, $\phi(V_i) = U$. □

Chapter 3

Non-Archimedean Components

In complex dynamics, it is often helpful to consider the set of topologically connected components of the Fatou set, with the original function ϕ acting on this set. We would like to develop a similar theory in the p-adic setting. However, it soon becomes clear that topological components will not do. The metric topology makes $\mathbb{P}^1(\mathbb{C}_p)$ a totally disconnected set; the largest connected components are singletons. Instead, we will define two alternative notions of "components" of the Fatou set.

Our first alternative notion will be that of *D-components*, defined in Section 3.1. D-components of the Fatou set will be our main objects of study; we will use them to state and prove a number of results, including a partial No Wandering D-Components Theorem (Theorem 5.3.5). For maps with at least two Julia points, Fatou D-components will always be disks; by contrast, Fatou components of complex maps usually have complicated fractal boundaries. However, the seemingly simpler idea of D-components can be viewed as a precise analogue of the Fatou components of complex dynamics; moreover, the dynamical theory which ensues will prove to be surprisingly rich.

Our second notion will be that of *analytic components*, defined in Section 3.2 using the affinoid domains of rigid analysis. Although most of our discussion will focus on D-components, our main results, including the No Wandering Domains Theorem, will also hold for analytic components. In addition, analytic components will be more appropriate for studying a few specific dynamical phenomena which will arise. The reader is therefore encouraged to view D-components as the fundamental notion but to keep analytic components in mind when they are needed.

3.1 D-Components

The non-Archimedean nature of the p-adics results in the happy fact that the image of a rational closed disk D under a power series convergent on D is itself a rational closed disk. (The analogous statement is almost true for irrational disks and rational open disks, except that the image could also be all of $\mathbb{C}_p$.) The proof of this fact is a straightforward exercise in non-Archimedean power series. More refined versions of this statement will appear in Lemmas 5.1.1 and 5.1.2.

However, we would like to work on $\mathbb{P}^1(\mathbb{C}_p)$, and so we need to generalize the notion of "disk" to the projective line. More precisely, we would like all disks in $\mathbb{C}_p$ to remain disks in $\mathbb{P}^1(\mathbb{C}_p)$, but we would also like the image of such a disk under any element of $\mathrm{PGL}(2, \mathbb{C}_p)$ to be a "disk" itself. As in [9], we are led to the following definition.

Definition 3.1.1.

1. *A* rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disk *is either a rational closed disk $\overline{D}_r(a)$ (for some $a \in \mathbb{C}_p$ and $r \in p^{\mathbb{Q}}$)* or *the complement of a rational open disk, $\mathbb{P}^1(\mathbb{C}_p) \setminus D_r(a)$ (for some $a \in \mathbb{C}_p$ and $r \in p^{\mathbb{Q}}$).*

2. *A* rational open $\mathbb{P}^1(\mathbb{C}_p)$-disk *is either a rational open disk $D_r(a)$ (for some $a \in \mathbb{C}_p$ and $r \in p^{\mathbb{Q}}$),* or *the complement of a rational closed disk, $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_r(a)$ (for some $a \in \mathbb{C}_p$ and $r \in p^{\mathbb{Q}}$).*

3. *An* irrational $\mathbb{P}^1(\mathbb{C}_p)$-disk *is either an irrational disk $D_r(a)$ (for some $a \in \mathbb{C}_p$ and $r > 0$ not in $p^{\mathbb{Q}}$)* or *the complement of an irrational disk, $\mathbb{P}^1(\mathbb{C}_p) \setminus D_r(a)$ (for some $a \in \mathbb{C}_p$ and $r > 0$ not in $p^{\mathbb{Q}}$).*

It is straightforward to show that the image of a $\mathbb{P}^1(\mathbb{C}_p)$-disk (rational closed, rational open, or irrational) under an element of $\mathrm{PGL}(2, \mathbb{C}_p)$ is another $\mathbb{P}^1(\mathbb{C}_p)$-disk (of the same type); see [9]. Furthermore, any $\mathbb{P}^1(\mathbb{C}_p)$-disk can be moved, by the action of an element of $\mathrm{PGL}(2, \mathbb{C}_p)$, to a disk in $\mathbb{C}_p$; in fact, it can be moved by an element of $\mathrm{PGL}(2, \mathcal{O})$ to a disk in $\mathbb{C}_p$ containing the origin.

Proposition 3.1.1.

1. *Let D_1 and D_2 be any two $\mathbb{P}^1(\mathbb{C}_p)$-disks with nonempty intersection. If $D_1 \cup D_2 \neq \mathbb{P}^1(\mathbb{C}_p)$, then either $D_1 \subset D_2$ or $D_2 \subset D_1$.*

2. *Let $\phi \in \mathbb{C}_p(z)$ be a rational function, and let D be a rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disk. Then the image of D under ϕ is either $\mathbb{P}^1(\mathbb{C}_p)$ or a rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disk.*

Proof. To prove the first statement, choose a point which does not lie in either disk; by a change of coordinates, we can move it to ∞. Then D_1 and D_2 become two intersecting disks in $\mathbb{C}_p$; therefore, one must contain the other.

To prove the second statement, begin by changing coordinates so that D does not contain ∞. Assume $\phi(D) \neq \mathbb{P}^1(\mathbb{C}_p)$, and choose $a \in \mathbb{P}^1(\mathbb{C}_p) \setminus \phi(D)$. Let $f(z) = (z - a)^{-1}$ (or, if $a = \infty$, let $f(z) = z$), and let $\psi(z) = f \circ \phi(z)$. Then ψ is a rational function with no poles on D; hence, it has a power series expansion on D, by Lemma 2.2.3. As mentioned earlier in this section, it follows that the image of D under ψ must be a rational closed disk; hence, the image under ϕ is a rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disk. □

In fact, the image of *any* $\mathbb{P}^1(\mathbb{C}_p)$-disk under a rational function is either $\mathbb{P}^1(\mathbb{C}_p)$ or a $\mathbb{P}^1(\mathbb{C}_p)$-disk; however, for rational open or irrational disks, some extra technical work must be done to exclude the possibility that the image is $\mathbb{C}_p$. We will not need this stronger result, and so we omit the proof.

In light of these facts, it is now natural to propose the following definition.

Definition 3.1.2. *Let $U \subseteq \mathbb{P}^1(\mathbb{C}_p)$ be an open set, and let $x \in U$. We define the* D-component of U containing x *to be the union of all rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disks containing x and contained in U.*

Note that it would be equivalent to define the D-component to be the union of all $\mathbb{P}^1(\mathbb{C}_p)$-disks (open, closed, or irrational) containing x and contained in U. In addition, the relationship "y is in the D-component containing x" is an equivalence relation; hence, the D-components can honestly be considered "components". Note also that a D-component must itself be either $\mathbb{P}^1(\mathbb{C}_p)$, $\mathbb{P}^1(\mathbb{C}_p) \setminus \{a\}$ for some $a \in \mathbb{P}^1(\mathbb{C}_p)$, or a $\mathbb{P}^1(\mathbb{C}_p)$-disk.

There are several ways to motivate the above definition. First and foremost, it turns out that we can view D-components as an analogue of the (topologically) connected components of an open subset of the complex plane, as follows. In the complex case, the component of an open set U containing a given point x is the set of all points $y \in U$ such that there is a path in U joining x and y (since $\mathbb{C}$ is locally path-connected). Because U is open and the path is compact, we can equivalently define the component to be the set of all $y \in U$ such that there exists a finite sequence of disks, each intersecting the previous one, with x in the first and y in the last. Carrying this reformulated definition over to the p-adic case, we see by Proposition 3.1.1 that it is equivalent to the definition of D-component given above.

There are other justifications for our definition, beginning with the far-sighted observation that D-components of the Fatou set will turn out to have interesting dynamnical properties. For instance, if the Julia set contains at least two points, then the Fatou set breaks into infinitely many D-components, but, as we shall see in Chapter 5, Sullivan's No Wandering Domains Theorem will still hold for (at least) a large class of rational functions. However, even before proving that, we note that disks are a natural setting for power series expansions, a key tool in both the p-adic and the complex theory. In addition, disks in $\mathbb{P}^1(\mathbb{C}_p)$ correspond to cosets of fractional ideals, and thus they exhibit some of the underlying algebraic structure.

Given a rational function $\phi \in \mathbb{C}_p(z)$ with Fatou set $\mathcal{F}$, and given a D-component V of $\mathcal{F}$, it is clear that $\phi(V)$ must be contained in a D-component of $\mathcal{F}$. After all, $\phi(V)$ is either a $\mathbb{P}^1(\mathbb{C}_p)$-disk, $\mathbb{P}^1(\mathbb{C}_p)$, or a set of the form $\mathbb{P}^1(\mathbb{C}_p) \setminus \{a\}$, all of which can be covered by disks containing any given point of $\phi(V)$. Thus, if we let S denote the set of D-components of $\mathcal{F}$, then ϕ induces a map $\Phi : S \to S$ by

$$\Phi(V) = \text{ the D-component containing } \phi(V).$$

Just as we could discuss fixed, periodic, and preperiodic points of $\mathbb{P}^1(\mathbb{C}_p)$ under the action of ϕ, we can discuss fixed, periodic, and preperiodic D-components, under the action of Φ. Similarly, we can discuss the forward orbit of a D-component. We will often abuse language and refer to D-components of $\mathcal{F}$ as D-components of ϕ.

In complex dynamics, a topological component which is not preperiodic is usually called a *wandering domain*; we will follow this tradition and refer to a non-preperiodic D-component as a *wandering D-component* or, when there is no ambiguity with other types of components, a *wandering domain*. Our conjectural analogue of Sullivan's theorem will state that every D-component of ϕ has finite forward orbit under this action of Φ; more colloquially, for any rational map $\phi \in \mathbb{C}_p(z)$, ϕ has no wandering domains. We will prove this result for a large class of rational maps.

Remark. In the complex case, components map onto components; however, this is not always true of D-components in the p-adic setting. For example, let p be an odd prime, and let $\phi(z) = p^{-1}(z^2 - 1)$, which has Julia set contained in $D_1(1) \cup D_1(-1)$. Then, denoting the D-component of x by U_x, we have $U_0 = D_1(0)$ and $\phi(U_0) = D_p(-p^{-1})$. On the other hand, $U_{-p^{-1}} = \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$, which contains but does not equal $D_p(-p^{-1})$.

Analytic components, defined in Section 3.2, map onto each other and thus do not exhibit this behavior. (In the above example, the entire Fatou set is a single analytic component.) However, even for D-components, this sort of behavior can occur only a bounded number of times for a given map, as the following theorems show.

Theorem 3.1.2. *Let $\phi \in \mathbb{C}_p(z)$ be a rational function of degree $d \geq 1$, and let $\mathcal{F}$ be the Fatou set of ϕ. Let S be the set of D-components of $\mathcal{F}$, and let Φ denote the action of ϕ on S. Define $T \subset S$ to be the subset*

$$T = \{V \in S : \exists\ U \in S \text{ s.t. } \Phi(U) = V \text{ and } \phi(U) \neq V\}.$$

Then T has at most $d - 1$ elements.

For polynomials, we have a stronger result:

Theorem 3.1.3. *Let $\phi \in \mathbb{C}_p[z]$ be a polynomial of degree $d \geq 1$, and let $\mathcal{F}$, S, Φ, and T be as in Theorem 3.1.2. If $\mathcal{F} = \mathbb{P}^1(\mathbb{C}_p)$, then T is empty; otherwise, T has one element, namely the D-component containing ∞.*

We will need the following lemma.

Lemma 3.1.4. *Let $\phi \in \mathbb{C}_p(z)$ be a rational function of degree $d \geq 1$. Let*

$$D_1, \dots, D_n \subset \mathbb{P}^1(\mathbb{C}_p)$$

be disjoint rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disks. Assume that for all $i = 1, \dots, n$, $\phi^{-1}(D_i)$ is not a finite union of disks. Then $n \leq d - 1$.

Proof of Lemma 3.1.4. By changing coordinates if necessary, assume that $\phi(\infty) \notin D_i$ for all i. Each $\phi^{-1}(D_i)$ is an affinoid and hence a finite union of disjoint connected affinoids; by hypothesis, at least one of these connected affinoids (call it V_i) must be a non-disk. Therefore, by Proposition 2.5.3, V_i is of the form

$$V_i = \overline{D}_{r_i}(a_i) \setminus \left(D_{s_{i1}}(b_{i1}) \cup D_{s_{i2}}(b_{i2}) \cup \cdots \cup D_{s_{im_i}}(b_{im_i})\right),$$

where $m_i \geq 1$, $a_i, b_{ij} \in \mathbb{C}_p$, and $r_i, s_{ij} \in p^{\mathbb{Q}}$.

Note that no $W_{ij} = D_{s_{ij}}(b_{ij})$ is affinoid, for it cannot be written as a disjoint union of closed annuli. Therefore, W_{ij} must contain some point x_{ij} which is not in

$\phi^{-1}(D_k)$ for any k. If W_{ij} contains some other $V_{i'}$, we can ensure that x_{ij} lies not only outside $V_{i'}$ but outside $\overline{D}_{r_{i'}}(a_{i'})$.

Let $C = \{x_{ij}\}_{i,j} \cup \{\infty\}$. We have

$$\phi(C) \cap (D_1 \cup D_2 \cup \cdots \cup D_n) = \emptyset.$$

Because C is a finite set, it is a simple exercise to show that there is a connected affinoid $U \subset \mathbb{P}^1(\mathbb{C}_p)$ with $U \cap D_i = \emptyset$ for all i, and $\phi(C) \subset U$. By Lemma 2.5.4, $\phi^{-1}(U)$ consists of at most d connected affinoids, none intersecting any V_i. However, by our choice of C, each point of C must lie in a different connected affinoid component of $\phi^{-1}(U)$. Now each $m_i \geq 1$, so C has at least $n+1$ elements (including the point ∞). Thus, $d \geq n+1$, and we are done. □

Proof of Theorem 3.1.2. We begin by claiming that for any $V \in T$, $\phi^{-1}(V)$ is not a finite disjoint union of $\mathbb{P}^1(\mathbb{C}_p)$-disks. For if it were such a finite union, then for any large enough rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disk D in V, $\phi^{-1}(D)$ would also be a finite union of disjoint rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disks, each mapping onto D. It follows that any D-component of $\phi^{-1}(V)$ would map onto V, and hence the Fatou D-component of any point in $\phi^{-1}(V)$ would map onto V, contradicting the definition of T.

Given any $V \in T$, it follows that there must be some rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disk $D \subset V$ such that $\phi^{-1}(D)$ is not a finite union of rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disks. By Lemma 3.1.4, there can be at most $d-1$ such disjoint D's. Hence, T can have at most $d-1$ elements. □

Proof of Theorem 3.1.3. The theorem will be a corollary of the following lemma, the proof of which is straightforward.

Lemma 3.1.5. *Let $D \subset \mathbb{C}_p$ be a disk containing the point $a \in \mathbb{C}_p$, and let*

$$D_1, \ldots, D_n \subset D$$

be smaller disks. Let $\phi \in \mathbb{C}_p[[z-a]]$ be a power series convergent on D, and suppose that

$$\phi\left(D \setminus (D_1 \cup \cdots \cup D_n)\right) \subseteq D'$$

is a disk. Then $\phi(D) \subseteq D'$.

To prove the theorem, note that the case $\mathcal{F} = \mathbb{P}^1(\mathbb{C}_p)$ is trivial. Otherwise, if D' is a D-component of $\mathcal{F}$ not containing ∞, then arguments similar to those in the proof of Theorem 3.1.2 show that $\phi^{-1}(D')$ is a finite union of sets of the form $D \setminus (D_1 \cup \cdots \cup D_n)$ mapping onto D'. But by Lemma 3.1.5, $\phi(D) = D'$; hence, $\phi^{-1}(D')$ is a finite union of disks. Therefore, the only D-component which could be in T is the one containing ∞; it suffices to show that this component is always in T.

Given any rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disk D containing ∞, $\phi^{-1}(D)$ is a finite disjoint union of connected affinoids, each of which maps onto D. However, the only point mapping to ∞ is ∞ itself; thus, $\phi^{-1}(D)$ must be a single connected affinoid. We will show that $\phi^{-n}(D)$ is not a disk for some $n \geq 1$.

Suppose that $\phi^{-n}(D)$ is always a disk, and let $D' = \bigcup \phi^{-n}(D)$, where the union is over all $n \geq 1$; by our supposition, D' is a disk containing ∞. Pick $x \in \mathcal{J}$ and

$r > 0$ such that $\overline{D}_r(x) \cap D' = \emptyset$. (This can be always be done, because D' is a closed set.) Then $\phi^n(\overline{D}_r(x)) \cap D = \emptyset$ for all $n \geq 1$. By Hsia's Theorem (Theorem 2.4.1), $\{\phi^n\}$ is equicontinuous on $\overline{D}_r(x)$, and hence $x \in \mathcal{F}$, and we have a contradiction. □

Thus, it is easy to produce examples of polynomials with a single D-component having the non-onto property; any polynomial with nonempty Julia set will do. We have as yet been unable to produce examples of rational functions with more than one such D-component. However, we present the following example of a rational function of degree 3 with two disjoint disks, each with inverse image not a finite union of disks.

Example. Let p be any prime, and let

$$\phi(z) = \frac{z^2\left(z - \frac{1}{p}\right)}{z-1}.$$

Let $D_1 = \overline{D}_p(0)$ and $D_2 = \mathbb{P}^1(\mathbb{C}_p) \setminus D_{p^2}(0)$; note that both are rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disks. Then

$$\phi^{-1}(D_1) = \overline{D}_1\left(\frac{1}{p}\right) \cup \left(\overline{D}_1(0) \setminus D_1(1)\right)$$

and

$$\phi^{-1}(D_2) = \overline{D}_{p^{-1}}(1) \cup \left(\mathbb{P}^1(\mathbb{C}_p) \setminus \left(D_p(0) \cup D_p\left(\frac{1}{p}\right)\right)\right).$$

3.2 Analytic Components

The second type of component we will define is based on the notion of *connected* affinoid sets in rigid analysis. Given a finite set of connected affinoids, each of which contains a given point $x \in \mathbb{P}^1(\mathbb{C}_p)$, their union is once again a connected affinoid. (Note: this is true only in dimension one; however, even in higher dimensions, the resulting union is in some sense "connected", even if it fails to be affinoid.) An arbitrary union of connected affinoids need not even be affinoid; however, it will serve to define a "component".

Definition 3.2.1. *Let $U \subseteq \mathbb{P}^1(\mathbb{C}_p)$ be an open set, and let $x \in U$. We define the* analytic component *of U containing x to be the union of all connected affinoids containing x and contained in U.*

As was the case for D-components, the relationship "y is in the analytic component containing x" is an equivalence relation. Furthermore, the analytic component containing x must contain the D-components containing x, since all rational closed disks are connected affinoids. In general, analytic components can be much more complicated sets than D-components; however, the analytic component of a given point is frequently no larger than the D-component of the same point, as the following proposition shows.

Proposition 3.2.1. *Let U be an open subset of $\mathbb{P}^1(\mathbb{C}_p)$. For $x \in U$, let V_x be the D-component of U containing x, and let W_x be the analytic component of U containing x. If V_x is not a rational open $\mathbb{P}^1(\mathbb{C}_p)$-disk, then $W_x = V_x$.*

Proof. If U is all but one point of $\mathbb{P}^1(\mathbb{C}_p)$, or simply all of $\mathbb{P}^1(\mathbb{C}_p)$, then $V_x = W_x = U$, and we are done. Thus, we only need to consider the case that V_x is a rational closed or irrational disk.

Let us fix a coordinate system such that $\infty \notin U$ and $V_x = \overline{D}_r(0)$, for some $r > 0$. We will show that any connected affinoid subset Y of U containing 0 must be contained in V_x. By Proposition 2.5.3, Y is of the form

$$Y = \overline{D}_s(0) \setminus (D_{t_1}(a_1) \cup \cdots \cup D_{t_n}(a_n))$$

for some $n \geq 0$, $a_i \in \overline{D}_s(0)$, and $s, t_i \in p^{\mathbb{Q}}$, with $t_i \leq s$. If $s \leq r$, then $Y \subset V_x$, and we are done; so we assume $s > r$. Since $0 \in Y$, we must also have $t_i \leq |a_i|$.

If all $a_i \in \overline{D}_r(0)$, then $\overline{D}_s(0) \subset U$, and hence $\overline{D}_r(0)$ would not be the full D-component of U. Thus, there is some $|a_i| > r$. Let $R = \min\{|a_i| : |a_i| > r\}$. Then $R > r$, and $D_R(0) \subset Y \subset U$, and once again, $\overline{D}_r(0)$ would not be the full D-component of U. □

We use analytic components in p-adic dynamics in much the same way that we used D-components. Given a rational function $\phi \in \mathbb{C}_p(z)$ with Fatou set $\mathcal{F}$, and given an analytic component W of $\mathcal{F}$, it can be shown that $\phi(W)$ is also an analytic component of $\mathcal{F}$. To show that $\phi(W)$ is contained in an analytic component, recall that for any connected affinoid W', $\phi(W')$ is also a connected affinoid. To show that $\phi(W)$ is a full analytic component, pick $x \in \phi(W)$. By Lemma 2.5.4, the inverse image of any connected affinoid W' in $\mathcal{F}$ containing x is a finite union of connected affinoids, each mapping onto W'; at least one must intersect W, and therefore, it is contained in W.

Thus, if we let S' denote the set of analytic components of $\mathcal{F}$, then ϕ induces a map $\Phi : S' \to S'$ by

$$\Phi(W) = \phi(W).$$

Under this action, we can define fixed, periodic, and preperiodic analytic components of ϕ, as well as the forward orbit of an analytic component. We can also define a *wandering analytic component* to be an analytic component with infinite forward orbit; when there is no danger of ambiguity, we call such a component a *wandering domain.*

Our analytic component version of Sullivan's theorem will state that every analytic component of $\mathcal{F}_\phi$ has finite forward orbit under the action of Φ. We shall see in Corollary 5.2.3 that any wandering analytic component must in fact be a disk; hence Sullivan's theorem for D-components and for analytic components will be equivalent, and there will be no danger of ambiguity when we speak of "wandering domains".

3.3 Quadratic Examples

In this section we present examples of various behaviors exhibited by quadratic maps, i.e. rational maps $\phi \in \mathbb{C}_p(z)$ of degree two. The most general form of such a map is

$$\phi(z) = \frac{az^2 + bz + c}{dz^2 + ez + f}.$$

We will classify ϕ into one of several types, based on the behavior of its fixed points; for each type, we will describe some of the dynamical behavior of the map. For the moment, our purpose is simply to demonstrate some of the phenomena which can occur in a p-adic dynamical system. We will therefore delay the proofs of most of the facts in this section to Appendix A.

A quadratic map $\phi \in \mathbb{C}_p(z)$ has either one, two, or three distinct fixed points. If it has only one (of multiplicity three), then it is conjugate to the map

$$z \mapsto z + \frac{1}{z},$$

which has good reduction and therefore empty Julia set. If ϕ has at least two distinct fixed points, then we can move one fixed point to 0 and another to ∞, and we get a map of the form

$$z \mapsto \frac{z^2 + \lambda z}{\mu z + 1}$$

for some $\lambda, \mu \in \mathbb{C}_p$ with $\lambda\mu \neq 1$; in fact, λ and μ are the multipliers of the fixed points at 0 and ∞, respectively. If one of λ or μ is zero, then ϕ is conjugate to a polynomial map of the form

$$z \mapsto z^2 + c$$

for some $c \in \mathbb{C}_p$.

3.3.1 $\mu = 0$ or $\lambda = 0$

In this case, ϕ is conjugate to a map of the form

$$\phi(z) = z^2 + c.$$

We begin by considering the case $|c| \leq 1$. Then ϕ has good reduction, and so $\mathcal{J}_\phi$ is empty; in fact,

$$\phi\left(\overline{D}_1(0)\right) = \overline{D}_1(0) \quad \text{and} \quad \phi\left(\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)\right) = \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0). \tag{3.1}$$

If $p = 2$, then the set of c such that $z^2 + c$ has good reduction is even larger. If $1 < |c|_2 \leq 4$, then ϕ is conjugate to a map of the form

$$z \mapsto z^2 + az,$$

where $a \in \mathbb{C}_p$ is a root of the equation $a^2 - 2a + 4c = 0$ (and therefore $|a|_2 = |2\sqrt{c}|_2 \leq 1$). This map now has good reduction, so the Julia set is empty, and, renaming $\phi(z) = z^2 + az$, equation (3.1) holds.

Finally, if $p \neq 2$ and $|c| > 1$, or if $p = 2$ and $|c| > 4$, then $\mathcal{J}_\phi$ is nonempty and has the structure of a Cantor set. For $p \neq 2$ (a case studied in detail in [34]), if we let $b = \sqrt{-c}$, the Julia set is contained in

$$D_{|b|}(b) \cup D_{|b|}(-b).$$

In fact, the Julia set is the iterated intersection of all backward iterates of these two disks (hence the Cantor set). For $p = 2$, the same phenomenon occurs, but the two disks are smaller. In both cases, all Fatou points are eventually attracted to the fixed point at ∞. There are infinitely many D-components, all of which are preperiodic (iterating eventually to the fixed D-component at ∞), and there is exactly one analytic component, which is the entire Fatou set.

Throughout all following cases, we will assume for simplicity that $p \neq 2$.

3.3.2 $|\lambda|, |\mu| < 1$, or $|\lambda|, |\mu| \leq 1$ with $|\lambda\mu - 1| = 1$

Given these conditions, we see that ϕ has good reduction, and $\mathcal{J}_\phi$ is empty. Note that this case includes the polynomial good reduction case.

3.3.3 $|\lambda| > 1$ and $|\mu| < 1$

As in the polynomial bad reduction case (which this case includes), the Julia set is a Cantor set, contained in

$$D_{|\lambda|}(0) \cup D_{|\lambda|}(-\lambda).$$

and all points in the Fatou set are attracted to the attracting fixed point at ∞.

3.3.4 $|\lambda| < 1$ and $|\mu| > 1$, or $|\lambda|, |\mu| > 1$

In this case, ϕ is conjugate to a map of the previous form, i.e., with $|\lambda| > 1$ and $|\mu| < 1$; we get a Cantor Julia set, and all Fatou points are attracted to the unique attracting fixed point.

3.3.5 $|\lambda| > 1$ and $|\mu| = 1$, (or $|\mu| > 1$ and $|\lambda| = 1$)

This is a more complicated case, and many behaviors are possible; we have not yet fully classified such dynamical systems. We will present a brief glimpse of some of the phenomena that occur.

We begin by noting that ϕ is conjugate to a map of the form

$$z \mapsto \frac{z^2 + z}{az + b}$$

with $|a| = 1$, $|b| < 1$; we shall abuse notation and refer to this new map as ϕ. There is a repelling fixed point at 0 and a neutral fixed point at ∞. If $|z| > 1$, then $|\phi(z)| = |z|$, and so ϕ takes $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$ into (and onto) itself. In fact, since $\phi(-1) = 0$, we have $0, -1 \in \mathcal{J}$, and so $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$ is the D-component of $\mathcal{F}$ containing ∞; however, it will not be the full analytic component.

If we let $\overline{x} \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$ denote the reduction of $x \in \mathbb{P}^1(\mathbb{C}_p)$, then for $z \in \mathbb{P}^1(\mathbb{C}_p)$ with $\overline{z} \neq 0$, we have

$$\overline{\phi(z)} = \frac{\overline{z}^2 + \overline{z}}{\overline{az}} = \frac{\overline{z} + 1}{\overline{a}}.$$

If $\overline{a} = 1$, then $\overline{\phi(z)} = \overline{z} + 1$, and so the disks $D_1(n)$ (for $n = 0, 1, \ldots p-1$) all contain Julia points (preimages of 0), while all other open disks of radius 1 in $\overline{D}_1(0)$ are p-periodic and Fatou. If $\overline{a} \neq 1$, then for $\overline{z} \neq 0$,

$$\overline{\phi(z)} = M \begin{pmatrix} 1 & 0 \\ 0 & \overline{a} \end{pmatrix} M^{-1} \overline{z},$$

where

$$M = \begin{pmatrix} 1 & 1 \\ 0 & \overline{a} - 1 \end{pmatrix}.$$

Thus, if n is the minimal positive integer such that $\overline{a}^n = 1$, and if none of

$$z, \phi(z), \ldots, \phi^{n-1}(z)$$

have reduction equal to 0, then $\overline{\phi^n(z)} = \overline{z}$. Once again, we will have finitely many disks $D_1(x)$ with Julia points, and the rest (for $|x| = 1$) will all be n-periodic.

Thus, for any value of a with $|a| = 1$, there are finitely many disks $D_1(x)$ in $\overline{D}_1(0)$ containing Julia points, and the rest are all Fatou. These disks (for $|x| = 1$) are D-components of the Fatou set, and they are all contained in the same analytic component, which also contains the fixed point at ∞ and hence $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$. In addition, all the Fatou disks $D_1(x)$ with $|x| = 1$ are periodic of the same period.

This phenomenon is by no means the only interesting feature of the dynamical system. For instance, there are, of course, many other Fatou D-components, including the preimages of the disks of the form $D_1(x)$. There are often still other periodic D-components and analytic components; some of these may be found by tracing iterates of the critical points, but it is conceivable that there may be still others.

It is also possible that a critical point lies in the Julia set. For example, if $p = 3$, $a = 7$, and $b = -9$, then 3 is a critical point, and

$$3 \mapsto 1 \mapsto -1 \mapsto 0 \mapsto 0.$$

Because 3 iterates to a repelling fixed point, it must be in the Julia set.

3.3.6 $|\lambda| = 1$, $|\mu| = 1$, and $|\lambda\mu - 1| < 1$

Setting $a = \mu^{-1}$ and $\varepsilon = \lambda - a$, we have

$$\phi(z) = az\left(1 + \frac{\varepsilon}{z + a}\right).$$

Note that $|a| = 1$ and $0 < |\varepsilon| < 1$. There are neutral fixed points at 0 and ∞, and for $|z| \neq 1$, $|\phi(z)| = |z|$. Thus, ϕ maps $D_1(0)$ and $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$ into (and onto) themselves.

If $\overline{a} \neq \pm 1$, then ϕ has no repelling fixed points, but it does have a repelling 2-cycle $\{x_1, x_2\}$, with $\overline{x_1} = \overline{-a}$ and $\overline{x_2} = \overline{-1}$. Note that if $\overline{z} \neq -a$, then $\overline{\phi(z)} = az$. Therefore, if n is the minimal integer such that $\overline{a}^n = 1$, then all but finitely many disks $D_1(x)$ with $|x| = 1$ are n-periodic. As in the previous example, we have an analytic component containing infinitely many periodic D-components of the same period, and also containing two fixed D-components, at 0 and ∞.

If $\overline{a} = \pm 1$, ϕ still has no repelling fixed points. If $|a^2 - 1|^2 > |\varepsilon|$, then the 2-cycle is repelling, and we have behavior similar to that described above. However, if $\overline{a} = 1$ and $|a - 1|^2 \leq |\varepsilon|$, then ϕ is conjugate to a map with good reduction, and hence the Julia set is empty. If $\overline{a} = -1$ and $|a + 1|^2 \leq |\varepsilon|$, then the 2-cycle is non-repelling, but it is unclear whether or not ϕ is conjugate to a map of good reduction, or whether the Julia set is even nonempty.

Chapter 4

Hyperbolic Maps

4.1 Definition and initial results

In complex dynamics, it is often useful to restrict one's attention to hyperbolic rational maps. These are maps which are everywhere expanding on the Julia set, with respect to some reasonable metric. The set of hyperbolic maps is conjectured to be open and dense in the moduli space of all complex rational maps (see, for example, [21]); thus, it seems that the study of hyperbolic maps could shed light on general maps. For a more detailed exposition on complex hyperbolic maps, we refer the interested reader to [6] or [23].

We would like to study an analogous class of maps in the p-adic setting. However, if we were to carry the complex definition of hyperbolicity into $\mathbb{C}_p$ verbatim, we would end up with too small a class of maps. Many of the useful properties of hyperbolic maps are proven using the fact that the Julia set of any complex rational map is compact; as we have seen, that is not the case for p-adic maps. Fortunately, all finite extensions of $\mathbb{Q}_p$ are locally compact; thus, the intersection of the Julia set with a finite extension of $\mathbb{Q}_p$ is a compact set. We can therefore propose the following definition of p-adic hyperbolicity.

Definition 4.1.1. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a rational function defined over K. Let $\mathfrak{J}$ and $\mathfrak{F}$ denote the Julia and Fatou sets, respectively, of ϕ. Assume (by a change of coordinates, if necessary) that $\infty \in \mathfrak{F}$. We say that ϕ is* hyperbolic *if for every finite extension L of K, there is an open set $V \subset L$ containing $\mathfrak{J} \cap L$, a continuous function $\sigma_L : V \to \mathbb{R}$, and positive constants $c_1, c_2 \in \mathbb{R}$ with $\sigma_L(V) \subset [c_1, c_2]$, such that for any $z \in V$ with $\phi(z) \in V$, we have*

$$\sigma_L(\phi(z))|\phi'(z)| \geq \sigma_L(z).$$

We remark that if ϕ is hyperbolic with respect to a coordinate z such that $\infty_z \notin \mathfrak{J}$, then ϕ is hyperbolic with respect to any other coordinate w such that $\infty_w \notin \mathfrak{J}$. Thus, the notion of hyperbolicity is independent of the choice of coordinate system. This can be proven directly without much difficulty, and we omit the proof. However, the following theorem gives a very useful characterization of hyperbolic maps which is independent of coordinate.

Theorem 4.1.1. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$. Let $\mathcal{J}$ and $\mathcal{F}$ denote the Julia and Fatou sets of ϕ. The following are equivalent:*

1. *ϕ is hyperbolic.*
2. *For any finite extension L of K, and for any coordinate z such that $\infty_z \in \mathcal{F}$, there is a positive integer m such that $|(\phi^m)'| > 1$ on $\mathcal{J} \cap L$.*
3. *$\mathcal{J}$ contains no critical points of ϕ.*

Proof. Clearly (1) implies (3). To show that (2) implies (1), we follow the standard proof from the complex theory as follows. (For the complex version, see, for example, [6].) Choose any coordinate z with $\infty_z \in \mathcal{F}$ and any finite extension L of K; let m be the positive integer given in (2). Note that by the compactness of $\mathcal{J} \cap L$, $|(\phi^n)'(z)|$ is bounded above on $\mathcal{J} \cap L$ for any $n = 1, \ldots, m$. Since $\infty_z \in \mathcal{F}$, there cannot be any poles in $\mathcal{J}$; in addition, by (2), there cannot be critical points in $\mathcal{J} \cap L$. Thus, we can choose a neighborhood V_1 of $\mathcal{J} \cap L$ containing no critical points or poles of $\phi, \phi^2, \ldots, \phi^m$ such that $|(\phi^m)'(z)| > 1$ on V_1.

We define

$$\sigma(z) = |\phi'(\phi^{m-2}(z))|^{1/m} |\phi'(\phi^{m-3}(z))|^{2/m} \cdots |\phi'(z)|^{(m-1)/m}$$

for $z \in V_1$. Let V be the set of all $z \in V_1$ such that $\phi^n(z) \in V_1$ for all $n = 1, \ldots, m$. It is then easy to verify that if $z, \phi(z) \in V$, we get

$$\frac{\sigma(\phi(z))}{\sigma(z)} = \frac{|(\phi^m)'(z)|^{1/m}}{|\phi'(z)|} > \frac{1}{|\phi'(z)|},$$

from which hyperbolicity follows.

The final step, to show that (3) implies (2), requires more work. Fix a finite extension L of K and a coordinate z with $\infty \notin \mathcal{J}$. By change of coordinates (specifically, multiplying by a constant of small absolute value), we can guarantee that the Julia set is contained in $\overline{D}_1(0)$. (Note that a change of coordinates fixing the point at infinity does not affect ϕ'.) Now because ϕ' is continuous on the compact set $\mathcal{J} \cap L$ with no zeros or poles, there are positive real numbers M_1 and M_2 such that for any $a \in \mathcal{J} \cap L$, $M_1 \leq |\phi'(a)| \leq M_2$. Without loss, assume that $M_1 < 1$.

We need the following lemma; the proof is straightforward, and we omit it.

Lemma 4.1.2. *Suppose $f \in \mathbb{C}_p[[z]]$ is a power series defined over $\mathbb{C}_p$ and convergent on $\overline{D}_r(0)$ for some $r > 0$. Suppose also that $|f'(0)| = c > 0$. Then there exists a positive real number s with $0 < s \leq r$ such that for any $x, y \in \overline{D}_s(0)$, $|f(x) - f(y)| = c|x - y|$.*

By Lemma 4.1.2, we can cover every point in $\mathcal{J} \cap L$ by a (rational closed) disk on which ϕ stretches by a constant factor; since $\mathcal{J} \cap L$ is compact, we may take a finite subcover. Let ε be the minimum of the radii of the resulting set of disks, and consider the (finite) cover of $\mathcal{J} \cap L$ by ε-disks. (Note that any larger disk in L is a finite union of ε-disks.)

Next we show that for any fixed $a \in \mathcal{J} \cap L$, the set

$$\{|(\phi^n)'(a)|\}_{n \geq 1}$$

is unbounded. For suppose that there were $\rho > 1$ such that $|(\phi^n)'(a)| \leq \rho$ for all $n \geq 0$. We claim that $\phi^n(\overline{D}_{\varepsilon/\rho}(a)) \subseteq \overline{D}_\varepsilon(\phi^n(a))$ for any $n \geq 0$. We will prove this claim by induction on n. It is clearly true for $n = 0$. Now assume we know it to be true for any $0 \leq i \leq n-1$. Note that ϕ^n stretches by a constant factor on $\overline{D}_{\varepsilon/\rho}(a)$, since each successive iteration of ϕ puts the image in a disk of radius at most ε, which is therefore a disk on which ϕ stretches by a constant factor. In fact, this constant factor must be $|(\phi^n)'(a)|$. Because ϕ^n stretches by at most ρ, the image under ϕ^n has radius no larger than ε, and we have proven our claim. However, the claim implies (by Hsia's Theorem) that $\{\phi^n\}$ is equicontinuous on $\overline{D}_{\varepsilon/\rho}(a)$, contradicting the fact that a is Julia. Thus, the derivatives are in fact unbounded.

Fix any $C > 1$. For any $a \in \mathcal{J} \cap L$, there is an integer $m_a \geq 1$ such that $|(\phi^{m_a})'(a)| \geq C$. By continuity, there is $\delta_a > 0$ such that $|(\phi^{m_a})'(a)| \geq C$ on $\overline{D}_{\delta_a}(a)$. We cover $\mathcal{J} \cap L$ with such δ_a-neighborhoods and take a finite subcover. On each of these finitely many disks, we have an integer m_a such that $|(\phi^{m_a})'| \geq C$ on the disk; we would like to find a single m which works for them all.

Let N be the maximum of the m_a. Choose an integer M which is a multiple of N large enough so that $C^{M/N} M_1^N > 1$. (Recall that M_1 was a lower bound for the derivative of ϕ.) We claim that any m larger than M works as the exponent we are looking for (i.e., that $|(\phi^m)'| > 1$ on $\mathcal{J} \cap L$).

To prove this, choose any $m \geq M$ and $z \in \mathcal{J} \cap L$. Then there exists $1 \leq m_1 \leq N$ with $|(\phi^{m_1})'(z)| \geq C$. Because any iterate of z is also in $\mathcal{J} \cap L$, there exists $1 \leq m_2 \leq N$ with $|(\phi^{m_2})'(\phi^{m_1}(z))| \geq C$, and $1 \leq m_3 \leq N$ with $|(\phi^{m_3})'(\phi^{m_1+m_2}(z))| \geq C$, and so on. Eventually, there is a positive integer j with $m_1 + \cdots + m_j > m$, but $m_1 + \cdots + m_{j-1} \leq m$. Note that $j > \frac{m}{N} \geq \frac{M}{N}$, since $m_i \leq N$. Also, $m - m_1 - \cdots - m_{j-1} \leq N$. Since $|\phi'| \geq M_1$ on $\mathcal{J} \cap L$, we see that

$$|(\phi^{m-m_1-\cdots-m_{j-1}})'(\phi^{m_1+\cdots+m_{j-1}}(z))| \geq M_1^{m-m_1-\cdots-m_{j-1}} \geq M_1^N.$$

Thus, it suffices to show that $|(\phi^{m_1+\cdots+m_{j-1}})'(z)| > M_1^{-N}$. But

$$\begin{aligned} &|(\phi^{m_1+\cdots+m_{j-1}})'(z)| = \\ &\quad |(\phi^{m_1})'(z)||(\phi^{m_2})'(\phi^{m_1}(z))| \cdots |(\phi^{m_{j-1}})'(\phi^{m_1+\cdots+m_{j-2}}(z))| \\ &\quad \geq C^{j-1} \geq C^{M/N} > M_1^{-N} \end{aligned}$$

and the proof is complete. □

Theorem 4.1.1 is much stronger than its complex analogue. In the complex case, a rational map is hyperbolic if and only if the closure of the postcritical set (i.e., the closure of the union of the forward orbits of all the critical points) is disjoint from the Julia set; in our case, we only need the critical set disjoint from the Julia set. However, if a p-adic rational map ϕ has no wandering D-components, then the postcritical set cannot accumulate at Julia points unless some critical point is actually in the Julia set; this statement follows from the fact that all disks are topologically closed. Thus, assuming no wandering domains, a p-adic map is hyperbolic if and only if its postcritical set is disjoint from its Julia set, just as in the complex case.

4.2 Theorems on hyperbolic maps

We are now prepared to prove the No Wandering Domains Theorem and a related result for hyperbolic rational maps. Both results will be corollaries of Lemma 4.2.1 below. Although the lemma and its corollaries will be superseded by the results of Chapter 5, the proofs for hyperbolic maps are simpler but still illustrate the main ideas of the proofs for more general maps.

Lemma 4.2.1. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a hyperbolic rational map with Fatou set $\mathcal{F}$. Let Φ denote the action of ϕ on the set of D-components of $\mathcal{F}$. Let L be any finite extension of K. Assume that ∞ is a Fatou fixed point. Then there exists $R > 0$ such that for any D-component U of $\mathcal{F}$ with $U \cap L \neq \emptyset$, there is an integer $n \geq 0$ such that*

$$\operatorname{rad}\left(\Phi^n(U)\right) \geq R.$$

For the purpose of Lemma 4.2.1, we will consider the D-component containing ∞ to have infinite radius.

Proof. Let $\mathcal{J}$ denote the Julia set of ϕ. By Theorem 4.1.1, there is some $m \geq 0$ such that for any $z \in \mathcal{J} \cap L$, $|(\phi^m)'(z)| > 1$. Thus, for any point $z \in \mathcal{J} \cap L$, there must be a disk $\overline{D}_\varepsilon(z)$ on which $|(\phi^m)'| > 1$; in fact, we can choose ε small enough (by Lemma 4.1.2) that $|(\phi^m)'|$ is constant, and that ϕ^m stretches uniformly by that factor, on that disk. Cover $\mathcal{J} \cap L$ with such disks and take a finite subcover. Let R_1 be the minimum radius of the disks in the subcover, and let W denote the union of all disks in the subcover. Let $C > 1$ denote the minimum of $|(\phi^m)'|$ on W.

We claim that there are finitely many D-components of $\mathcal{F}$ which are contained in $\mathbb{P}^1(\mathbb{C}_p) \setminus W$ and contain points of L. If there were infinitely many, then we could construct a sequence by selecting one L-point from each; since $\mathbb{P}^1(L)$ is compact, this sequence would accumulate at a point $a \in \mathbb{P}^1(L)$. However, a would have to be Julia, for if it were Fatou, then its D-component would include infinitely many of the points in the sequence. Thus,

$$a \in \mathcal{J} \cap \mathbb{P}^1(L) \subset W;$$

but $\mathbb{P}^1(L) \setminus W$ is closed, so the sequence could not accumulate at a point in W. Our claim follows from this contradiction.

Let R_2 be the minimum radius of D-components which are contained in $\mathbb{P}^1(\mathbb{C}_p) \setminus W$, and let R be the minimum of R_1 and R_2. With this choice of R, if there is a D-component U which is a counterexample to the lemma, then U and all its forward iterates must intersect W. Because W is a finite union of disks, all of which contain Julia points, it follows that U and all its forward iterates are contained in W.

Let $r = \operatorname{rad}(U) > 0$. Pick a positive integer M such that $C^M r \geq R$. Because all iterates of U are contained in W, every application of ϕ^m will stretch U by a factor of at least C; hence,

$$\operatorname{rad}\left(\phi^{Mm}(U)\right) \geq C^M r \geq R$$

as desired. □

Theorem 4.2.2. *(No Wandering Domains for Hyperbolic Rational Maps) Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a hyperbolic rational map. Then ϕ has no wandering D-components.*

Proof. Let $\mathcal{F}$ denote the Fatou set of ϕ, and let Φ denote the action of ϕ on the set of D-components of $\mathcal{F}$. Without loss of generality, we assume that the Julia set of ϕ is nonempty. By Corollary 2.4.4, we can change coordinates so that ϕ has a non-repelling fixed point at ∞. The Fatou set therefore contains ∞, and the D-component of ∞ is of the form $\mathbb{P}^1(\mathbb{C}_p) \setminus D$, for some finite disk D.

Now suppose U is a wandering D-component of ϕ. U must contain a point $a \in \overline{\mathbb{Q}}_p$; let $L = K(a)$. Note that all iterates of U must also contain L-points. Furthermore, since U is wandering, none of its iterates is the (fixed) D-component at ∞. Let $U_0 = U$.

Pick $R > 0$ to satisfy Lemma 4.2.1. Some iterate $V_0 = \Phi^{n_0}(U_0)$ must have radius at least R. Let $U_1 = \Phi(V_0)$. Then some iterate $V_1 = \Phi^{n_1}(U_1)$ has radius at least R. Continuing in this fashion, we have an infinite sequence of disks $\{V_i\}$, all disjoint (or else they would contain each other and be periodic), all contained in D (since they cannot intersect the D-component at ∞), all with radius at least R, and all containing points of L.

However, $L \cap D$ is compact, since L is a locally compact metric space and D is closed and bounded. Thus, there could not be such a sequence of disjoint disks, and we have a contradiction. □

By a similar argument, we can also prove the following related theorem. We omit the details.

Theorem 4.2.3. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a hyperbolic rational map with Fatou set $\mathcal{F}$. Let L be any finite extension of K. Then there are only finitely many periodic D-components of $\mathcal{F}$ which contain points of L.*

Our proof of Sullivan's theorem for hyperbolic maps is based on Norton's proof ([27]) of the same result in the complex case. The basic idea is that the iterates of a wandering domain would occupy infinite area; that idea will still be the foundation of our more general results in Chapter 5.

For our proof, it is crucial that we limit ourselves to functions defined over $\overline{\mathbb{Q}}_p$ rather than over $\mathbb{C}_p$, because finite extensions are locally compact, whereas transcendental extensions need not be. However, the dynamical systems arising from number theory will all be defined over $\overline{\mathbb{Q}}$, and hence over $\overline{\mathbb{Q}}_p$.

4.3 Non-hyperbolic examples

By Theorem 4.1.1, a map is hyperbolic if and only if there are no critical points in its Julia set. In some sense, one would expect there to be no Julia critical points; after all, the map contracts very strongly in the neighborhood of a critical point. However, it is possible to force a critical point to lie in the Julia set. The simplest way would be to map a critical point to a repelling fixed point. For example, the function

$$\phi(z) = \frac{1}{p}(z^3 - z^2) + 1,$$

defined over $\mathbb{Q}_p$, has a critical point at 0 which maps to a repelling fixed point at 1, and hence ϕ is not hyperbolic. Recall also the example in Section 3.3.5 of a quadratic rational map with a critical point which mapped, after three iterations, to a repelling fixed point.

It is also possible to have Julia critical points which are not preperiodic, as the following example shows.

Example. Let $p = 2$, and let

$$\phi(z) = \frac{31}{4}(z^3 - z^2) + 1.$$

Then ϕ has critical points at 0 and $\frac{2}{3}$. The critical point at 0 maps to the repelling fixed point at 1. However, we are more interested in the other critical point. Note that

$$\phi\left(\frac{2}{3}\right) = -\frac{4}{27} \in \overline{D}_{|8|}(4) \cap \mathbb{Q}_2.$$

Now if $w \in \mathbb{Z}_2$, then

$$\phi(4 + 8w) \equiv -11 + 16w + 16w^2 \equiv -11 (\mathrm{mod}\ 32).$$

Therefore, $\phi(\overline{D}_{|8|}(4) \cap \mathbb{Q}_2) \subset \overline{D}_{|32|}(-11)$. Furthermore, for $|w| \leq |32|$,

$$\phi(-11 + 32w) \equiv 4 (\mathrm{mod}\ 8),$$

and so we also have $\phi(\overline{D}_{|32|}(-11)) \subset \overline{D}_{|8|}(4)$. Thus, ϕ maps $\mathbb{Q}_2$-points of $\overline{D}_{|8|}(4)$ to $\mathbb{Q}_2$-points of $\overline{D}_{|32|}(-11)$, and vice versa.

It is easy to show that for any two points $x, y \in \overline{D}_{|8|}(4)$ such that $|x - y| \leq |16|$,

$$|\phi(x) - \phi(y)| = \frac{1}{2}|x - y|,$$

and for any two points $x, y \in \overline{D}_{|32|}(-11)$,

$$|\phi(x) - \phi(y)| = 4|x - y|.$$

It follows that, given any two points $x, y \in \overline{D}_{|8|}(4) \cap \mathbb{Q}_2$, there is some $n \geq 0$ such that

$$|\phi^n(x) - \phi^n(y)| \geq |16|.$$

Since x and y could have been arbitrarily close at the start, this implies that

$$\overline{D}_{|8|}(4) \cap \mathbb{Q}_2 \subset \mathcal{J}_\phi.$$

In particular, our critical point at $\frac{2}{3}$ must also be Julia.

On the other hand, our map ϕ can also be viewed as a rational function in $\mathbb{Q}_3(z)$. If $|z|_3 \geq 1$, then $|\phi(z)|_3 = |z|_3^3$. In particular,

$$\left|\phi^n\left(\frac{2}{3}\right)\right|_3 \to \infty \quad \text{as} \quad n \to \infty,$$

but $\phi^n(\frac{2}{3})$ never equals ∞ itself. Thus, $\frac{2}{3}$ is not preperiodic. Since preperiodicity is independent of metric, it is also not preperiodic in $\mathbb{Q}_2$.

Thus, ϕ is an example of a non-hyperbolic map with a Julia critical point which is not preperiodic.

Chapter 5

Main Theorems

In this chapter we will state and prove our strongest results; in particular, Theorem 5.3.5 will be our strongest No Wandering Domains Theorem. We will begin with a series of lemmas on power series in Section 5.1; later, in Section 5.3.2, there will be more technical results used in the proof of our main lemma, Lemma 5.3.1. However, the reader is encouraged to read the other sections first, and then return to the technical lemmas after their application to broader results has been made clear.

5.1 Preliminary Lemmas

In this section we state and prove several lemmas regarding the action of p-adic power series on disks. We begin with two basic results which are more detailed versions of the facts stated at the beginning of Section 3.1.

Lemma 5.1.1. *Let $V = \overline{D}_r(a)$ be a rational closed disk in $\mathbb{C}_p$, and let $f \in \mathbb{C}_p[[z-a]]$ be a non-constant power series convergent on V. Then the image $f(V)$ is a rational closed disk $\overline{D}_s(f(a))$. Furthermore, if K is any extension of $\mathbb{Q}_p$, $f \in K[[z-a]]$, and $r \in |K^*|$, then $s \in |K^*|$.*

Proof. We can write $f(z) = g(z-a)$ where $g \in \mathbb{C}_p[[z]]$. Thus, without loss, we may assume that $a = 0$. Write

$$f(z) = \sum_{i=0}^{\infty} c_i z^i,$$

where $c_i \in \mathbb{C}_p$; by the convergence of f, we have

$$\lim_{i\to\infty} |c_i| r^i = 0.$$

Note that, by hypothesis, $r \in p^{\mathbb{Q}}$; and because f is non-constant, there must be some $i > 0$ with $c_i \neq 0$.

Let $s > 0$ be given by

$$s = \max_{i\geq 1} |c_i| r^i;$$

the maximum must be achieved and must be finite, by the convergence of the limit. Let j be the smallest integer such that $s = |c_j|r^j$. Clearly, if $c_j \in K$ and $r \in |K^*|$, then $s \in |K^*|$. By ultrametricity, $f(V) \subset \overline{D}_s(c_0)$; thus, it suffices to show that every point of $\overline{D}_s(c_0)$ is in $f(V)$.

Pick $b \in \mathbb{C}_p$ with $|b| \leq s$. We only need to show that the equation

$$-b + \sum_{i=1}^{\infty} c_i z^i = 0$$

has a solution z with $|z| \leq r$. This follows from Theorem 2.2.1, because $|c_j|r^j = s \geq |b|$. □

Lemma 5.1.2. *Let $V = D_r(a)$ be a rational open or irrational disk in $\mathbb{C}_p$, and let $f \in \mathbb{C}_p[[z-a]]$ be a non-constant power series convergent on V. Then the image $f(V)$ is either all of $\mathbb{C}_p$ or an open disk $D_s(f(a))$. Furthermore, in the latter case, if K is any finite extension of $\mathbb{Q}_p$, $f \in K[[z-a]]$, and $r \in |K^*|$, then $s \in |K^*|$.*

Proof. Once again, we may assume that $a = 0$. As in the previous proof, let $\{c_i\}$ be the coefficients of f. By the convergence of f, we have, for any $0 < \rho < r$,

$$\lim_{i \to \infty} |c_i|\rho^i = 0.$$

Let $s > 0$ be given by

$$s = \sup_{i \geq 1} |c_i|r^i.$$

If $s = \infty$, then given any $b \in \mathbb{C}_p$, there is some ρ with $0 < \rho < r$ and a positive integer j such that $|c_j|\rho^j \geq |b|$. Thus, by Theorem 2.2.1, we can solve

$$-b + \sum_{i=1}^{\infty} c_i z^i = 0$$

with $|z| \leq \rho$. Hence, $f(V) = \mathbb{C}_p$. On the other hand, if s is finite, then by arguments similar to those of the proof of Lemma 5.1.1, we can prove that $f(V)$ is an open disk of radius s. □

In addition to the statements of the preceding lemmas, keep in mind the equation

$$s = \max_{i \geq 1} |c_i|r^i,$$

relating the coeffients $\{c_i\}$ and the radii r and s of the domain and image disks. The following lemmas will use this formula to find the radius of the image disk, given information about the power series.

Lemma 5.1.3. *Let $V = \overline{D}_r(a)$ be a rational closed disk in $\mathbb{C}_p$, and let*

$$f(z) = \sum_{i=0}^{\infty} c_i(z-a)^i, \quad c_i \in \mathbb{C}_p$$

be convergent on V*. Then* f *is one-to-one on* V *if and only if for all* $i > 1$*,* $|c_i|r^i < |c_1|r$*. In this case,* $|f'(z)| = |c_1|$ *for all* $z \in V$*, and* $\mathrm{rad}(f(V)) = |c_1|r$*; furthermore, for any* $x, y \in V$*,*

$$|f(x) - f(y)| = |c_1||x - y|.$$

Proof. Assume without loss that $a = 0$. Suppose first that for all $i > 1$, $|c_i|r^i < |c_1|r$. By the proof of Lemma 5.1.1, $f(V)$ is a closed disk of radius $|c_1|r$. We need to show that, given $b \in \mathbb{C}_p$ with $|b| \leq |c_1|r$, the equation

$$-b + \sum_{i=1}^{\infty} c_i z^i = 0$$

has exactly one solution z with $|z| \leq r$. This follows immediately from Theorem 2.2.1, with $m = 0$ and $n = 1$, and we have one direction of the equivalence.

Conversely, suppose there is some $j > 1$ with $|c_j|r^j \geq |c_1|r$. We may assume that j is the smallest such integer (at least 2). Then the inverse images of c_0 are roots of

$$\sum_{i=1}^{\infty} c_i z^i = 0.$$

However, by Theorem 2.2.1, this equation has at least j roots in $\overline{D}_r(0)$, and thus f is not one-to-one on V. The equivalence is proven.

Assume from now on that f is one-to-one on V. The derivative of f is

$$f'(z) = \sum_{i=1}^{\infty} i c_i z^{i-1}.$$

Note that for $z \in V$ and $i \geq 2$,

$$|i c_i z^{i-1}| \leq |c_i| r^{i-1} < |c_1|,$$

by our assumption that f is one-to-one. Thus, by ultrametricity, $|f'(z)| = |c_1|$. As we have seen, $\mathrm{rad}(f(V)) = |c_1|r$; we only need to prove the final statement of the lemma.

Pick $x, y \in V$. Then

$$f(x) - f(y) = (x - y) \sum_{i=1}^{\infty} c_i \frac{x^i - y^i}{x - y} = c_1(x - y) \left(1 + \sum_{i=2}^{\infty} \frac{c_i}{c_1} \cdot \frac{x^i - y^i}{x - y} \right).$$

However,

$$\left| \frac{x^i - y^i}{x - y} \right| = |x^{i-1} + x^{i-2}y + \cdots + y^{i-1}| \leq r^{i-1}.$$

Again by ultrametricity, and the fact that $|c_i|r^{i-1} < |c_1|$, it follows that

$$|f(x) - f(y)| = |c_1||x - y|$$

as desired. □

The reader should be cautioned that a power series may lack critical points on a disk but fail to be one-to-one; this situation is in sharp contrast with the complex setting, where an analytic function on a simply connected domain with no critical points is automatically one-to-one. Lemma 5.1.7 will help us understand the action of p-adic power series on disks near critical points; such disks are often mapped multiply-to-one without containing critical points themselves.

Before stating that lemma, we will need the following three lemmas on p-adic absolute values of binomial coefficients. Recall that $v(x)$ denotes the p-adic valuation of x. If r is a rational number, let $\lfloor r \rfloor$ denote the greatest integer less than or equal to r. $|x| = |x|_p$ will, as usual, denote the p-adic absolute value of x.

Lemma 5.1.4. *Let n be a positive integer, and let $m = \left\lfloor \frac{n}{p} \right\rfloor$. Then*

$$|n!|_p = |p^m|_p |m!|_p.$$

Proof. Note that

$$|n(n-1)(n-2)\cdots(pm+1)| = 1,$$

since none of the factors is divisible by p. Thus, we may assume without loss that $n = pm$. Keeping in mind that the only terms of a product of integers which contribute to its absolute value are those divisible by p, we have

$$|n!| = |(pm)(pm-1)\cdots 1| = |(pm)(p(m-1))(p(m-2))\cdots(p)| = |p^m||m!|.$$

□

Lemma 5.1.5. *Let m be a positive integer, let $n = pm$, let $1 \le i \le n$, and let $j = \left\lfloor \frac{i}{p} \right\rfloor$. Then*

$$\left|\binom{n}{i}\right| = \begin{cases} \left|\binom{m}{j}\right|, & \textit{if } p|i. \\ |p||m-j|\left|\binom{m}{j}\right|, & \textit{otherwise.} \end{cases}$$

Proof. Suppose first that $p|i$. Then by Lemma 5.1.4,

$$\begin{aligned} |i!| &= |p^j||j!|, \\ |n!| &= |p^m||m!|, \text{ and} \\ |(n-i)!| &= |p^{m-j}||(m-j)!|, \end{aligned}$$

from which the result follows. On the other hand, if $p \nmid i$, we still have $|i!| = |p^j||j!|$ and $|n!| = |p^m||m!|$. However,

$$\left\lfloor \frac{n-i}{p} \right\rfloor = m - j - 1,$$

and so

$$|(n-i)!| = |p^{m-j-1}||(m-j-1)!|,$$

from which the result follows. □

Lemma 5.1.6. *Let $n \geq 1$ be a positive integer, $0 < r \leq 1$ a real number, and $l = v(n)$. Then*

$$\max_{i=1,\ldots,n} \left\{ \left| \binom{n}{i} \right| r^i \right\} = \max_{e=0,\ldots,l} \left\{ |p^{l-e}| r^{p^e} \right\}.$$

Proof. We proceed by induction on l. If $l = 0$, then the right hand side is simply r. On the left hand side, we have (for $i \geq 1$)

$$\left| \binom{n}{i} \right| r^i \leq r^i \leq r;$$

furthermore, for $i = 1$, the value of $|n|r = r$ is achieved, and so the left hand side is also r.

Next, assume the statement is true for any positive integer with valuation $l - 1$; we will prove it for n with $v(n) = l$. Write $n = pm$ with $v(m) = l - 1$.

For any integer i, we can write $i = pj + k$, with $k \in \{0, 1, \ldots, p-1\}$. For fixed j, by Lemma 5.1.5, we have

$$\max_{k=0,1,\ldots,p-1} \left\{ \left| \binom{n}{pj+k} \right| r^{pj+k} \right\}$$

$$= \max \left\{ \left| \binom{m}{j} \right| r^{pj}, \max_{k=1,\ldots,p-1} \left\{ \left| \binom{m}{j} \right| |p||m-j| r^{pj+k} \right\} \right\}.$$

However, $|p||m-j|r^k < 1$, and so

$$\max_{k=0,1,\ldots,p-1} \left\{ \left| \binom{n}{pj+k} \right| r^{pj+k} \right\} = \left| \binom{m}{j} \right| r^{pj}.$$

Applying this to the left hand side of the equation in the statement of the lemma, we have

$$\max_{i=1,\ldots,n} \left\{ \left| \binom{n}{i} \right| r^i \right\} =$$

$$\max \left\{ \max_{i=1,\ldots,p-1} \left\{ \left| \binom{n}{i} \right| r^i \right\}, \max_{j=1,\ldots,m-1} \left\{ \max_{k=0,1,\ldots,p-1} \left\{ \left| \binom{n}{pj+k} \right| r^{pj+k} \right\} \right\}, r^n \right\} =$$

$$\max \left\{ |n|r, \max_{j=1,\ldots,m-1} \left\{ \left| \binom{m}{j} \right| r^{pj} \right\}, \left| \binom{m}{m} \right| r^{pm} \right\} =$$

$$\max \left\{ |p^l|r, \max_{j=1,\ldots,m} \left\{ \left| \binom{m}{j} \right| r^{pj} \right\} \right\}. \tag{5.1}$$

By our inductive hypothesis, using r^p in place of r, note that

$$\max_{j=1,\ldots,m} \left\{ \left| \binom{m}{j} \right| r^{pj} \right\} = \max_{e=0,\ldots,l-1} \left\{ |p^{l-e-1}| r^{p^{e+1}} \right\} = \max_{e=1,\ldots,l} \left\{ |p^{l-e}| r^{p^e} \right\}.$$

Thus, 5.1 becomes simply

$$\max_{e=0,\ldots,l} \left\{ |p^{l-e}| r^{p^e} \right\},$$

as desired. □

Our final technical lemma, along with its corollaries, is of crucial importance for understanding the behavior of a power series near a critical point.

Lemma 5.1.7. *Let $V = \overline{D}_r(a)$ be a closed disk in $\mathbb{C}_p$, and let*

$$f(z) = c_0 + \sum_{i=d}^{\infty} c_i (z-a)^i, \quad c_i \in \mathbb{C}_p \tag{5.2}$$

be convergent on V, with $d \geq 1$ and $c_d \neq 0$. Suppose that for all $i > d$, $|c_i| r^i < |d! c_d| r^d$. Let $b \in V$, $\sigma = |b-a|$, and $0 < \rho \leq \sigma$. Then

$$\operatorname{rad}\left(f\left(\overline{D}_\rho(b)\right)\right) = |dc_d|\sigma^d \max_{e=0,\dots,v(d)} \left\{ |p^{-e}| \left(\frac{\rho}{\sigma}\right)^{p^e} \right\}.$$

Proof. Without loss, we assume throughout that $a = 0$. Expand f as a power series centered at b. Writing z as $z = b + x$, with $0 \leq |x| \leq \rho$, we have

$$f(b+x) = f(b) + f(b+x) - f(b) = f(b) + \sum_{n=d}^{\infty} c_n \left((b+x)^n - b^n\right)$$

$$= f(b) + \sum_{n=d}^{\infty} c_n \left(\sum_{i=1}^{n} \binom{n}{i} b^{n-i} x^i \right) = f(b) + \sum_{i=1}^{\infty} \left(\sum_{n=i}^{\infty} c_n b^{n-i} \binom{n}{i} \right) x^i,$$

where we let $c_n = 0$ for $n = 1, \dots, d-1$. Because the original power series for f is convergent on V (and, in particular, at b), the above sums converge, and the exchange of summation signs is justified.

For $i \geq 1$, let C_i denote the coefficient of x^i in the above power series expansion, i.e.,

$$C_i = \sum_{n=i}^{\infty} c_n b^{n-i} \binom{n}{i}.$$

Then the radius of $f(\overline{D}_\rho(b))$ is $\max\{|C_i|\rho^i\}$. By our assumption that $|c_j| r^j < |d! c_d| r^d$ for any $j \neq d, 0$, and by the fact that $\rho \leq |b| \leq r$, it follows that for $i = 1, \dots, d$,

$$|C_i| = \left| c_d b^{d-i} \binom{d}{i} \right|,$$

and for $i > d$,

$$|C_i|\rho^i \leq \max_{n \geq i} \left\{ \left| c_n b^{n-i} \binom{n}{i} \right| \rho^i \right\} \leq \max_{n \geq i} \left\{ \left| c_n b^{n-d} \right| \rho^d \right\} < |c_d|\rho^d = \left| c_d \binom{d}{d} \right| \rho^d.$$

Thus,

$$\max_{i \geq 1} \left\{ |C_i|\rho^i \right\} = \max_{i=1,\dots,d} \left\{ \left| c_d b^{d-i} \binom{d}{i} \right| \rho^i \right\}.$$

However,

$$|b^{d-i}|\rho^i = \sigma^d \left(\frac{\rho}{\sigma}\right)^i,$$

and so, by Lemma 5.1.6, the radius is

$$\max_{i=1,\ldots,d}\left\{\sigma^d\left|c_d\binom{d}{i}\right|\left(\frac{\rho}{\sigma}\right)^i\right\} = |c_d|\sigma^d \max_{e=0,\ldots,v(d)}\left\{\left|p^{v(d)-e}\right|\left(\frac{\rho}{\sigma}\right)^{p^e}\right\}.$$

Our result follows from the observation that $|d| = |p^{v(d)}|$. □

The following two corollaries will be very useful for proving the results of this chapter. In fact, we will not directly use Lemma 5.1.7 nearly as much as we will quote its corollaries.

Corollary 5.1.8. *Let V, f, b, ρ, and σ be as in Lemma 5.1.7, and suppose p does not divide d. Then*

$$\operatorname{rad}\left(f\left(\overline{D}_\rho(b)\right)\right) = |c_d|\rho\sigma^{d-1}.$$

Proof. Immediate from Lemma 5.1.7. □

It should be noted that, for a map satisfying the hypotheses of Corollary 5.1.8, the radius of the image of the larger disk is

$$\operatorname{rad}\left(f\left(\overline{D}_\sigma(b)\right)\right) = |c_d|\sigma^d,$$

and so the ratio of the radii of the two image disks is the same as the original ratio of radii, $\dfrac{\rho}{\sigma}$. Later, in the proof of Lemma 5.3.1, we will take a to be a Julia point and $\overline{D}_\rho(b)$ to be a D-component of the Fatou set of a rational map ϕ. Any disk containing a will eventually get large under iteration of ϕ, while any D-component must stay bounded in size; Corollary 5.1.8 will help to provide a contradiction, provided the power series expansion at a satisfies the required hypotheses.

Corollary 5.1.9. *Let V, f, b, ρ, and σ be as in Lemma 5.1.7, and let $\alpha \in \mathbb{R}$ be the value $\alpha = |p|^{(p-1)^{-1}} < 1$. Suppose $\rho/\sigma \le \alpha$. Then*

$$\operatorname{rad}\left(f\left(\overline{D}_\rho(b)\right)\right) = |dc_d|\rho\sigma^{d-1}.$$

Furthermore, if $\rho/\sigma < \alpha$, then f is one-to-one on $\overline{D}_\rho(b)$.

Proof. For the first statement, it suffices to show that for any nonnegative integer e,

$$|p^{-e}|\alpha^{p^e-1} \le 1.$$

However, the left hand side of this inequality is

$$|p|^{-e}|p|^{(p^e-1)/(p-1)} = |p|^{-e}|p|^{p^{e-1}+p^{e-2}+\cdots+1},$$

so it suffices to show that

$$p^{e-1} + p^{e-2} + \cdots + 1 \ge e.$$

There are e terms on the left hand side, all of which are at least 1; hence, the inequality holds.

To show the second statement, expand f as a power series about b. Let $\{C_i\}$ be the coefficients. As we saw in the proof of Lemma 5.1.7, $|C_1| = |dc_d|\sigma^{d-1}$, and for $i = 2, \ldots, d$,

$$|C_i| = \left| c_d \binom{d}{i} \right| \sigma^{d-i}.$$

f has degree d on V, and therefore it cannot have degree larger than d on $\overline{D}_\rho(b)$; thus, we only need to show that $|C_i|\rho^{i-1} < |C_1|$ for $i = 2, \ldots, d$. However, by Lemma 5.1.6 and the first half of this proof,

$$|C_i|\alpha^{i-1} \leq |C_1|.$$

Because $\rho < \alpha$, the strict inequality holds for $i \geq 2$ and with ρ in place of α. □

Corollary 5.1.9 will be used in much the same way as Corollary 5.1.8, except that it can be used at points a where Corollary 5.1.8 may not apply. In this case, the radius of the image of the larger disk is

$$\mathrm{rad}\left(f\left(\overline{D}_\sigma(b)\right)\right) = |c_d|\sigma^d,$$

and so the ratio of the radii of the two image disks is

$$|d|\frac{\rho}{\sigma}.$$

While this ratio may be smaller than the original ratio of radii, we have at least some control over it if we know something about d.

5.2 Initial Results

The results of the previous section concerning the radii of power series images of disks have some surprising consequences for dynamics. The following theorem restricts the set of disks which can be preperiodic D-components.

Theorem 5.2.1. *Let $\phi \in \mathbb{C}_p(z)$ be a rational map with Fatou set $\mathcal{F}$, and let $U \subset \mathcal{F}$ be a D-component of the Fatou set of ϕ. If U is preperiodic, then U is not an irrational $\mathbb{P}^1(\mathbb{C}_p)$-disk.*

Proof. We begin by assuming that the Julia set $\mathcal{J}$ contains at least two points; otherwise, the statement is trivial. Select a coordinate system so that $\infty \in \mathcal{J}$, and suppose that U is an irrational preperiodic D-component. In particular, $U = \overline{D}_r(a)$ with $r \notin p^{\mathbb{Q}}$. Let Φ denote the action of ϕ on D-components of $\mathcal{F}$.

We begin by claiming that $\phi(U)$ is also an irrational disk, and that $\Phi(U) = \phi(U)$. To prove this, pick some $s > r$ such that $V = \overline{D}_s(a)$ contains no poles of ϕ; such an s must exist, since there are only finitely many poles, none of which lie in U. By Lemma 2.2.3, we can expand ϕ as a power series on V:

$$\phi(z) = \phi(a) + \sum_{i=1}^{\infty} c_i(z-a)^i,$$

with $|c_i|s^i \to 0$ as $i \to \infty$. Let $s' = \max\{|c_i|s^i\}$, and let $r' = \mathrm{rad}(\phi(U))$. Then $r' = \max\{|c_i|r^i\} \notin p^{\mathbb{Q}}$. Furthermore, for any t' with $r' < t' \leq s'$, there must be some t with $r < t < s$ such that $t' = \max\{|c_i|t^i\}$. Since $\overline{D}_t(a)$ contains Julia points, so does its image, $\overline{D}_{t'}(\phi(a))$. It follows that $\overline{D}_{r'}(\phi(a))$ is the largest disk containing $\phi(a)$ and contained in the Fatou set; hence, $\Phi(U) = \phi(U)$, which we have already seen to be an irrational disk.

Applying the claim inductively, we see that for any nonnegative integer n, $\phi^n(U)$ is an irrational disk and coincides with $\Phi^n(U)$. Because U is preperiodic, there is some $m \geq 0$ and $n \geq 1$ such that $\phi^{n+m}(U) = \phi^m(U)$. Thus, by considering the map ϕ^n on the D-component $\phi^m(U)$, it suffices to show our theorem in the case that U is fixed.

Assume that U is fixed by ϕ; we have $U = \overline{D}_r(a)$ and the above power series expansion of ϕ on U. Since $\phi(U) = U$, it follows that

$$r = \max_{i \geq 1} \left\{|c_i|r^i\right\}.$$

If this maximum is achieved for any $i > 1$, then we would have $r^{1-i} = |c_i|$, and therefore $r \in p^{\mathbb{Q}}$. Thus, the maximum is achieved only at $i = 1$, and so $|c_i|r^i < |c_1|r$ for any $i \geq 2$. Furthermore, $|c_1| = 1$.

As before, since there are no poles in a slightly larger disk centered at a, we can write ϕ as a power series on a larger disk. Because the terms of the power series approach zero, we can in fact guarantee that there must be some $s > r$ such that

$$\lim_{i \to \infty} |c_i|s^i = 0 \quad \text{and} \quad \max_{i \geq 1} \left\{|c_i|s^i\right\} = s.$$

Thus, letting $V = \overline{D}_s(a)$, we have $\phi(V) = V$. By Hsia's Theorem (Theorem 2.4.1), V must be Fatou, and hence we have a Fatou disk larger than U and containing a. This contradicts the assumption that U is a D-component, and the proof is complete. □

Assuming that wandering domains do not exist, Theorem 5.2.1 would imply that all D-components are rational disks, provided the Julia set contains at least two points. In that theme, the following proposition restricts wandering domains to closed disks. Of course, it has already been superseded for hyperbolic maps by Theorem 4.2.2 and will be superseded for a larger class of maps by Theorem 5.3.5. However, it has the advantage of holding for all maps defined over $\overline{\mathbb{Q}}_p$, and its corollary equates the No Wandering D-Component and No Wandering Analytic Component conjectures (see Conjecture 1).

Proposition 5.2.2. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$. Let $\mathcal{F}$ be the Fatou set of ϕ, and let Φ denote the action of ϕ on D-components of $\mathcal{F}$. Suppose U is a wandering D-component of $\mathcal{F}$. Then there is some $N \geq 0$ such that for all $n \geq N$, $\Phi^n(U)$ is either a rational closed or irrational $\mathbb{P}^1(\mathbb{C}_p)$-disk.*

Before proving Proposition 5.2.2, we make the following definition for the sake of future convenience.

Definition 5.2.1. *Let $\phi \in \mathbb{C}_p(z)$ be a rational function. We say ϕ is* normalized *if ∞ is a non-repelling fixed point of ϕ, and $\phi(\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)) \subseteq \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$.*

By Corollary 2.4.4, any ϕ has a non-repelling fixed point. Thus, by a change of coordinates, we can move this point to ∞. Then, by another change of the form $z \mapsto cz$, the second condition of the above definition will also hold. Therefore, any rational function is conjugate to a normalized function; and if the original function was defined over $\overline{\mathbb{Q}}_p$, we can guarantee that the normalized version is as well.

Proof of Proposition 5.2.2. By changing coordinates if necessary, we can assume that ϕ is normalized. Note that U is a disk of positive radius around a point in $\mathbb{C}_p$ and therefore contains a point b in $\overline{\mathbb{Q}}_p$. Extend K if necessary so that b is defined over K; it follows that for any $n \geq 0$, $\phi^n(U)$ contains a K-point.

Let $\mathfrak{J}$ denote the Julia set of ϕ, and let A denote the set of accumulation points of $\{\phi^n(b)\}_{n\geq 0}$. Since K is a finite extension of $\mathbb{Q}_p$, A is a compact subset of K. Furthermore, $A \subset \mathfrak{J}$. For if $x \in A \cap \mathfrak{F}$, then the D-component of x intersects infinitely many iterates $\phi^n(U)$, contradicting the hypothesis that U is wandering.

Now for any $x \in A$, there is some disk centered at x which contains no poles; by Lemma 2.2.3, we can write ϕ as a power series on this disk. Cover A by such disks and take a finite subcover; let R be the minimum radius of the disks in the subcover. We can assume that $R \leq 1$. Let W denote the union of all closed disks of radius R containing points of A, i.e.

$$W = \bigcup_{x\in A} \overline{D}_R(x).$$

Note in particular that W contains no poles of ϕ.

Because $\{\phi^n(b)\} \subset K$ and K is locally compact, we see that, by definition of A, there must be some $N \geq 0$ such that for all $n \geq N$, $\phi^n(b) \in W$. In particular, for any such n, there exists $x \in A$ with $\phi^n(b) \in \overline{D}_R(x)$. The disks $\Phi^n(U)$ and $\overline{D}_R(x)$ therefore intersect, and since $x \in A \subset \mathfrak{J}$, it follows that in fact $\Phi^n(U) \subset \overline{D}_R(x)$.

Now suppose that there is some $n \geq N$ such that $\Phi^n(U)$ is of the form $D_r(a)$, with $r \in p^{\mathbb{Q}}$. Because $\Phi^n(U) \subset \overline{D}_R(a)$, we have $r \leq R$. Note that $\phi(D_r(a))$ is contained in $\overline{D}_1(0)$, since U is wandering and therefore cannot have iterates intersecting the fixed component at infinity. Thus, by Lemma 5.1.2, $\phi(D_r(a))$ is of the form $D_s(\phi(a))$; note also that $s \leq R$, since $\Phi^{n+1}(U) \subset \overline{D}_R(\phi(a))$. However, ϕ is defined as a power series on $\overline{D}_R(a)$, and in particular on $\overline{D}_r(a)$; we would like to know what the image of this latter disk is.

The radii of both images, $\phi(D_r(a))$ and $\phi(\overline{D}_r(a))$, are the same, namely,

$$\text{rad}\,(\phi\,(D_r(a))) = \text{rad}\left(\phi\left(\overline{D}_r(a)\right)\right) = \max\{|c_i|r^i\},$$

where $\{c_i\}$ are coefficients of the power series at a. In particular, the radius of $\phi(\overline{D}_r(a))$ is s. By Lemma 5.1.1, this image is a closed disk, and therefore $\phi(\overline{D}_r(a)) = \overline{D}_s(\phi(a))$.

Repeating this argument with $D_s(\phi(a))$ in place of $D_r(a)$, we see that

$$\text{rad}(\phi^2(\overline{D}_r(a))) \leq R,$$

and in general, for any $k \geq 0$,

$$\text{rad}(\phi^k(\overline{D}_r(a))) \leq R.$$

Thus, all forward iterates of $\overline{D}_r(a)$ are contained in W. By Theorem 2.4.1, $\overline{D}_r(a)$ is contained in the Fatou set. But because $r \in p^{\mathbb{Q}}$, $\overline{D}_r(a) \setminus D_r(a) \neq \emptyset$, and we have contradicted our assumption that $D_r(a)$ was a full D-component. For $n \geq N$, then, all $\Phi^n(U)$ must be closed disks (either rational or irrational). □

Corollary 5.2.3. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$. Let Φ denote the action of ϕ on the set of D-components of the Fatou set $\mathcal{F}$. Suppose V is a wandering analytic component of ϕ. Then there is some $N \geq 0$ such that for all $n \geq N$, $\Phi^n(U)$ is a rational closed or irrational $\mathbb{P}^1(\mathbb{C}_p)$-disk. In particular, ϕ has wandering analytic components if and only if it has wandering D-components.*

Proof. If V is a wandering analytic component, then any D-component U contained in V is also wandering. If none of the iterates of V are disks, then by Proposition 3.2.1, all of the iterates of U are rational open disks. By Proposition 5.2.2, such a D-component cannot exist. Thus, all iterates of V after a certain point must be disks; again by Proposition 5.2.2, they must be closed disks.

To prove the final statement of the corollary, it is clear that a wandering analytic component contains wandering D-components; but because wandering analytic components must eventually be disks, they are eventually D-components. □

We close this section with a related proposition on periodic components defined over $\overline{\mathbb{Q}}_p$. Like Proposition 5.2.2, it has already been superseded by Theorem 4.2.3 for hyperbolic maps and will be further superseded by Theorem 5.3.6 for a larger class of maps. Once again, however, while it only applies to rational open disks, it has the advantage of applying to all maps defined over $\overline{\mathbb{Q}}_p$.

Proposition 5.2.4. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$. Let L be a finite extension of K. Then there are at most finitely many periodic rational open D-components of the Fatou set $\mathcal{F}$ which contain points of L.*

Proof. Assume without loss that ϕ is normalized. Let $D_{s_1}(b_1), \ldots, D_{s_m}(b_m)$ be Fatou disks which together contain all poles of ϕ. Let $s = \min\{s_i\} > 0$.

Suppose U is a rational open periodic D-component of $\mathcal{F}$; assume that U is not the fixed component at ∞. Let $\{U, \Phi(U), \Phi^2(U), \ldots, \phi^{n-1}(U)\}$ be all the forward iterates of U. All must be rational open disks; for if one were closed, then by the same argument as in the proof of Theorem 5.2.1, all would be closed.

For any $k \geq 0$, $\Phi^k(U) = D_{r_k}(a_k)$, with $r_k \in p^{\mathbb{Q}}$ and $a_k \in \mathbb{C}_p$. If none of the $\overline{D}_{r_k}(a_k)$ contain poles, then by the same argument as in the proof of Proposition 5.2.2, each $\overline{D}_{r_k}(a_k)$ would be Fatou, contradicting the fact that $\Phi^k(U)$ is a D-component. Thus, there is some k such that $\overline{D}_{r_k}(a_k)$ contains a pole. In particular, $r_k \geq s$.

Thus, any periodic cycle of rational open D-components includes one of radius at least s. Because L is locally compact, and because all iterates of such a D-component are contained in $\overline{D}_1(0)$, there can be only finitely many which contain points of L. □

Proposition 5.2.4 implies the following corollary, which can be proved in the same way as Corollary 5.2.3.

Corollary 5.2.5. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$. Let L be a finite extension of K. There are infinitely many periodic analytic components of the Fatou set $\mathfrak{F}$ which contain points of L if and only if there are infinitely many D-components of $\mathfrak{F}$ containing points of L.*

5.3 Rational maps with no wild recurrent Julia critical points

5.3.1 Definitions and Main Lemma

Definition 5.3.1. *Let $\phi \in \mathbb{C}_p(z)$ be a rational map. We say that a point $x \in \mathbb{P}^1(\mathbb{C}_p)$ is* recurrent *if x is not periodic but is contained in the closure of the set $\{\phi^n(x) : n \geq 0\}$.*

We will often abuse language and say that x *accumulates at* y if for any $N \geq 0$, y is contained in the closure of $\{\phi^n(x) : n \geq N\}$. Thus, we could define a recurrent point as a non-periodic point that accumulates at itself.

Definition 5.3.2. *Let $\phi \in \mathbb{C}_p(z)$ be a rational map. We say that a critical point $x \in \mathbb{P}^1(\mathbb{C}_p)$ of ϕ is* wild *if the index of ramification of ϕ at x is divisible by p. If x is not wild, we say x is* tame.

For example, if $\phi(z) = z^p$, then 0 is a wild critical point. From the standpoint of algebraic geometry, a critical point is wild if it is a point of wild ramification of the map $\phi : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p)$.

In addition to the two preceding "standard" definitions, we will also need the following nonstandard definition. It is useful only for simplifying the language needed for proving our main lemma (Lemma 5.3.1).

Definition 5.3.3. *Let $\phi \in \mathbb{C}_p(z)$ be a rational map with Fatou set $\mathfrak{F}$ and Julia set $\mathfrak{J}$; let Φ denote the action of ϕ on D-components of $\mathfrak{F}$. Let $x \in \mathfrak{J}$ with $x \neq \infty$. Let $K \subset \mathbb{C}_p$ be a complete extension of $\mathbb{Q}_p$. Given a real number $\varepsilon > 0$, we say that x has property $P(\varepsilon, K)$ if there exist positive real numbers $M, r > 0$ (which depend on K, ϕ, and ε) such that the following condition holds:*

For any D-component U of $\mathfrak{F}$ with $U \subset \overline{D}_r(x)$, $U \cap K \neq \emptyset$, and

$$\frac{\mathrm{rad}(U)}{\mathrm{dist}(U, x)} \geq \varepsilon,$$

there is a nonnegative integer k such that

$$\mathrm{rad}\left(\Phi^k(U)\right) \geq M.$$

For the purpose of the above definition, we will consider the radius of a D-component containing ∞ to be infinite.

The idea of Definition 5.3.3 is that if a D-component containing a K-point is large relative to its distance from a Julia point with property P, then some iterate of the D-component is large in a global sense. Thus, property P will be very useful for proving our main lemma, which we are now prepared to state.

Lemma 5.3.1. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a normalized rational map with Fatou set $\mathcal{F}$ and Julia set $\mathcal{J}$. Let Φ denote the action of ϕ on the set of D-components of $\mathcal{F}$. Let L be any finite extension of K. Then there exist positive constants $M, R > 0$ with the following property:*

If U is a D-component of $\mathcal{F}$ with $U \cap L \neq \emptyset$ and $\mathrm{dist}(U, \mathcal{J} \cap L) \leq R$, then there is some $k \geq 0$ such that

$$\mathrm{rad}\left(\Phi^k(U)\right) \geq M.$$

As in Definition 5.3.3, we will consider the D-component containing ∞ to have infinite radius.

Lemma 5.3.1 is a partial generalization of Lemma 4.2.1, and its corollaries are the main theorems of this thesis. To prove it, we will need several technical lemmas. The reader is encouraged to skip to the aforementioned theorems, found in Section 5.3.3, before delving into Section 5.3.2, which contains the technical lemmas and the proof of Lemma 5.3.1.

5.3.2 Technical Lemmas

Lemma 5.3.2. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be normalized, with Julia set $\mathcal{J}$. Let $x \in \mathcal{J}$ and $N \geq 0$ such that there are no critical points in $\{\phi^n(x) : n \geq N\}$, and for any $\varepsilon > 0$, $\phi^N(x)$ has property $P(\varepsilon, K)$. Then for any $\varepsilon > 0$, x has property $P(\varepsilon, K)$.*

Proof. Expand ϕ^N as a power series

$$\phi^N(z) = c_0 + \sum_{i=d}^{\infty} c_i (z-x)^i$$

centered at x, with $c_d \neq 0$. Pick $s > 0$ so that the series converges on $\overline{D}_s(x)$, and $|c_i| s^i < |d! c_d| s^d$ for any $i > d$. By hypothesis, given $\varepsilon > 0$, $\phi^N(x)$ has property $P(|d|\varepsilon, K)$. Let r be the radius around $\phi^N(x)$ in Definition 5.3.3. Decrease s if necessary so that $\phi^N(\overline{D}_s(x)) \subseteq \overline{D}_r(\phi^N(x))$. Let M be the lower bound from Definition 5.3.3 for $\phi^N(x)$.

Let $\mathcal{F}$ be the Fatou set of ϕ, and let $\mathcal{U}_K$ be the set of all D-components of $\mathcal{F}$ containing points of K. By Lemma 5.1.7, it follows that if $U \in \mathcal{U}_K$ such that $U \subset \overline{D}_s(x)$, then

$$\mathrm{rad}\left(\Phi^N(U)\right) \geq \mathrm{rad}\left(\phi^N(U)\right) \geq |dc_d| \mathrm{dist}(U, x)^{d-1} \mathrm{rad}(U),$$

and

$$\mathrm{dist}\left(\Phi^N(U), \phi^N(x)\right) = |c_d| \mathrm{dist}(U, x)^d.$$

Therefore, if

$$\frac{\mathrm{rad}(U)}{\mathrm{dist}(U,x)} \geq \varepsilon,$$

then

$$\frac{\mathrm{rad}\left(\Phi^N(U)\right)}{\mathrm{dist}\left(\Phi^N(U),\phi^N(x)\right)} \geq |d|\frac{\mathrm{rad}(U)}{\mathrm{dist}(U,x)} = |d|\varepsilon.$$

Since $\phi^N(x)$ has property $P(|d|\varepsilon, K)$ with lower bound M, it follows that some iterate of $\Phi^N(U)$ has radius at least M, and we are done. □

Before stating the next lemma, we need the following notation. Given K a finite extension of $\mathbb{Q}_p$ and $\phi \in K(z)$ with Julia set $\mathcal{J}$, define

$$C_{\mathcal{J}} = \{y \in \mathcal{J} : \phi'(y) = 0\}$$

to be the set of all Julia critical points, and let $S_0 = \emptyset$ and $T_0 = C_{\mathcal{J}}$. Then, define S_i and T_i inductively for $i \geq 1$ by

$$\begin{aligned} S_i &= \{C_{\mathcal{J}} \text{ points not accumulating at any wild } T_{i-1} \text{ points}\} \\ T_i &= C_{\mathcal{J}} \setminus S_i. \end{aligned}$$

Lemma 5.3.3. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be normalized. Let $x \in S_i$ for some $i \geq 0$. Then for any $\varepsilon > 0$, x has property $P(\varepsilon, K)$.*

Proof. We will proceed by induction on i. The statement is vacuous for $i = 0$; for positive i, assume that it is known for $i - 1$, and we will prove it for i.

Pick $x \in S_i$. Pick $N \geq 0$ such that there are no critical points in the set $\{\phi^n(x) : n \geq N\}$. Such an N must exist; otherwise, since there are only finitely many critical points, some iterate of x would be a periodic critical point and hence Fatou. By Lemma 5.3.2, it suffices to show that $\phi^N(x)$ has property $P(\varepsilon, K)$ for any positive ε. Thus, we may assume that x has no critical points in its forward orbit.

Pick $\varepsilon > 0$; we can assume that $\varepsilon \in |K^*|$ and $\varepsilon < 1$. Let $\mathcal{F}$ be the Fatou set and $\mathcal{J}$ the Julia set of ϕ. Let $\alpha = |p|^{(p-1)^{-1}} < 1$. Let $C_t \subset C_{\mathcal{J}}$ denote the set of tame Julia critical points. Extend K if necessary to contain $C_{\mathcal{J}}$, and also so that $\alpha \in |K^*|$. Let π be a uniformizer of K. Note that $x \in K$. Let $\mathcal{U}_K$ denote the set of all D-components of $\mathcal{F}$ which contain points of K.

We will now cover $\mathcal{J} \cap K$ with a finite set of disks, similar to the covering of the set A in the proof of Proposition 5.2.2. For any $z_0 \in \mathcal{J} \cap K$, there is some $s > 0$ such that $\phi(z)|_{\overline{D}_s(z_0)}$ is of the form

$$c_0 + \sum_{i=d}^{\infty} c_i (z - z_0)^i$$

where $d \geq 1$, $c_d \neq 0$, and $|c_i| s^{i-d} < |d! c_d|$ for all $i > d$. Cover $\mathcal{J} \cap K$ by such disks and take a finite subcover. Let R be the minimum radius of the disks in the subcover; we may assume that $R \leq 1$. Let W be the union of all closed disks of radius R centered at points of $\mathcal{J} \cap K$.

Pick $r > 0$ such that for any critical point a at which x does not accumulate,

$$|\phi^n(x) - a| > \frac{r}{\alpha} \tag{5.3}$$

for all $n \geq 0$. Note that equation (5.3) implies that all accumulation points of x must also be at least distance r/α from such critical points. Decrease r if necessary so that $r < R$. Let M be the minimum of the lower bounds required in the definition of property $P(\alpha\varepsilon, K)$ for each of the (finitely many) points in S_{i-1}. Decrease M if necessary so that $0 < M \leq r\varepsilon$.

Pick $U \in \mathfrak{U}_K$ with $U \subset \overline{D}_r(x)$ and

$$\frac{\operatorname{rad}(U)}{\operatorname{dist}(U, x)} \geq \varepsilon.$$

Pick $b \in U \cap K$. Let ρ_0 be the largest value in $|K^*|$ such that $\overline{D}_{\rho_0}(b) \subseteq U$, and let $\sigma_0 = \operatorname{dist}(U, x)$. Note that $\sigma_0 = |b - x| \in |K^*|$. For $k \geq 1$, define ρ_k and σ_k inductively, as follows. Given ρ_{k-1}, let

$$\rho_k = \operatorname{rad}\left(\phi\left(\overline{D}_{\rho_{k-1}}\left(\phi^{k-1}(b)\right)\right)\right) \leq \operatorname{rad}(\Phi^k(U)).$$

Given σ_{k-1}, if $\operatorname{dist}(\phi^{k-1}(b), C_t) \geq \sigma_{k-1}$, let

$$\sigma_k = \operatorname{rad}\left(\phi\left(\overline{D}_{\sigma_{k-1}}\left(\phi^{k-1}(b)\right)\right)\right).$$

Otherwise, if $y \in C_t$ with $|\phi^{k-1}(b) - y| = \operatorname{dist}(\phi^{k-1}(b), C_t) < \sigma_{k-1}$, let

$$\sigma'_{k-1} = |\phi^{k-1}(b) - y|$$

and

$$\sigma_k = \operatorname{rad}\left(\phi\left(\overline{D}_{\sigma'_{k-1}}\left(\phi^{k-1}(b)\right)\right)\right).$$

Note that for any $k \geq 0$, $\overline{D}_{\sigma_k}(\phi^k(b))$ contains a point of $\mathfrak{J} \cap K$. This is because $\overline{D}_{\sigma_0}(b)$ contains such a point (namely x_0), and therefore all of its forward iterates contain points of $\mathfrak{J} \cap K$. In addition, when we shrink σ_k to σ'_k, we do so because $\overline{D}_{\sigma'_k}(\phi^k(b))$ contains $y \in \mathfrak{J} \cap K$; hence, the new disk and all its iterates contain $\mathfrak{J} \cap K$ points.

Let e_k denote the ratio ρ_k/σ_k. By our choice of U, note that $\varepsilon \leq e_0 < 1$. Also note that $\rho_k, \sigma_k, e_k \in |K^*|$.

Claim 5.3.1. *If $\sigma_k \leq r$ and $S_{i-1} \cap \overline{D}_{\sigma_k/\alpha}(\phi^k(b)) = \emptyset$, then*

1. *if $\operatorname{dist}(\phi^k(b), C_t) \geq \sigma_k$, then $e_{k+1} = e_k$.*

2. *otherwise, $e_{k+1} \geq |\pi^{-1}|e_k$.*

Assume the claim is true for a moment. $\overline{D}_{\sigma_0}(b)$ intersects the Julia set (at x), so its iterates have arbitrarily large radii. Using the claim repeatedly, we see that at some step k, either $\sigma_k \geq r$, or there is some $x' \in S_{i-1} \cap \overline{D}_{\sigma_k/\alpha}(\phi^k(b))$, or some $y \in C_t$ is close to $\phi^k(b)$. In the first case, $\rho_k \geq r\varepsilon \geq M$ (since $e_k \geq \varepsilon$), and so $\Phi^k(U)$ has radius at least M, and we are done. In the second case, $\Phi^k(U)$ satisfies

$$\frac{\mathrm{rad}\left(\Phi^k(U)\right)}{\mathrm{dist}\left(\Phi^k(U), x'\right)} \geq \alpha\varepsilon,$$

(again, because $e_k \geq \varepsilon$), and by the inductive hypothesis, some iterate of $\Phi^k(U)$ has radius at least M. In the third case, we note that $e_{k+1} > e_k$, and that $\overline{D}_{\sigma'_k}(\phi^k(b))$ intersects the Julia set (at x'); therefore, its iterates must eventually have large radii. Thus, we can start our process again by iterating $\overline{D}_{\sigma'_k}(\phi^k(b))$.

Provided the radii stay smaller than r and the iterates stay away from S_{i-1} points, we can continue this process indefinitely. At each stage, we either produce an iterate of U with radius at least M (and the process stops), or we increase e_k. However, $e_k < 1$, since the disk of radius σ_k contains Julia points, and the disk of radius ρ_k does not. Furthermore, when e_k increases, it increases by a factor of at least $|\pi^{-1}|$; thus, it can only increase a bounded number of times. Thus, at some stage, we must produce an iterate of U with radius at least $r\varepsilon$. To prove the lemma, then, it suffices to prove the claim.

Fix $k \geq 0$, and suppose $\sigma_k \leq r$ and $S_{i-1} \cap \overline{D}_{\sigma_k/\alpha}(\phi^k(b)) = \emptyset$. Let

$$V_\sigma = \overline{D}_{\sigma_k}(\phi^k(b)) \quad \text{and} \quad V_\rho = \overline{D}_{\rho_k}(\phi^k(b)).$$

As we saw above, V_σ contains some point z of $\mathcal{J} \cap K$; because $\sigma_k \leq r < R$, we have $V_\sigma \subset \overline{D}_R(z)$. By our choice of R, we know that $\overline{D}_R(z)$ contains at most one critical point; and if there is a critical point, it must be in $\mathcal{J} \cap K$.

If there is no critical point in $\overline{D}_R(z)$, then by Lemma 5.1.3, our choice of R guarantees that ϕ is one-to-one on $\overline{D}_R(z)$ and hence on V_σ; thus, the ratio of radii of $\phi(V_\sigma)$ to $\phi(V_\rho)$ is the same as that of V_σ to V_ρ, and we are done.

If there is a wild critical point $a \in \overline{D}_R(z)$, then it must be outside $\overline{D}_{\sigma_k/\alpha}(\phi^k(b))$. This is because x does not accumulate at any wild points besides those in S_{i-1}; and by our definition of r, the ratio of σ_k to the distance between b and a is less than α. The reader may object that, by our choices of $\{\sigma_j\}$, we cannot assume that some iterate of x lies in $\overline{D}_{\sigma_k/\alpha}(\phi^k(b))$. However, if at some point we decreased σ_i to σ'_i, the resulting disk contained a critical point $y \in C_t$ which was within r of an iterate of x. By our choice of r, y must be an accumulation point of the iterates of x, and therefore some iterate of x must be nearby.

Thus, in the case of a wild critical point $a \in \overline{D}_R(z)$, we can apply Corollary 5.1.9 to the power series expansion of ϕ about a, and we see that ϕ preserves the ratio of the radii of V_σ and V_ρ. By Corollary 5.1.8, the same is true if there is a tame critical point in $\overline{D}_R(z)$ which is not in V_σ.

The only case that remains to be considered is that V_σ contains a tame critical point y. As before, y must in fact be an accumulation point of x_0.

If $\mathrm{dist}(\phi^k(b), y) = \sigma_k$, then applying ϕ to V_σ and V_ρ, we see by Corollary 5.1.8, that

$$e_{k+1} = \frac{\rho_{k+1}}{\sigma_{k+1}} = \frac{\mathrm{rad}\left(\phi\left(\overline{D}_{\rho_k}\left(\phi^k(b)\right)\right)\right)}{\mathrm{rad}\left(\phi\left(\overline{D}_{\sigma_k}\left(\phi^k(b)\right)\right)\right)} = \frac{\rho_k}{\sigma_k} = e_k.$$

On the other hand, if $\sigma'_k = \mathrm{dist}(\phi^k(b), y) < \sigma_k$, we can apply Corollary 5.1.8 to $\overline{D}_{\sigma'_k}\left(\phi^k(b)\right)$ to show that

$$e_{k+1} = \frac{\rho_{k+1}}{\sigma_{k+1}} = \frac{\mathrm{rad}\left(\phi\left(\overline{D}_{\rho_k}\left(\phi^k(b)\right)\right)\right)}{\mathrm{rad}\left(\phi\left(\overline{D}_{\sigma'_k}\left(\phi^k(b)\right)\right)\right)} = \frac{\rho_k}{\sigma'_k} > \frac{\rho_k}{\sigma_k} = e_k.$$

Furthermore, $e_k, e_{k+1} \in |K^*|$, so if $e_{k+1} > e_k$, then $e_{k+1} \geq |\pi^{-1}|e_k$. The proof of the claim is complete, and the lemma follows. □

Lemma 5.3.4. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ have no wild recurrent Julia critical points. Then there exists some $m \geq 0$ such that $T_m = \emptyset$.*

Proof. Note that $T_0 = C_{\mathcal{J}}$, and

$$T_{i+1} = \{C_{\mathcal{J}} \text{ points accumulating at wild } T_i \text{ points}\}.$$

Therefore, we can write

$$T_i = \left\{ a_0 \in C_{\mathcal{J}} \;\middle|\; \begin{array}{l} \exists a_1, \ldots, a_i \in C_{\mathcal{J}} \text{ wild, and} \\ \forall j = 0, \ldots, i, \;\; a_j \text{ accumulates at } a_{j+1} \end{array} \right\}.$$

Let $m-1$ be the number of wild Julia critical points. If T_m were nonempty, then there would be wild Julia critical points $a_1, \ldots, a_m$ with a_j accumulating at a_{j+1}. Thus, there must be j and k with $j < k$ and $a_j = a_k$. Note that accumulation is transitive; if a accumulates at b and b accumulates at c, then a accumulates at c. Thus, a_j accumulates at $a_k = a_j$; it follows that a_j is a wild recurrent Julia critical point, contradicting the hypotheses of the lemma. So $T_m = \emptyset$. □

We can now prove our main lemma.

Proof of Lemma 5.3.1. Extend K to contain L. Let $C_{\mathcal{J}}$ denote the set of Julia critical points, and let $\alpha = |p|^{(p-1)^{-1}}$; extend K again to contain $C_{\mathcal{J}}$ and so that $\alpha \in |K|$. Let π be a uniformizer of K. Define the radius R and the set W as in the proof of Lemma 5.3.3. Let $\mathcal{U}_K$ denote the set of all D-components of $\mathcal{F}$ which contain points of K.

By Lemmas 5.3.3 and 5.3.4, we know that for any $\varepsilon > 0$, all Julia critical points have property $P(|\pi|\alpha, K)$. Let M be the minimum of the lower bounds required in Definition 5.3.3 for each of the (finitely many) Julia critical points to have property $P(|\pi|\alpha, K)$. Decrease M if necessary so that $M \leq R$.

Claim 5.3.2. *For any $U \in \mathcal{U}_K$ with $U \subset W$, there exists $k \geq 0$ such that either*

1. $\mathrm{rad}(\Phi^k(U)) \geq R$, *or*

2. there is $y \in C_{\mathcal{J}}$ with

$$\frac{\operatorname{rad}\left(\Phi^k(U)\right)}{\operatorname{dist}\left(\Phi^k(U), y\right)} \geq |\pi|\alpha.$$

The key observation used in the proof of the claim is that for any disk $V \subset W$ with $\operatorname{rad}(V) < R$ and

$$\frac{\operatorname{rad}(V)}{\operatorname{dist}(V, C_{\mathcal{J}})} < \alpha,$$

ϕ must be one-to-one on V. To see this, pick $a \in V$, and consider the disk $\overline{D}_R(a) = \overline{D}_R(x)$ for some $x \in \mathcal{J} \cap K$. If $\overline{D}_R(x)$ contains no critical points, then by Lemma 5.1.3 and our choice of R, ϕ is one-to-one on $\overline{D}_R(x)$ and hence on V. On the other hand, if $\overline{D}_R(x)$ does contain critical points, then it contains exactly one, which lies in $\mathcal{J} \cap K$; we can assume that x is this critical point. By Corollary 5.1.9, ϕ is one-to-one on V, because the radius of V is less than a factor of α times the distance of V to x.

We prove the claim by contradiction. Pick $U \in \mathfrak{U}_K$ for which the claim fails. Pick $b \in U \cap K$. Let $r = \operatorname{rad}(U)$, and let $s \in |K^*|$ be the smallest value in $|K^*|$ which is strictly larger than r. By definition of D-components, $\overline{D}_s(b)$ contains Julia points.

Since the claim fails for $k = 0$, we see that $r < R$ and

$$\frac{r}{\operatorname{dist}(U, C_{\mathcal{J}})} < |\pi|\alpha.$$

Because $|\pi|\alpha, R \in |K^*|$, it follows that $s \leq R$ and

$$\frac{s}{\operatorname{dist}(U, C_{\mathcal{J}})} \leq |\pi|\alpha < \alpha.$$

As we saw above, ϕ must be one-to-one on $\overline{D}_s(b)$, and so, by Lemma 5.1.3,

$$\frac{\operatorname{rad}\left(\phi(U)\right)}{\operatorname{rad}\left(\phi\left(\overline{D}_s(b)\right)\right)} = \frac{r}{s}.$$

Similarly, by choosing $k = 1$, it follows that

$$\frac{\operatorname{rad}\left(\phi^2(U)\right)}{\operatorname{rad}\left(\phi^2\left(\overline{D}_s(b)\right)\right)} = \frac{r}{s},$$

and, continuing the process, for any $k \geq 0$,

$$\frac{\operatorname{rad}\left(\phi^k(U)\right)}{\operatorname{rad}\left(\phi^k\left(\overline{D}_s(b)\right)\right)} = \frac{r}{s}.$$

In particular, every $\phi^k\left(\overline{D}_s(b)\right)$ has radius at most $R|\pi|^{-1}$ and is therefore contained in $\overline{D}_{|\pi^{-1}|}(0)$; by Theorem 2.4.1, $\overline{D}_s(b)$ is contained in the Fatou set. But we saw before that it contains Julia points. We have a contradiction, and so the claim follows.

The claim tells us that given any U as in the statement of Lemma 5.3.1, some iterate $\Phi^k(U)$ either has radius at least R, or there is $y \in C_{\mathcal{J}}$ with

$$\frac{\operatorname{rad}\left(\Phi^k(U)\right)}{\operatorname{dist}\left(\Phi^k(U), y\right)} \geq |\pi|\alpha. \tag{5.4}$$

In the former case, we have an iterate of radius at least M, as desired. In the latter case, because y has property $P(|\pi|\alpha, K)$ with lower bound M, we know that some later iterate of U has radius M. Either way, the proof is complete. □

5.3.3 Theorems

The strongest results of this thesis now follow relatively easily from the main lemma. First and foremost, we have the following partial analogue of Sullivan's No Wandering Domains Theorem; it is also a partial generalization of Theorem 4.2.2.

Theorem 5.3.5. *(No Wandering Domains.) Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ have no wild recurrent Julia critical points. Then ϕ has no wandering D-components.*

Proof. Given $\phi \in K(z)$ with no recurrent Julia critical points, we can assume that ϕ is normalized. We do so by conjugating the original ϕ by some element of $\mathrm{PGL}(2, \overline{\mathbb{Q}}_p)$; the resulting normalized function is defined over a finite extension of K, so we replace K by this finite extension.

Suppose U is a wandering D-component. Then U must contain some point $b \in \overline{\mathbb{Q}}_p$. Let $L = K(b)$; hence, U and all its iterates contain points of L. Since $\mathbb{P}^1(L)$ is compact, the sequence $\{\phi^n(b)\}_{n\geq 0}$ has accumulation points in $\mathbb{P}^1(L)$. Let $A \subset \mathbb{P}^1(L)$ denote the set of such accumulation points. Then $A \subset \mathcal{J} \cap L$, for if an accumulation point were Fatou, then its D-component would contain infinitely many iterates of U, contradicting the hypothesis that U was wandering.

Select $M, R > 0$ by Lemma 5.3.1. There must be some integer $N \geq 0$ such that for all $n \geq N$, $\operatorname{dist}(\Phi^n(U), A) \leq R$; thus, $\operatorname{dist}(\Phi^n(U), \mathcal{J} \cap L) \leq R$. By Lemma 5.3.1, for any $n \geq N$, there must be some iterate $\Phi^{n+k_0}(U)$ of radius at least M. Then, starting with $\Phi^{n+k_0+1}(U)$, there is some further iterate $\Phi^{n+k_0+k_1}(U)$ of radius at least M. We can continue this process to produce an infinite sequence of iterates of U, all of radius at least M, all containing points of L, and, because U is wandering, all distinct. Since they are all full D-components, they cannot even intersect.

However, none of the iterates of U can be the D-component at ∞ (which is fixed), and therefore they are all contained in $\overline{D}_1(0)$. Thus, we have infinitely many non-intersecting disks of radius $M > 0$ centered at points of $L \cap \overline{D}_1(0)$. Because L is locally compact, this is impossible; we have the desired contradiction. □

By a similar argument, we can also prove the following generalization of Theorem 4.2.3.

Theorem 5.3.6. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a rational map with Fatou set $\mathcal{F}$. Suppose that ϕ has no wild recurrent Julia critical points.*

Let L be any finite extension of K. Then there are only finitely many periodic D-components of $\mathcal{F}$ which contain points of L.

Sullivan's proof of the complex No Wandering Domains Theorem (see [32]) is completely general; it uses the theory of quasi-conformal maps to generate too many functions in the moduli space of all rational maps of a given degree. Such a theory is not currently available in the p-adic setting; however, it seems likely that Sullivan's theorem should still hold in full generality:

Conjecture 1. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a rational function. Then ϕ has no wandering D-components. Furthermore, if L is a finite extension of K, then the Fatou set of ϕ has only finitely many periodic D-components containing points of L.*

Recall that, by Corollaries 5.2.3 and 5.2.5, the truth of the above conjecture would not be changed by substituting "analytic components" for "D-components".

While Theorem 5.3.5 is not as strong as Conjecture 1, it is very strong in its own right. The examples of non-hyperbolic maps given at the end of Section 4.3 have no recurrent Julia critical points. In fact, it is not currently known whether there exist maps with wild recurrent Julia critical points. However, in Section 7.2, we will see an example of a function which might have this property.

5.4 Restrictions on Conjectures

At first glance it may appear that in the preceding theorems and conjectures on the finite number of periodic D-components, the reference to a finite extension L of K is an annoyance which should be removable. After all, in the complex setting, there are always at most finitely many periodic components of the Fatou set; there is no need to refer to finite extensions. The standard proof of this result is essentially to associate critical points with each periodic cycle (see, for example, [6] or [23]). It is very easy in the complex case to associate critical points to attracting cycles, and with some more work, critical points can be associated to other types of periodic components. However, this association fails in the p-adic case; in fact, the reference to the finite extension L in Conjecture 1 cannot be removed.

In Section 3.3.5, we presented an example of a map with infinitely many periodic D-components. However, in that example, the infinite set of D-components were all contained in a single analytic component; thus, it is still conceivable that there must be only finitely many analytic components. Unfortunately, this hope also fails to be true.

We now give two examples to illustrate the situation. In the first, we present a function with infinitely many periodic analytic components. In the second, we demonstrate that analytic components containing attracting points need not have associated critical points.

Example. Let p be an odd prime, and let

$$\phi(z) = \frac{z^3 + (1+p)z^2}{z+1} = z^2 + \frac{pz^2}{z+1}.$$

Let $\mathcal{F}$ and $\mathcal{J}$ be the Fatou and Julia sets of ϕ. One easily checks that the fixed points of ϕ are ∞, 0, α_1, and α_2, where α_1 and α_2 are the roots of $z^2 + pz - 1$. By Hensel's Lemma, one can verify that $\alpha_1 \in 1 - \frac{p}{2} + p^2\mathbb{Z}_p$, and $\alpha_2 \in -1 - \frac{p}{2} + p^2\mathbb{Z}_p$. Note that

$$\phi'(z) = 2z + \frac{pz(z+2)}{(z+1)^2};$$

thus, $|\phi'(\alpha_2)| = p > 1$, so α_2 is repelling and hence Julia.

If $|z| > 1$, then $|\phi(z)| = |z^2|$, so $\phi^n(z) \to \infty$, and we have a fixed D-component of $\mathcal{F}$. Similarly, if $|z| < 1$, then $|\phi(z)| = |z^2|$, so $\phi^n(z) \to 0$, and we have another fixed D-component. If $|z| = 1$ and $|z+1| = 1$, then it is easy to verify that $\phi(D_1(z)) = D_1(z^2)$.

Now if $n \geq 1$ is a positive integer, and $|z| = 1$ with

$$\overline{z}^{2^n-1} = 1,$$

(where $\overline{z}$ denotes the reduction of z modulo the maximal ideal of $\mathcal{O}$), then by applying ϕ^i to $D_1(z)$ as above, we will never hit $D_1(-1)$; this is simply because, in the residue field, no integer power of $\overline{z}$ is -1. Thus, $\phi^n(D_1(z)) = D_1(z)$; if n is the smallest integer such that $\overline{z}^{2^n-1} = 1$, then this disk has exact period n.

On the other hand, if $|w| = 1$ with

$$\overline{w}^{2^n} = -1,$$

then $\phi^n(D_1(w)) = D_1(-1)$; and because $\alpha_2 \in D_1(-1)$, there must be a Julia point in $D_1(w)$.

Thus, we have infinitely many Julia points w with $|w| = 1$, and in fact, infinitely many distinct residue classes of such Julia points. As a result, given any of the periodic disks $D_1(z)$ (where $\overline{z}^{2^n-1} = 1$), there is no larger disk, or even connected affinoid, containing $D_1(z)$ and contained in $\mathcal{F}$. Thus, each such $D_1(z)$ is a full analytic component. As we have seen, they are all periodic, and there are infinitely many of them.

Example. In the previous example, the only components containing attracting points were the ones at 0 and ∞; those components contained critical points (at 0 and ∞). In this example, we will exhibit attracting cycles with no corresponding critical points.

Let $p = 2$, and let

$$\phi(z) = 4z^3 + z^2 + \frac{1}{2}.$$

So

$$\phi'(z) = 12z^2 + 2z.$$

One easily verifies that, besides the superattracting fixed point at ∞, the only fixed points are the three roots $\{\alpha_i\}_{i=1,2,3}$ of $8z^3 + 2z^2 - 2z + 1$. By Thereom 2.2.1, one can check that

$$|\alpha_1| = 4 > 1, \quad |\alpha_2| = |\alpha_3| = \frac{1}{\sqrt{2}} < 1.$$

Thus, ϕ has attracting fixed points (namely, α_2 and α_3) besides the one at ∞. Because α_1 is repelling, the Julia set of ϕ is nonempty. We will see in Theorem 7.1.2 that in this situation, α_2 and ∞ must lie in distinct analytic components.

Meanwhile, besides the critical point at ∞, the only other critical points are at 0 and $-\frac{1}{6}$. Now

$$0 \mapsto \frac{1}{2} \mapsto \frac{5}{4} \mapsto \frac{67}{8}$$

and then is attracted to ∞, while

$$-\frac{1}{6} \mapsto \frac{55}{108} = \frac{1}{4} \cdot \frac{55}{27} \mapsto \frac{202757}{157464} = \frac{1}{8} \cdot \frac{202757}{19683}$$

and then is also attracted to ∞. Thus, the analytic component at α_2 does not attract any critical points, even though α_2 is attracting.

Question. Is it possible for a rational map with nonempty Julia set to have infinitely many distinct analytic components which contain attracting periodic points?

5.5 Entire Maps

Sullivan's original proof of the No Wandering Domains theorem and our proof in the p-adic case both rely heavily on the fact that the map in question is rational. Sullivan works in the space of rational functions of a given degree, and our p-adic proof uses the lack of essential singularities. Thus, it is not surprising that the theorem fails in both cases when the function in question is entire, rather than rational. Baker constructed an entire complex function with a wandering domain in [2]; his wandering domain accumulated only at ∞. Eremenko and Lyubich ([8]) produced entire functions with wandering domains having more accumulation points. In this section, we follow Baker's model to construct a p-adic power series converging on $\mathbb{C}_p$ (i.e., a p-adic entire function) and having a wandering analytic component of the Fatou set accumulating at ∞.

5.5.1 A p-adic analogue of Baker's function

Fix $\gamma_1 \in \mathbb{C}_p$ with $|\gamma_1| < 1$. We define the sequence $\{\gamma_n\}$ inductively by

$$\gamma_{n+1} = \gamma_n^2 \prod_{i=1}^{n-1} \frac{\gamma_n}{\gamma_i}.$$

Because $\gamma_2 = \gamma_1^2$ and $|\gamma_1| < 1$, induction shows that $|\gamma_{n+1}| \leq |\gamma_1 \gamma_n|$, and so the sequence approaches zero, with absolute values strictly decreasing. Thus, we can define the function

$$g(z) = z^2 \prod_{n=1}^{\infty} (1 + \gamma_n z).$$

The product converges for any z because the $\{\gamma_n\}$ approach zero; it follows that g is entire. In fact, the z^{n+2} term of the power series expansion around zero has coefficient

$$c_{n+2} = \sum \left(\prod_{j=1}^{n} \gamma_{i_j} \right)$$

where the sum is taken over all unordered n-tuples of distinct positive integers $(i_1, \ldots, i_n)$. (Note that $c_2 = 1$ and $c_0 = c_1 = 0$.) We will show that the Fatou set $\mathcal{F}$ of g has wandering analytic components.

Let $r_n = |\gamma_n|^{-1} > 1$. From the infinite product definition of g and the inductive definition of γ, it is clear that if $r_n < |z| < r_{n+1}$, then $r_{n+1} < |g(z)| < r_{n+2}$. Thus, any z with absolute value in such a range must be Fatou, since its iterates, and the iterates of points in a small disk around x, approach ∞. We claim that the annulus

$$A_n = D_{r_{n+1}}(0) \setminus \overline{D}_{r_n}(0)$$

is in fact an analytic component of $\mathcal{F}$.

We begin by noting that for any $n \geq 1$, g must have a fixed point of absolute value r_n. To see this, note that g certainly has a zero (namely $-\gamma^{-1}$) of that absolute value. By Theorem 2.2.1, there must be distinct integers $j, k \geq 2$ with $|c_j| r_n^j = |c_k| r_n^k = \max_{i \geq 2} |c_i| r_n^i$. However, this is the same as the condition that $g(z) - z$ has a root of absolute value r_n, and so we have the desired fixed point. Furthermore, this fixed point is repelling, since $|g'(z)| > 1$ for z large enough. Let a_n be this repelling fixed point; in particular, $a_n \in \mathcal{J}$, where $\mathcal{J}$ denotes the Julia set of g.

If $|z| = r_n$, then $|\gamma_n z|, |\gamma_{n+1} g(z)| \leq 1$; in particular, $\gamma_n z, \gamma_{n+1} g(z) \in \mathcal{O}$. Thus, it makes sense to talk about the reductions $\overline{\gamma_n z}$ and $\overline{\gamma_{n+1} g(z)}$ modulo the maximal ideal. In fact,

$$\overline{\gamma_{n+1} g(z)} = \overline{\left(\gamma_n^2 \prod_{i=1}^{n-1} \frac{\gamma_n}{\gamma_1} \right) z^2 \prod_{i=1}^{\infty} (1 + \gamma_i z)}$$

$$= (\overline{\gamma_n z})^2 \left(\prod_{i=1}^{n-1} \overline{\gamma_n z} \right) \left(\overline{1 + \gamma_n z} \right) = (\overline{\gamma_n z})^{n+2} + (\overline{\gamma_n z})^{n+1}.$$

In particular, $\overline{\gamma_n a_n} = -1$. On the other hand, if $\overline{\gamma_n z} \neq 0, -1$, then

$$g\left(D_{|z|}(z)\right) = D_{|g(z)|}(g(z)).$$

For fixed $\overline{y} \in \overline{\mathbb{F}}_p^*$ and a fixed integer $n \geq 1$, we would like to find all $z \in \mathbb{C}_p$ with $|\gamma_n z| = 1$ and $\overline{\gamma_{n+1} g(z)} = \overline{y}$. By the preceding paragraph, this is the set of all $z \in \mathbb{C}_p$ with $|\gamma_n z| = 1$ and

$$(\overline{\gamma_n z})^{n+2} + (\overline{\gamma_n z})^{n+1} - \overline{y} = 0.$$

The equation

$$\overline{x}^{n+2} + \overline{x}^{n+1} - \overline{y} = 0$$

does not have 0 or -1 as a root, and it has $\overline{x}$-derivative $(n+2)\overline{x}^{n+1} + (n+1)\overline{x}^n$. Thus, it has no multiple roots if $p | (n+2)$, and it might have a double root otherwise.

In any case, it has at least $n+1$ distinct roots, none of which are 0 or -1. Therefore, there are at least $n+1$ distinct disks $D_{\gamma_n^{-1}}(z_i)$ which map onto $D_{\gamma_{n+1}^{-1}}(\gamma_{n+1}^{-1}y)$ under g.

If we let $y = \gamma_{n+1}a_{n+1}$, it follows that there are $n+1$ different values $\{z_i\}$ with $|\gamma_n z_i| = 1$, $\{\overline{\gamma_n z_i}\}$ all distinct, and $g(z_i) = a_{n+1}$. Similarly, each of these z_i has n preimages, each in a different residue class, and so on. For fixed $m \geq 1$, then, there are at least $(m+n)(m+n-1)\cdots(m+1)$ distinct disks $D_{\gamma_m^{-1}}(z_i)$ with $|\gamma_m z_i| = 1$ and $g^n(z_i) = a_{m+n}$. Thus, there are infinitely many distinct disks $D_{\gamma_m^{-1}}(z)$ with $|\gamma_m z| = 1$ which contain points of $\mathcal{J}$.

If we now fix $n \geq 1$, then any Fatou annulus which has only a finite number of holes and intersects A_n must be contained in A_n. A_n is therefore a full analytic component, as claimed. We have seen that $g(A_n) = A_{n+1}$; thus, A_n is in fact a wandering analytic component.

5.5.2 A question

As mentioned previously, Eremenko and Lyubich ([8]) found several other examples of complex entire functions with wandering domains. However, in all of their examples, as in Baker's, the wandering domain has an accumulation point at ∞. This led them to ask whether it is possible for an entire function to have a wandering domain which remains bounded. This question has been answered in the negative in various special cases (see, for example, [4] and [10]); as in our rational p-adic case, problems arise when there are recurrent critical points. For p-adic entire functions, the same methods we used to prove our version of Sullivan's theorem can be used to prove the following theorem.

Theorem 5.5.1. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K[[z]]$ be a power series convergent on all of $\mathbb{C}_p$. Suppose that ϕ has no wild recurrent Julia critical points. Then any wandering analytic component W of ϕ must have unbounded iterates $\{\phi^n(W)\}_{n\geq 0}$.*

We also ask the same question that Eremenko and Lyubich asked: is it possible for a p-adic entire function to have a wandering analytic component with bounded iterates?

Chapter 6

Dynamics on Fixed D-Components

In Chapter 5, we saw that for at least a very large class of p-adic rational maps, all D-components of the Fatou set are preperiodic. The study of dynamics on D-components of the Fatou set then quickly reduces to dynamics on periodic D-components. Furthermore, if a D-component has period n under the map ϕ, then it is fixed under the map ϕ^n. In view of the No Wandering Domains Theorem, then, it becomes very useful to study the possible dynamics of a rational map on a fixed D-component of its Fatou set.

In complex dynamics, there are only four possible types of dynamics on a fixed component of the Fatou set. The component can be *attracting*, *parabolic*, a *Siegel disk*, or a *Herman ring*. An attracting component is one which contains a unique attracting fixed point to which all points of the component are attracted. A parabolic component is one which has a neutral Julia fixed point in its boundary, to which all points of the component are attracted. A Siegel disk is a simply-connected component with a unique neutral Fatou fixed point; the function is holomorphically conjugate on the component to an irrational rotation. Finally, a Herman ring is a component which is conformally equivalent to an annulus; the function is holomorphically conjugate on the component either to a rotation or to the composition of a rotation and an inversion. (For more details on this classification, see [6], for example.) Moreover, the total number of periodic cycles of components is bounded by a constant depending only on the degree of the map.

As we have seen in the examples at the end of Section 5.4, no such bound can exist for p-adic rational functions. However, we are still able to say something about the dynamics which are possible on a fixed D-component.

6.1 Closed D-components

In this section, we prove two results about the action of a p-adic rational map on a rational closed and fixed D-component of its Fatou set. Proposition 6.1.1 will state that such D-components map multiply-to-one onto themselves (whereas all fixed complex non-attracting components map one-to-one onto themselves). Proposition 6.1.2, which will follow as a corollary, will guarantee that all such D-components actually contain fixed points.

Proposition 6.1.1. *Let $\phi \in \mathbb{C}_p(z)$ be a rational function, and let U be a fixed D-component of the Fatou set $\mathcal{F}$. If U is a rational closed $\mathbb{P}^1(\mathbb{C}_p)$-disk, then $\phi : U \to U$ is onto and d-to-one for some integer $d > 1$.*

Proof. Because U is rational closed, we can change coordinates so that $U = \overline{D}_1(0)$. Expanding ϕ as a power series on U, we have

$$\phi(z) = \sum_{i=0}^{\infty} c_i z^i$$

with $|c_i| \leq 1$ (because $\phi(U) \subset U$) and $c_i \to 0$ as $i \to \infty$. We claim that there is some $i \geq 2$ such that $|c_i| = 1$.

Suppose not. Because ϕ has finitely many poles, we can apply Lemma 2.2.3 to produce some $r > 1$ such that the above series converges on $\overline{D}_r(0)$. By our supposition, we can decrease r if necessary (keeping $r > 1$) so that $|c_i| r^{i-1} \leq 1$ for all $i \geq 2$. Then

$$\max_{i \geq 1} \{|c_i| r^i\} \leq r,$$

and so $\phi(\overline{D}_r(0)) \subset \overline{D}_r(0)$. Therefore, by Theorem 2.4.1, $\overline{D}_r(0) \subset \mathcal{F}$, contradicting the hypothesis that U was a D-component. Our claim is proven.

Let d be the greatest integer such that $|c_d| = 1$. By the claim, $d > 1$. Furthermore, for any $b \in \overline{D}_1(0)$, there are exactly d solutions (counting multiplicity) to $\phi(z) = b$ in $\overline{D}_1(0)$, by Theorem 2.2.1. □

Proposition 6.1.2. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a rational function. Let U be a fixed D-component of the Fatou set $\mathcal{F}$. Suppose U is not a rational open $\mathbb{P}^1(\mathbb{C}_p)$-disk. Then U contains a fixed point.*

Proof. If the Julia set contains fewer than two points, then $\mathcal{F} = U$ and the statement is trivial. Thus, we may assume that U is a disk. By Theorem 5.2.1, U is either a rational closed or rational open disk; by assumption, it is rational closed.

By Proposition 6.1.1, ϕ maps U onto U d-to-one; in particular, $\phi|_U$ is

$$\phi(z) = \sum_{i=0}^{\infty} c_i z^i$$

with $|c_i| \leq 1$, $c_i \to 0$, and $|c_d| = 1$ for some $d > 1$. The power series $\phi(z) - z$ has all coefficients bounded by 1, with $|c_d| = 1$; therefore, by Theorem 2.2.1, ϕ has a fixed point in $\overline{D}_1(0)$. □

In contrast to the complex case, a fixed D-component can have more than one fixed point, and possibly infinitely many periodic points. In [14], Hsia gives the example of the function $\phi(z) = pz^3 + z^2 + 1$ (for p an odd prime); it is easy to verify that $\overline{D}_1(0)$ is a D-component of the Fatou set of this function, and yet it contains two fixed points and infinitely many periodic points.

The situation for rational open disks is a little more complicated. If an open disk maps into itself multiply-to-one, then by essentially the same argument as in the proof of Proposition 6.1.2, the disk must contain a fixed point. The same is true if the disk maps into but not onto itself. However, if a rational open D-component maps onto itself one-to-one, then it need not contain fixed points. For instance, if p is an odd prime, the function

$$\phi(z) = p + z + \frac{p^3 z}{z^2 - 1}$$

has Fatou set with a fixed D-component $D_1(0)$ containing no fixed points. On the other hand, in this case, the analytic component containing 0 also contains ∞, which is a fixed point. It is therefore natural to ask whether it is possible for a map to have a fixed analytic component with no fixed points; this question is currently open.

6.2 Attracting periodic points

While we cannot yet fully classify the dynamics on a D-component containing an attracting fixed point, the following proposition demonstrates what happens on the largest open Fatou disk containing an attracting fixed point.

Proposition 6.2.1. *Let $f(z) = \sum c_i z^i \in \mathbb{C}_p[[z]]$ be a power series converging on $D_1(0)$, with $c_0 = 0$, $|c_1| < 1$, and $|c_i| \leq 1$ for all i. Then for any $a \in D_1(0)$,*

$$\lim_{n\to\infty} f^n(a) = 0.$$

In particular, f has no periodic points in $D_1(0)$ besides 0.

Proof. Fix $0 < r < 1$. Let j be the largest positive integer such that

$$\max_{i\geq 1}\{|c_i| r^i\} = |c_j| r^j;$$

such j must exist, by the convergence of the series. Then for any $a \in \overline{D}_r(0)$, $|f(a)| \leq |c_j a^j|$. If $j > 1$, then $|c_j| \leq 1$, so clearly $f^n(a)$ approaches zero. On the other hand, if $j = 1$, then $|c_j| < 1$, and so again, $f^n(a)$ approaches zero. □

Thus, if U is a fixed D-component which is a disk of radius r, and if U contains an attracting periodic point a, then all points in $D_r(a)$ will be attracted to a. Of course, if U is closed, not all points of U qualify. For example, in Hsia's example $\phi(z) = pz^3 + z^2 + 1$, if we let $p = 13$, then there is an attracting 3-cycle contained in $\overline{D}_1(0)$; however, it does not attract any points in the disk $D_1(-3)$, which contains a neutral fixed point.

It should be noted that, while attracting points do not allow other periodic points closer than the furthest reaches of the closed D-component in which they lie, the same is not true of neutral points. However, for most (and conjecturally all) neutral points, there is some neighborhood with no periodic points. More precisely, Herman and

Yoccoz ([12]) have shown that for a function ϕ with a fixed point a with multiplier λ not a root of unity, there is a neighborhood about a on which ϕ is conjugate to the map $f(z) = a + \lambda(z - a)$. In fact, their results apply to higher dimensional maps as well. More detailed study of such conjugations in the one-dimensional setting can be found in [1, 19, 35, 37].

Chapter 7

Polynomials

In much the same way that polynomials stand out in the theory of complex dynamics, p-adic polynomials have dynamical properties which make them worthy of special study. In this chapter we investigate some of these properties, and we give a partial analysis of a certain family of 2-adic cubic polynomials.

7.1 General polynomials

Throughout this chapter, we will be concerned only with p-adic polynomials having nonempty Julia set. If $\phi \in \mathbb{C}_p[z]$ is such a polynomial,

$$\phi(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0,$$

we can make a change of coordinates of the form $z \mapsto az$ to guarantee that $c_n = 1$. If all other coordinates are p-adic integers, then

$$\phi(\overline{D}_1(0)) \subset \overline{D}_1(0) \quad \text{and} \quad \phi(\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)) \subset \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0);$$

the entire projective line is therefore Fatou. Any monic polynomial with integral coefficients is an example of good reduction, a phenomenon which will be studied in greater detail in Chapter 8. For now, however, we are interested in bad reduction; all of our interesting polynomials, when put in monic form, will have non-integral coefficients.

From a dynamical standpoint, the identifying feature of any polynomial ϕ is that there is one invariant point, namely ∞. It is necessarily a superattracting fixed point and a critical point of ramification degree equal to the degree of the map. Given $\phi \in \mathbb{C}_p[z]$, let Φ denote the action of ϕ on the set of analytic components of the Fatou set $\mathcal{F}$, and let W_∞ denote the analytic component of $\mathcal{F}$ containing ∞. If W' is another analytic component such that $\Phi(W') = W_\infty$, then $\phi(W') = W_\infty$, and therefore there is some $x \in W'$ with $\phi(x) = \infty$; then $x = \infty$, and $W' = W_\infty$. Thus, $\phi^{-1}(W_\infty) = \phi(W_\infty) = W_\infty$, exactly as is the case for the connected Fatou component at ∞ for complex polynomials.

Taking the complex analogy a step further, we may refer to $\mathbb{P}^1(\mathbb{C}_p) \setminus W_\infty$ as the *filled Julia set* of ϕ. In the complex case, the filled Julia set consists of precisely those points which are not attracted to ∞. At the moment, it is not clear whether this characterization holds in the p-adic case as well; fortunately, as we will see in Theorem 7.1.2, it does hold, at least for polynomials with nonempty Julia set.

The following proposition will be superseded by Corollary 7.1.3; however, it will be needed to prove the theorem.

Proposition 7.1.1. *Let $\phi \in \mathbb{C}_p[z]$ be a polynomial, and let W_∞ denote the analytic component of the Fatou set at ∞. Let ∂W_∞ denote the topological boundary of W_∞, and let $\mathfrak{J}$ denote the Julia set of ϕ. Then $\mathfrak{J} \subset \partial W_\infty$.*

Proof. Pick any $x \in \mathfrak{J}$ and $r > 0$. Suppose $D_r(x) \cap W_\infty = \emptyset$. Then for any $n \geq 0$,

$$\phi^n\left(D_r(x)\right) \cap W_\infty = \emptyset,$$

because W_∞ is invariant under ϕ; by Theorem 2.4.1, $D_r(x)$ is contained in the Fatou set, and we have a contradiction. Thus, for any $r > 0$, $D_r(x) \cap W_\infty$ is nonempty; hence, $\mathfrak{J} \subset \partial W$. □

In fact, $\mathfrak{J} = \partial W_\infty$, but before we can prove the other inclusion, we will need the following theorem.

Theorem 7.1.2. *Let $\phi \in \mathbb{C}_p[z]$ be a polynomial, and let W_∞ denote the analytic component of the Fatou set containing ∞. Suppose that the Julia set of ϕ is not empty. Then W_∞ consists of precisely those points in $\mathbb{P}^1(\mathbb{C}_p)$ which are attracted to ∞ under iteration of ϕ.*

Proof. The case that $\deg \phi = 1$ is straightforward, and so we will assume throughout that $\deg \phi \geq 2$.

Let $U \subset W_\infty$ denote the D-component of the Fatou set containing ∞. W_∞ cannot be a $\mathbb{P}^1(\mathbb{C}_p)$-disk, for if it were, then $\partial W_\infty = \emptyset$, contradicting Proposition 7.1.1. Thus, by Proposition 3.2.1, all D-components in W_∞ are rational open $\mathbb{P}^1(\mathbb{C}_p)$-disks. Let V be such a D-component; we claim that some iterate $\phi^n(V)$ is contained in U.

Suppose not. Then no iterate of $\phi^n(V)$ can even intersect U. Write

$$U = \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_R(a) \quad \text{and} \quad V = D_r(b),$$

where $r, R \in p^{\mathbb{Q}}$ and $a, b \in \mathbb{C}_p$; let $V' = \overline{D}_r(b)$. Then $\phi(V) = D_s(\phi(b))$ and $\phi(V') = \overline{D}_s(\phi(b))$ for some $s \in p^{\mathbb{Q}}$. Since ϕ has no finite poles, it follows by induction that for any iterate $\phi^n(V)$, we have

$$\operatorname{rad}\left(\phi^n(V)\right) = \operatorname{rad}\left(\phi^n(V')\right).$$

In particular, since all iterates of V are contained in $\overline{D}_R(a)$, so are all iterates of V'. By Theorem 2.4.1, V' is contained in the Fatou set, contradicting the fact that V is a D-component. Thus, some iterate of V must be contained in U.

Now U is a rational open $\mathbb{P}^1(\mathbb{C}_p)$-disk containing ∞, and every point in W_∞ is eventually mapped into U; furthermore, $\phi(U) \subset U$. It suffices to show that for any $z \in U$, $\phi^n(z)$ approaches ∞.

Change coordinates so that ∞ becomes 0 and U becomes $D_1(0)$. Now ϕ maps U into U with an attracting fixed point at 0; by Proposition 6.2.1, all points of U are attracted to the fixed point. The proof is complete. □

Corollary 7.1.3. *Let $\phi \in \mathbb{C}_p[z]$ be a polynomial, and let W_∞ denote the analytic component of the Fatou set at ∞. Let ∂W_∞ denote the topological boundary of W_∞, and let $\mathcal{J}$ denote the Julia set of ϕ. Then $\mathcal{J} = \partial W_\infty$.*

Proof. If $\mathcal{J} = \emptyset$, then the statement is trivially true. Thus, we may assume that $\mathcal{J} \neq \emptyset$.

By Proposition 7.1.1, it suffices to show that $\partial W_\infty \subseteq \mathcal{J}$. Pick $R \in p^{\mathbb{Q}}$ so that $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_R(0) \subset W_\infty$; such R must exist, since W_∞ is an open set containing ∞. Change coordinates (by a change of the form $z \mapsto cz$) so that $R = 1$.

Pick $x \in \partial W_\infty$ and $r > 0$. Then for any $C > 0$, $\overline{D}_r(x)$ contains some point $y \in W_\infty$ with

$$|x - y| \leq \frac{1}{C}.$$

Furthermore, by Theorem 7.1.2, there is some $n \geq 0$ with $|\phi^n(y)| > 1$. However, $x \notin W_\infty$, since W_∞ is open; therefore, $|\phi^n(x)| \leq 1$. Thus,

$$|\phi^n(x) - \phi^n(y)| = |\phi^n(y)| > 1 \geq C|x - y|.$$

By definition of equicontinuity, $\{\phi^n\}$ is not equicontinuous on $\overline{D}_r(x)$; since this is true for all $r > 0$, it follows that $x \in \mathcal{J}$. □

Proposition 7.1.4. *Let $\phi \in \mathbb{C}_p[z]$ be a polynomial. Let V be any analytic component of the Fatou set. Then V is a $\mathbb{P}^1(\mathbb{C}_p)$-disk if and only if $\infty \notin V$. Furthermore, if V is a disk, then it is a closed disk.*

Proof. If the Julia set is empty, then the Fatou set is $\mathbb{P}^1(\mathbb{C}_p)$, and the statement is trivially true. We will therefore assume that the Julia set $\mathcal{J}$ is nonempty. Let $\mathcal{F}$ denote the Fatou set.

As we saw in the proof of Theorem 7.1.2, if V is the component at ∞, then V is not a disk. So we may assume that $\infty \notin V$. Thus, by Theorem 7.1.2, the iterates $\{\phi^n(V)\}$ are bounded. In particular, there is some disk $D' = \overline{D}_R(0)$ which contains all iterates of V.

Let U be any connected affinoid contained in V; write

$$U = D \setminus (D_1 \cup \cdots \cup D_m),$$

where D is a rational closed disk in $\mathbb{C}_p$, and each D_i is a rational open disk in D. Then for any $n \geq 0$, ϕ is a power series on D (in fact, it is a polynomial); furthermore, $\phi^n(U) \subset D'$. By Lemma 3.1.5, $\phi(D) \subset D'$ Since this is true for any $n \geq 0$, it follows by Theorem 2.4.1 that $D \subset \mathcal{F}$; hence, $D \subset V$.

If x_0 is some point of V, then V is the union of all Fatou connected affinoids containing x_0. However, we just saw that any such affinoid is contained in a Fatou disk; thus, V is the union of all Fatou disks containing x_0. It follows that V is a disk. It only remains to show that V is a closed disk.

Suppose not. Then V is a rational open disk $D_r(a)$. Let $V' = \overline{D}_r(a)$. Recall that all iterates of V are contained in D'. As in the proof of Theorem 7.1.2, it follows that all iterates of V' are contained in D'; thus, V' is Fatou, contradicting the assumption that V is a D-component. □

7.2 The family $\phi_c(z) = \frac{1}{2}(z^3 + z^2) + c$

In the complex setting, the family $f_c(z) = z^2 + c$ exhibits a wide variety of dynamical behaviors, as c varies over $\mathbb{C}$. The unique finite critical point 0 provides a useful starting point for studying the family. For example, any attracting cycle must attract 0; in fact, one of the iterates of any attracting periodic point must lie in the same Fatou component as 0. In particular, there can be only one attracting cycle. In addition, if 0 is attracted to ∞, then the Fatou set consists of a single component, and the Julia set is a totally disconnected Cantor set; if the iterates of 0 are all bounded, then the Julia set is connected. (The celebrated Mandelbrot set is the set of all $c \in \mathbb{C}$ for which the iterates of 0 are bounded. For more information on the Mandelbrot set, see [6] or [23], for example.)

We would like to carry out a similar study in the p-adic case. There are some hurdles, however. As we have seen, attracting periodic points need not attract critical points, and there is no bound on the number of periodic cycles of components. Because $\mathbb{C}_p$ is totally disconnected, the Julia set is always totally disconnected as well. Finally, the family $f_c(z) = z^2 + c$, as we saw in Section 3.3.1, has disappointingly simple dynamics: if $|c| \leq 1$, then the iterates of the critical point 0 are bounded and the Julia set is empty; if $|c| > 1$, then the iterates of 0 approach ∞, and the Julia set is a Cantor set.

We would like to find another family with more interesting dynamics which can still be studied with sufficient effort. We propose the family

$$\phi_c(z) = \frac{1}{2}(z^3 + z^2) + c,$$

for $p = 2$ and $c \in \mathbb{Q}_2$. ϕ_c has derivative

$$\phi_c'(z) = \frac{3}{2}z^2 + z,$$

and therefore has critical points at ∞, 0, and $-\frac{2}{3}$. Solving the equation $\phi_c(z) = z$ by Theorem 2.2.1, we see that ϕ_c has at least one fixed point α of absolute value at least 1. Thus, $|\phi_c'(\alpha)| \geq 2$, and so ϕ_c has a repelling fixed point and hence nonempty Julia set.

If $|c| > 1$, then all critical points are attracted to ∞; however, if $c \in \mathbb{Z}_2$, then $\phi_c(\mathbb{Z}_2) \subset \mathbb{Z}_2$. Thus, $\mathbb{Z}_2$ is somewhat analogous to the Mandelbrot set; and fortunately, unlike the p-adic family $z^2 + c$, there is a wide variety of possible dynamics for $c \in \mathbb{Z}_2$. In this section, we will present a case-by-case analysis of some of the behaviors that arise for some c. We will do so by tracing the iterates of the critical points. While this method may not find all periodic Fatou cycles, it will at least unveil some of the interesting phenomena that occur.

For the rest of this section, we will consider only $c \in \mathbb{Z}_2$. We begin by noting that the change of coordinates

$$z \mapsto -\frac{3}{2} - z$$

	minimal m with $\phi^m(0)$ in periodic D-component	period of D-component	period of attracting cycle
$c \equiv 0(4)$	0	1	1
$c \equiv 2(4)$	0	1	2
$c \equiv 1$ or $25(32)$	1	8	8
$c \equiv 9$ or $17(32)$	0	5	5
$c \equiv 5(16)$	0	2	2
$c \equiv 13(16)$	0	2	4
$c \equiv 3$ or $11(32)$	0	8	8
$c \equiv 19(32)$	0	5	5
$c \equiv 59(64)$	0	3	3
$c \equiv 27(64)$	0	3	6
$c \equiv 7(32)$	3	2	4
$c \equiv 55(256)$	0	3	3
$c \equiv 183(256)$	0	3	6
$c \equiv -9(128)$	12	2	4
$c = 23$	30	4	–
general $c \equiv -1(8)$	?	?	?

Table 7.1: iterates of 0 under ϕ_c

exchanges the two finite critical points while leaving ∞ fixed. Conjugating by this coordinate change, our function ϕ_c becomes

$$\tilde{\phi}_c(z) = \frac{1}{2}(z^3 + z^2) - c - \frac{20}{27}.$$

Thus, switching the two critical points has the effect of subjecting c to the inversion

$$\tilde{c} = -c - \frac{20}{27}.$$

In particular, we only need consider what happens to the critical point at 0 if we want to trace the iterates of all the critical points. We will now compute these iterates in specific cases; the results of our computations are summarized in Table 7.1. These computations were done with the aid of PARI/GP.

Before presenting the computations, we make note of three facts concerning a polynomial map f with nonempty Julia set and Fatou set $\mathfrak{F}$. First, if D is a disk containing the point a, and $f^n(D) \subset D$ for some $n \geq 1$, then $D \subset \mathfrak{F}$, and D is contained in a periodic D-component of period dividing n. Second, if D is a closed disk and $f^n(D) = D$, then we claim D is actually a D-component. Its points stay bounded, and therefore (by Theorem 7.1.2) D does not intersect the analytic component at ∞; we are using the fact that the Julia set is nonempty. It is certainly contained in a D-component, but any larger disk would not map into itself; hence, it is actually a D-component, as claimed. Third, if D is an open disk, and n is the minimal positive integer such that $f^n(D) \subset D$, and there is some point $a \in D$ with $|(f^n)'(a)| < 1$, then D contains a unique periodic point, which is attracting of period n and which attracts all points of D. This is because we can solve $f^n(z) - z$ on D by Theorem 2.2.1 by examining the power series of f^n centered at a; the rest follows from Proposition 6.2.1. Note that the same is true if D is a closed disk but $f^n(D)$ is a proper subset of D.

If $c \equiv 0 (\mathrm{mod}\ 2)$, then it is easy to verify that

$$\phi_c\left(\overline{D}_{|2|}(0)\right) = \overline{D}_{|2|}(0),$$

and therefore $\overline{D}_{|2|}(0)$ is the D-component of the Fatou set containing 0. Solving $\phi_c(z) = z$ by Theorem 2.2.1, there are two attracting fixed points of ϕ_c in $\overline{D}_{|2|}(0)$; however, there may be many other periodic points as well. If $c \equiv 0 (\mathrm{mod}\ 4)$, then also $\phi_c(D_{|2|}(0)) = D_{|2|}(0)$, and so by our third fact above, 0 is attracted to an attracting fixed point. However, if $c \equiv 2 (\mathrm{mod}\ 4)$, then

$$D_{|2|}(0) \to D_{|2|}(2) \to D_{|2|}(0).$$

Thus, 0 is attracted to an attracting 2-periodic point.

If $c \equiv 1 (\mathrm{mod}\ 32)$, then

$$\overline{D}_{|2^{5/2}|}(0) \to \overline{D}_{|2^4|}(1) \to \overline{D}_{|2^3|}(2) \to \overline{D}_{|2^5|}(7) \to \overline{D}_{|2^4|}(5) \to$$

$$\overline{D}_{|2^3|}(4) \to \overline{D}_{|2^5|}(9) \to \overline{D}_{|2^4|}(6) \to \overline{D}_{|2^6|}(-1) \to \overline{D}_{|2^5|}(1) \subset \overline{D}_{|2^4|}(1).$$

Thus, all of $\overline{D}_{|2^4|}$ is attracted to a unique attracting 8-periodic point. A closer analysis shows that the actual D-component is $\overline{D}_{2^e}(1)$, where $e = \dfrac{27}{14}$. Similar reasoning (albeit with a slightly different sequence of disks) applies in the case that $c \equiv 25 (\mathrm{mod}\ 32)$.

If $c \equiv 9 (\mathrm{mod}\ 32)$, then

$$\overline{D}_{|2^3|}(0) \to \overline{D}_{|2^5|}(9) \to \overline{D}_{|2^4|}(-2) \to \overline{D}_r(c-2) \subset \overline{D}_{|2^5|}(7) \to \overline{D}_{|2^4|}(-3) \to \overline{D}_{|2^3|}(0),$$

where $r > |2^5|$ depends on c. In particular, $\phi^5(\overline{D}_{|2^3|}(0))$ is a proper subset of $\overline{D}_{|2^3|}(0)$, and so all of the disk is attracted to a unique attracting 5-periodic point inside. The actual D-component is a closed disk of slightly larger radius. Similar reasoning applies to the case $c \equiv 17 (\mathrm{mod}\ 32)$.

If $c \equiv 5 (\mathrm{mod}\ 8)$, then

$$\overline{D}_{|2^2|}(0) \to \overline{D}_{|2^3|}(5) \to \overline{D}_{|2^2|}(0).$$

If in fact $c \equiv 5 (\mathrm{mod}\ 16)$, then

$$D_{|2^2|}(0) \to D_{|2^3|}(5) \to D_{|2^2|}(0);$$

however, if $c \equiv 13 (\mathrm{mod}\ 16)$, then

$$D_{|2^2|}(0) \to D_{|2^3|}(-3) \to D_{|2^2|}(4) \to D_{|2^3|}(5) \to D_{|2^2|}(0).$$

Thus, in the first case we have an attracing 2-cycle, while in the second we have an attracting 4-cycle.

The other cases listed in Table 7.1 can be proven similarly. The table exhausts the case $c \not\equiv -1 (\bmod 8)$. Note that some of the behaviors for $c \equiv -1 (\bmod 8)$ are quite different from the rest. For instance, it is suddenly possible for the component at 0 to wander for quite a few iterations before finding a periodic component (as for $c \equiv -9 (\bmod 128)$ or $c = 23$). In addition, for $c = 23$, the periodic component contains no attracting points; all periodic points in the component are neutral. One possible reason for this less predictable behavior is that for $c = -1$, the critical point at 0 is Julia. In fact, $\phi_{-1}(0) = -1$ is a repelling fixed point. It seems likely that for $c \in \overline{D}_{|8|}(-1)$, a wide variety of dynamical behaviors is possible.

We will give one more example of a map in this special set. If we set $c = -33$, then even after many iterations, the forward orbit of 0 does not appear to follow any periodic pattern; assuming there is truly no periodicity, and assuming the No Wandering Domains Conjecture, it would follow that 0 is in the Julia set. Furthermore, calculations by PARI/GP show that, for instance:

$$\begin{aligned} v_2\left(\phi_{-33}^{39}(0)\right) &= 9, \\ v_2\left(\phi_{-33}^{2204}(0)\right) &= 12, \\ v_2\left(\phi_{-33}^{2836}(0)\right) &= 13, \\ v_2\left(\phi_{-33}^{24210}(0)\right) &= 16. \end{aligned}$$

Thus, it seems plausible that 0 could be recurrent; since its ramification index is 2, that would make it a wild recurrent Julia critical point. However, it is unclear how one might prove such a statement.

Chapter 8

Reduction of Rational Maps

It has been mentioned several times in this thesis that a rational map $\phi \in \mathbb{C}_p(z)$ of good reduction has empty Julia set; however, we have not yet defined the notion of "good reduction". In this chapter, we will correct this omission and then discuss the relationship between reduction and Julia sets.

8.1 Background

Recall that a map $\phi(z) \in \mathbb{C}_p(z)$ can be written in homogeneous coordinates as

$$\phi([x, y]) = [f(x, y), g(x, y)],$$

where $f, g \in \mathcal{O}[x, y]$ are relatively prime homogeneous polynomials of degree $d = \deg \phi$. We can ensure that at least one coefficient of either f or g has absolute value 1. Also recall the reduction map $\mathcal{O} \to \overline{\mathbb{F}}_p$, which we will denote $a \mapsto \overline{a}$. It induces a map

$$\mathcal{O}[x, y] \to \overline{\mathbb{F}}_p[x, y];$$

we shall denote the reduction of a polynomial f by $\overline{f}$. Following [25], we make the following definition.

Definition 8.1.1. *Let $\phi \in \mathbb{C}_p(z)$ be a map with homogenous presentation*

$$\phi([x, y]) = [f(x, y), g(x, y)],$$

where $f, g \in \mathcal{O}[x, y]$ are relatively prime homogeneous polynomials of degree $d = \deg \phi$, and at least one coefficient of f or g has absolute value 1. We say that ϕ has good reduction *if $\overline{f}$ and $\overline{g}$ have no common zeros in $\overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p$ besides $(x, y) = (0, 0)$. If ϕ does not have good reduction, we say it has* bad reduction.

Thus, ϕ has good reduction if and only if the resultant $\operatorname{Res}(f, g) \in \mathcal{O}$ has absolute value 1 (that is, $\operatorname{Res}(\overline{f}, \overline{g})$ does not vanish). Equivalently, if we view the projective line as a scheme, $\phi : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p)$ has good reduction if and only if it extends to a morphism $\psi : \mathbb{P}^1_{\mathcal{O}} \to \mathbb{P}^1_{\mathcal{O}}$ of schemes.

We have the following theorem, proved in [25].

Theorem 8.1.1. *(Morton, Silverman) Let $\phi \in \mathbb{C}_p(z)$ have good reduction. Then the Julia set of ϕ is empty.*

Morton and Silverman actually prove the stronger result that the spherical distance between two points in $\mathbb{P}^1(\mathbb{C}_p)$ cannot increase under application of a map of good reduction. The idea to keep in mind is that each residue class in $\mathbb{P}^1(\overline{\mathbb{F}}_p)$ corresponds to an open $\mathbb{P}^1(\mathbb{C}_p)$-disk: either $D_1(a)$ for some $a \in \mathcal{O}$, or $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$. If ϕ has good reduction, then it maps any such disk into (and onto) another such disk.

It follows from Theorem 8.1.1 that if ϕ is a rational map, and $f \in \mathrm{PGL}(2, \mathcal{O})$ such that the conjugated map

$$f^{-1} \circ \phi \circ f$$

has good reduction, then the Julia set of ϕ is empty. For example, the map

$$\phi(z) = \frac{z^2}{p}$$

has bad reduction as written. But the change of coordinates $f(z) = pz$ gives

$$f^{-1} \circ \phi \circ f(z) = z^2,$$

which has good reduction and therefore empty Julia set; thus, ϕ also has empty Julia set.

On the flip side of the coin, all of the maps of bad reduction that we have seen so far have had nonempty Julia set. It is therefore natural to ask whether having empty Julia set is equivalent to having good reduction in some coordinate system. We shall soon answer this question in the negative.

We have seen that the Julia set of a map ϕ is the same as that of ϕ^n, for any $n \geq 1$. It is also true that if ϕ has good reduction, then ϕ^n does as well. Given the negative answer to the question in the preceding paragraph, one may ask whether the converse is true: if ϕ^n has good reduction, then must ϕ also have good reduction? In this case, for maps of degree at least 2, the answer will be yes.

8.2 Results and examples

In this section we will prove several results on good reduction which address the questions raised in the previous section. Many of our statements will concern change of coordinates, and so we will need a clear definition of what we mean by a "coordinate". We will say that a coordinate w on $\mathbb{P}^1(\mathbb{C}_p)$ is an isomorphism $w : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p)$; the w-coordinate $w(P)$ of a point P is simply its image under the isomorphism. Here, we are viewing the target space $\mathbb{P}^1(\mathbb{C}_p)$ as $\mathbb{C}_p \cup \{\infty\}$, so that we may say $w(P) = \infty$ or $w(Q) = 0$, for instance. We will say that a rational map $\phi \in \mathbb{C}_p(z)$ has good reduction *with respect to the coordinate* w if $w \circ \phi \circ w^{-1}$ has good reduction according to Definition 8.1.1.

8.2.1 A map of bad reduction and empty Julia set

In this section we will prove a series of lemmas which will help us find an explicit coordinate system in which a map ϕ has good reduction, given that it has good reduction in some coordinate. In the end, we will use them negatively, to prove that a given map has bad reduction in all cordinates.

Lemma 8.2.1. *Let $\phi \in \mathbb{C}_p(z)$ have good reduction with respect to some coordinate w. Let $f \in \mathrm{PGL}(2, \mathcal{O})$. Then $f \circ \phi \circ f^{-1}$ has good reduction with respect to w.*

Proof. Since w is an automorphism of $\mathbb{P}^1(\mathbb{C}_p)$, $w \in \mathrm{PGL}(2, \mathbb{C}_p)$. Let $\tilde{f} = w \circ f \circ w^{-1}$. The determinant of $\tilde{f}$ lies in $\mathcal{O}^*$, so by clearing denominators, we may assume that $\tilde{f} \in \mathrm{PGL}(2, \mathcal{O})$. If we let $\psi = w \circ \phi \circ w^{-1}$, then we wish to show that $\tilde{f} \circ \psi \circ \tilde{f}^{-1}$ has good reduction, given that ψ does. Thus, we may assume without loss that w is the identity. In other words, ϕ has good reduction, and we want to show that $f \circ \phi \circ f^{-1}$ does as well.

It suffices to show that the statement is true for the maps $f(z) = cz$ (for $c \in \mathcal{O}^*$), $f(z) = 1/z$, and $f(z) = z + c$ (for $c \in \mathcal{O}$). Write ϕ in homogeneous coordinates as

$$\phi[x, y] = [g(x, y), h(x, y)],$$

where g and h are homogeneous of degree d. By assumption, $\overline{g}$ and $\overline{h}$ have no nontrivial common zeros in $\overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p$.

If $c \in \mathcal{O}^*$, then

$$c\phi[x, cy] = [cg(x, cy), h(x, cy)].$$

Now $\overline{cg(x, cy)}$ and $\overline{h(x, cy)}$ have a nontrivial common zero if and only if $\overline{g(x, cy)}$ and $\overline{h(x, cy)}$ do; by the substitution $y' = cy$ and our assumption, they do not. Thus, we have good reduction in the case $f(z) = cz$.

If $f(z) = 1/z$, then the conjugated version of ϕ is

$$[x, y] \mapsto [h(y, x), g(y, x)].$$

By our assumption, it is immediate that $\overline{h(y, x)}$ and $\overline{g(y, x)}$ have no nontrivial common zeros.

It remains to show the lemma in the case that $f(z) = z + c$, for $c \in \mathcal{O}$. The conjugated function is

$$[x, y] \mapsto [g(x - cy, y) + ch(x - cy, y), h(x - cy, y)].$$

If $(\overline{x}, \overline{y})$ is a nontrivial common root of $\overline{h}(\overline{x} - \overline{cy}, \overline{y})$ and $\overline{g}(\overline{x} - \overline{cy}, \overline{y}) + \overline{c}\overline{h}(\overline{x} - \overline{cy}, \overline{y})$, then clearly $(\overline{x} - \overline{cy}, \overline{y})$ is a nontrivial common root of $\overline{g}$ and $\overline{h}$. By assumption, this is impossible, and so the conjugated function has good reduction. □

Lemma 8.2.2. *Let $\phi \in \mathbb{C}_p(z)$ have good reduction with respect to some coordinate z. Let $P \in \mathbb{P}^1(\mathbb{C}_p)$ be any point. Then there is some coordinate w such that $w(P) = \infty$ and ϕ has good reduction with respect to w.*

Proof. Without loss, we may assume that the isomorphism z is the identity. If $P = \infty$, then we can choose $w = z$ and we are done. Thus, we may assume $P \in \mathbb{C}_p$.

If $|P| \leq 1$, then the transformation

$$w(z) = \frac{1}{z - P}$$

is an element of $\mathrm{PGL}(2, \mathcal{O})$. Clearly, $w(P) = \infty$, and by Lemma 8.2.1, ϕ has good reduction with respect to w.

If $|P| \geq 1$, then $P^{-1} \in \mathcal{O}$. Let

$$w(z) = \frac{z}{P^{-1}z - 1} \in \mathrm{PGL}(2, \mathcal{O}).$$

By Lemma 8.2.1, ϕ has good reduction with respect to w, and clearly $w(P) = \infty$. □

Our next lemma has a somewhat more technical statement, but it should be viewed as a continuation of Lemma 8.2.2. That lemma allowed us to specify the point at infinity; the following lemma then allows us to specify the point at 0 without moving the point at ∞.

Lemma 8.2.3. *Fix a coordinate z for $\mathbb{P}^1(\mathbb{C}_p)$, and let $P = z^{-1}(\infty)$. Let $\phi \in \mathbb{C}_p(z)$ be some rational function. Let w be some other coordinate, and suppose that $w(P) = \infty$ and ϕ has good reduction with respect to w. Let Q be another point with $w(Q) \in \mathcal{O}$, and let $x = z(Q)$. For any $c \in \mathbb{C}_p^*$, let*

$$f_c(z) = cz + x.$$

Then there is some $c \in \mathbb{C}_p^*$ such that $f_c^{-1} \circ \phi \circ f_c$ has good reduction with respect to z.

Proof. We can assume without loss that z is the identity isomorphism. Because $w(\infty) = \infty$, it follows that $w(z) = az + b$ for some $a \in \mathbb{C}_p^*$ and $b \in \mathbb{C}_p$. By definition of x, we have $ax + b \in \mathcal{O}$.

Let $\psi(z) = w \circ \phi \circ w^{-1}(z)$, which has good reduction, by hypothesis. We wish to find $c \in \mathbb{C}_p^*$ such that

$$f_c^{-1} \circ w^{-1} \circ \psi \circ w \circ f_c$$

has good reduction. By Lemma 8.2.1, then, it suffices to show that $w \circ f_c \in \mathrm{PGL}(2, \mathcal{O})$.

However,

$$w \circ f_c(z) = a(cz + x) + b = acz + ax + b.$$

By hypothesis, $ax + b \in \mathcal{O}$; thus, we can choose $c = a^{-1}$, and we are done. □

Proposition 8.2.4. *Let $\phi \in \mathbb{C}_p(z)$ be a rational map with an attracting fixed point at ∞. Suppose that the iterates $\{\phi^n(0)\}$ are bounded. Suppose also that for any $c \in \mathbb{C}_p^*$, $c^{-1}\phi(cz)$ has bad reduction. Then ϕ has bad reduction in any coordinate.*

Proof. Suppose that ϕ had good reduction with respect to some coordinate w. Then by Lemma 8.2.2, we can assume that $w(\infty) = \infty$. Because $w \circ \phi \circ w^{-1}$ has good reduction and maps ∞ to ∞, it must map $V = \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$ into itself as well. Because the fixed point is attracting, Proposition 6.2.1 tells us that all points of V are attracted to ∞.

Let $x = 0$ and $Q = w^{-1}(0)$. Because the iterates of 0 are bounded, we have $Q \notin V$. Thus, $Q \in \overline{D}_1(0) = \mathcal{O}$. By Lemma 8.2.3, $c^{-1}\phi(cz)$ has good reduction for some $c \in \mathbb{C}_p^*$, contradicting the hypotheses. □

Until now, the only way we could know for certain that a map had bad reduction in all coordinates was to find a repelling periodic point. Proposition 8.2.4, on the other hand, is useful for producing examples of functions with empty Julia set having bad reduction in all coordinates.

Example. Let $p = 2$, and let

$$\phi(z) = z^8 + \frac{1}{2}z^4.$$

Clearly, ∞ is an attracting fixed point, and 0 is fixed. Furthermore, for any $c \in \mathbb{C}_p^*$,

$$\frac{1}{c}\phi(cz) = c^7 z^8 + \frac{c^3}{2}z^4.$$

If $|c| \neq 1$, then we will have bad reduction because of the z^8 term. On the other hand, if $|c| = 1$, then we have bad reduction because of the z^4 term. We have bad reduction for every c, and therefore, by Proposition 8.2.4, ϕ has bad reduction in all coordinates.

We will now show that ϕ has empty Julia set. Pick $z \in \mathbb{P}^1(\mathbb{C}_p)$. If $|z| > |2^{-1/4}|$, then $|\phi(z)| = |z^8|$; therefore, $\phi^n(z) \to \infty$, and z is Fatou. To prove that all z are Fatou, it suffices to show the following claim:

Claim 8.2.1. *Let $p = 2$ and $\phi(z) = z^8 + z^4/2$. If If $|z| \leq |2^{-1/4}|$, then for any $x \in \overline{D}_{|2|}(z)$,*

$$|\phi(z) - \phi(x)| \leq |z - x|.$$

Proof of Claim. Let $w = z - x$; so $|w| \leq |2|$. So

$$|\phi(z) - \phi(x)| = \left| \left[z^8 - (z-w)^8\right] + \frac{1}{2}\left[z^4 - (z-w)^4\right] \right|$$

$$= \left| z^4 + (z-w)^4 + \frac{1}{2} \right| \left| z^4 - (z-w)^4 \right| \leq \left| \frac{1}{2} \right| \left| z^4 - (z-w)^4 \right|$$

$$= 2\left|4z^3 w - 6z^2 w^2 + 4zw^3 - w^4\right| = |w| \left| 2z^3 - 3z^2 w + 2zw^2 - \frac{1}{2}w^3 \right| \leq |w|,$$

the last inequality holding because of ultrametricity and because of the bounds on $|z|$ and $|w|$. The claim is proven. □

8.2.2 Iterates and reduction

The preceding example answers the first question at the end of Section 8.1: there are maps with empty Julia set and bad reduction in all coordinates. The following proposition will answer the second question.

Proposition 8.2.5. *Let $\phi \in \mathbb{C}_p(z)$ be a rational map of degree $d \geq 2$, let w be some coordinate on $\mathbb{P}^1(\mathbb{C}_p)$, and let $n \geq 1$ be any positive integer. Then ϕ has good reduction with respect to w if and only if ϕ^n has good reduction with respect to w.*

To prove Proposition 8.2.5, we will need the following lemma. The proof is straightforward, and we omit it.

Lemma 8.2.6. *Let $f \in \mathbb{C}_p[[z-a]]$ be a power series converging on the rational open disk $D_{r_1}(a)$ with image $D_{r_2}(b)$. Let $0 < s_1 < r_1$, and let*

$$s_2 = \operatorname{rad}\left(f\left(D_{s_1}(a)\right)\right) = \operatorname{rad}\left(f\left(\overline{D}_{s_1}(a)\right)\right).$$

Then $r_1 s_2 \leq r_2 s_1$. Futhermore, if $\deg f \geq 2$, then $r_1 s_2 < r_2 s_1$.

Proof of Proposition 8.2.5. We can assume without loss that the coordinate w is the identity isomorphism. The composition of two maps of good reduction has good reduction; thus, if ϕ has good reduction, so does ϕ^n. To prove the converse, assume that ϕ^n has good reduction but ϕ does not.

Let $\mathcal{D}$ be the set of all $\mathbb{P}^1(\mathbb{C}_p)$-disks which are inverse images of points of $\mathbb{P}^1(\overline{\mathbb{F}}_p)$ under the reduction map; in other words,

$$\mathcal{D} = \left\{\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)\right\} \cup \left\{D_1(a)\right\}_{a \in \mathcal{O}}.$$

(Note that the representation of any of the finite disks as $D_1(a)$ is not unique. For instance, $D_1(0) = D_1(p)$.) For notation, write $W = \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$.

Because ϕ^n has good reduction, then for any $D \in \mathcal{D}$, the image $\phi^n(D)$ is also a disk in $\mathcal{D}$. In particular, if $\phi(D) \supset D$, then $\phi(D) = D$; otherwise, we would have $\phi^n(D) \supsetneq D$, so $\phi^n(D) \notin \mathcal{D}$.

On the other hand, because ϕ has bad reduction, there is some $D_0 \in \mathcal{D}$ such that for any $D \in \mathcal{D}$, $\phi(D_0) \not\subset D$. By a $\mathrm{PGL}(2, \mathcal{O})$ change of coordinates, we may assume that $D_0 = W$. Now $\phi(D_0)$ is an open $\mathbb{P}^1(\mathbb{C}_p)$-disk, and it cannot contain D_0. Thus, by another $\mathrm{PGL}(2, \mathcal{O})$ change of coordinates, we may assume that $\infty \notin \phi(D_0)$. It follows that $\phi(D_0)$ is a rational open disk $D_r(a)$. Now $D_r(a) \not\subset D_0$ (by definition of D_0), so $a \in \overline{D}_1(0)$. Furthermore, $r > 1$, or else $\phi(D_0) \subset D_1(a) \in \mathcal{D}$. Therefore, $\phi(D_0) = D_r(0)$ for $r > 1$; in particular, for any $D \in \mathcal{D} \setminus \{D_0\}$, $\phi(D_0) \supsetneq D$.

Pick $D \in \mathcal{D} \setminus \{D_0\}$. If $\phi(D) \subset \phi(D_0)$, then let $f = \phi^{n-1}|_{\phi(D_0)}$. Now $f(\phi(D_0))$ is a $\mathbb{P}^1(\mathbb{C}_p)$-disk, so (by composition with an appropriate element of $\mathrm{PGL}(2, \mathbb{C}_p)$, if needed), f may be written as a power series on $\phi(D_0)$. If $\phi(D) \subsetneq \phi(D_0)$, then $\operatorname{rad}(\phi(D)) < \operatorname{rad}(\phi(D_0))$, and by Lemma 8.2.6, $\phi^n(D) \subsetneq \phi^n(D_0)$. However, since ϕ^n has good reduction, both images under ϕ^n must be elements of $\mathcal{D}$; in particular, if they are different, then they are disjoint. Thus, we must have had $\phi(D) = \phi(D_0)$.

On the other hand, by the above, $\phi(D_0) \supset D$; hence, $\phi(D) \supsetneq D$, which we have seen is impossible. Therefore, $\phi(D) \not\subset \phi(D_0)$.

Furthermore, if $\phi(D) \cap \phi(D_0) \neq \emptyset$, then since $\phi(D) \not\subset \phi(D_0)$, it must be that either $\phi(D) \cup \phi(D_0) = \mathbb{P}^1(\mathbb{C}_p)$ or else $\phi(D) \supsetneq \phi(D_0)$. The latter cannot happen because $\phi(D_0) \supset D$, and the former cannot happen, because then $\phi^n(D \cup D_0) = \mathbb{P}^1(\mathbb{C}_p)$. Thus, for any $D \in \mathcal{D} \setminus \{D_0\}$, $\phi(D) \cap \phi(D_0) = \emptyset$. Since the union of all such D is $\overline{D}_1(0)$, and because $D_0 = W = \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$, we have

$$\phi(\overline{D}_1(0)) \cap \phi(W) = \emptyset.$$

Because ϕ is onto and d-to-one,

$$\phi : \overline{D}_1(0) \twoheadrightarrow \mathbb{P}^1(\mathbb{C}_p) \setminus D_r(0) \quad d\text{-to-one,} \quad \text{and} \quad \phi : W \twoheadrightarrow D_r(0) \quad d\text{-to-one.}$$

Now define $g(z)$ on $D_1(0)$ to be

$$g(z) = \phi\left(\frac{1}{z}\right),$$

so $g(D_1(0)) = \phi(W) = D_r(0)$. By Lemma 8.2.6,

$$\phi\left(\mathbb{P}^1(\mathbb{C}_p) \setminus D_r(0)\right) = g\left(\overline{D}_{r^{-1}}(0)\right) = \overline{D}_s(0)$$

with $s < r^{-1}r = 1$. In other words,

$$\phi^2\left(\overline{D}_1(0)\right) = \overline{D}_2(0) \subset D_1(0).$$

Thus, for any $D \in \mathcal{D} \setminus \{D_0\}$, $\phi^2(D) \subset D_1(0)$. By iterating,

$$\phi^{2n}(D) \subset D_1(0).$$

However, ϕ^{2n} has good reduction, because ϕ does. Thus, $\overline{\phi^{2n}}$ is a degree d^{2n} rational map in $\overline{\mathbb{F}}_p(z)$. On the other hand, by the above argument, for any $\overline{a} \in \overline{\mathbb{F}}_p$, $\overline{\phi^{2n}}(\overline{a}) = 0$; it follows that $\overline{\phi^{2n}} = 0$, and we have a contradiction. □

The assumption that $\deg \phi \geq 2$ is crucial in Proposition 8.2.5. For example, if

$$\phi(z) = \frac{p^2}{z},$$

then ϕ has bad reduction as written; however, $\phi^2(z) = z$, which has good reduction in any coordinate. On the other hand, we can change coordinates by $w = p^{-1}z$ to get

$$\phi(w) = \frac{1}{w},$$

which has good reduction.

In fact, any degree one rational function ϕ either has good reduction in some coordinate, or else it has a repelling fixed point; in the latter case, all of its iterates also have bad reduction in all coordinates. To see this, we consider two cases. If ϕ has a unique fixed point, then it is conjugate to a map of the form $z \mapsto z + a$; by a change of coordinates $w = cz$, we can ensure that $a \in \mathcal{O}$, and we have good reduction. If ϕ has two distinct fixed points, we can put one at 0 and one at ∞; the map is now of the form $z \mapsto bz$. If $|b| \neq 1$, then one of the points is repelling; if $|b| = 1$, we have good reduction.

Appendix A

Proofs for Quadratic Examples

The purpose of this appendix is to prove the facts stated in Section 3.3. Sections 3.3.5 and 3.3.6 are self-contained; we will therefore restrict our attention to the earlier examples.

First, suppose ϕ is a quadratic map with a unique fixed point. Then by a change of coordinates, we may assume that this fixed point is ∞. There must be some other point x such that $\phi(x) = \infty$; otherwise, ϕ would be a quadratic polynomial and therefore have a finite fixed point. By another change of coordinates, we may assume that $x = 0$. Thus, ϕ must be of the form

$$\phi(z) = az + b + \frac{c^2}{z}.$$

Solving $\phi(z) = z$, we should have no finite solutions; thus, $a = 1$ and $b = 0$. Let $z = cw$; then

$$\phi(w) = w + \frac{1}{w},$$

as claimed in Section 3.3. This map has good reduction; by Theorem 8.1.1, the Julia set is empty.

The other case is that there are two or more fixed points; we can place one at 0 and one at ∞. As a result,

$$\phi(z) = \frac{az^2 + bz}{ez + f}.$$

Because $\deg \phi = 2$, neither a nor f can be zero. By a change of coordinates of the form $z \mapsto cz$, we can specify that $a = f = 1$, producing the desired form

$$\phi(z) = \frac{z^2 + \lambda z}{\mu z + 1}, \tag{A.1}$$

where λ and μ are not multiplicative inverses. Clearly, if $\mu = 0$, then we have a polynomial; and if $\lambda = 0$, then ϕ is conjugate to a polynomial via $z \mapsto 1/z$.

For most of this section, we will assume that ϕ is of the form shown in equation (A.1). The derivative is

$$\phi'(z) = \frac{\mu z^2 + 2z + \lambda}{(\mu z + 1)^2}.$$

Note that 0 and ∞ are fixed points with multipliers λ and μ, respectively. There is also a third fixed point at

$$x_0 = \frac{1-\lambda}{1-\mu}$$

with multiplier

$$\nu = \frac{2-\lambda-\mu}{1-\lambda\mu}.$$

(Note that if one of λ or μ is 1, then the third fixed point coincides with 0 or ∞.)

If λ and μ satisfy the conditions of Section 3.3.2 (namely, $|\lambda|, |\mu| \leq 1$ with $|\lambda\mu - 1| = 1$), then ϕ has good reduction and therefore empty Julia set. If $|\lambda| < 1$ and $|\mu| > 1$, then the conjugation $z \mapsto 1/z$ exchanges 0 and ∞. The result is a quadratic map with multipliers $\lambda' = \mu$ and $\mu' = \lambda$ at fixed points 0 and ∞; we have with $|\lambda'| > 1$ and $|\mu'| < 1$. Similarly, if $|\lambda|, |\mu| > 1$, then the third fixed point x_0 has multiplier ν with $|\nu| < 1$. Thus, by a change of coordinates exchanging x_0 and ∞ while leaving 0 fixed, one can verify that we get a quadratic map fixing 0 and ∞ with multipliers λ and ν. Thus, all cases with $|\lambda|, |\mu| \neq 1$ either are already understood or reduce to the case that $|\lambda| > 1$ and $|\mu| < 1$.

In Section A.1, we will consider the case $|\lambda| > 1$ and $|\mu| < 1$ for odd primes; note that this case includes polynomials of bad reduction for odd primes. In Section A.2, we will study only the polynomial case for $p = 2$. For both sections, we will need the following lemma on the power series expansion of a square root.

Lemma A.0.7. *Let $g_2(w) \in \mathbb{Q}[[w]]$ be the binomial power series expansion for $\sqrt{1+w}$, i.e.,*

$$g_2(w) = 1 + \sum_{n=1}^{\infty} C_n w^n,$$

where

$$C_n = \frac{(-1)^{n-1}}{2^{2n-1} n} \binom{2n-2}{n-1}.$$

Then for any $n \geq 1$, $4^n C_n \in \mathbb{Z}$. In particular, for $p \neq 2$, $g_2 \in \mathbb{C}_p[[w]]$ converges on $D_1(0)$ with image contained in $D_1(0)$; and for $p = 2$, $g_2 \in \mathbb{C}_2[[w]]$ converges on $D_{|4|}(0)$ with image contained in $D_1(0)$.

Lemma A.0.7 may be proven by showing that $4^n C_n \in \mathbb{Z}_p$ for any prime p. We omit the details.

A.1 Case 1: $p \neq 2$, $|\lambda| > 1$, and $|\mu| < 1$

Let K denote the completion of the field $\mathbb{Q}_p(\lambda, \mu)$. We may write

$$\phi(z) = \left(\frac{z}{\mu}\right)\left(\frac{z+\lambda}{z+\mu^{-1}}\right).$$

Let $\mathcal{F}$ and $\mathcal{J}$ denote the Fatou and Julia sets of ϕ. Note that ∞ is an attracting fixed point. In fact, if $|z| > |\lambda|$, then

$$|\phi(z)| \geq \frac{|z|}{|\mu|} \frac{|z|}{\max(|z|, |\mu^{-1}|)} = \min\left(\frac{|z|}{|\mu|}, |z|^2\right).$$

In particular, a point outside $\overline{D}_{|\lambda|}(0)$ will increase in absolute value by squaring until it is outside $D_{|\mu^{-1}|}(0)$, when it starts increasing by multiples of $|\mu^{-1}|$. Thus, $\phi^n(z) \to \infty$ as $n \to \infty$; since this happens whenever $|z| > |\lambda|$, it follows that $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_{|\lambda|}(0) \subset \mathcal{F}$. In other words, $\mathcal{J}$ is contained in $\overline{D}_{|\lambda|}(0)$.

We shall see that in fact, all of the Fatou set is attracted to ∞. To prove this, we will need the following claim.

Claim A.1.1. *There are power series*

$$\psi_1 : \overline{D}_{|\lambda|}(0) \to D_{|\lambda|}(0) \quad \textit{and} \quad \psi_2 : \overline{D}_{|\lambda|}(0) \to D_{|\lambda|}(-\lambda)$$

which converge on $\overline{D}_{|\lambda|}(0)$ and which are local inverses of ϕ. In other words, if $D_i = \psi_i(\overline{D}_{|\lambda|}(0))$ $(i = 1, 2)$, then $\psi_i \circ \phi(z) = z$ for $z \in D_i$, and $\phi \circ \psi_i(z) = z$ for $z \in \overline{D}_{|\lambda|}(0)$. Furthermore, ψ_1 and ψ_2 have integer coefficients in the field $\mathbb{Q}_p(\lambda, \mu)$.

Remark. The ψ_i map into $D_{|\lambda|}(0)$ and $D_{|\lambda|}(-\lambda)$ but not onto. However, the important thing is that their images, D_1 and D_2, are disjoint.

Proof. For any $y \in \mathbb{C}_p$, note that $\phi(z) = y$ if and only if $z^2 + (\lambda - \mu y)z - y = 0$, i.e., if and only if

$$z = \frac{(\mu y - \lambda) \pm \sqrt{\lambda^2 + (4 - 2\lambda\mu)y + \mu^2 y^2}}{2},$$

which we can rewrite as

$$z = \frac{1}{2}\left[(\mu y - \lambda) \pm \lambda\sqrt{1 + \left(\frac{4}{\lambda^2} - \frac{2\mu}{\lambda}\right)y + \frac{\mu^2}{\lambda^2}y^2}\right].$$

Now if we are given $|y| \leq |\lambda|$, then

$$\left|\left(\frac{4}{\lambda^2} - \frac{2\mu}{\lambda}\right)y + \frac{\mu^2}{\lambda^2}y^2\right| \leq \max\left(\left|\frac{4}{\lambda}\right|, |2\mu|, |\mu^2|\right) < 1$$

and therefore

$$\sqrt{1 + \left(\frac{4}{\lambda^2} - \frac{2\mu}{\lambda}\right)y + \frac{\mu^2}{\lambda^2}y^2} = 1 + \frac{1}{2}\left[\left(\frac{4}{\lambda^2} - \frac{2\mu}{\lambda}\right)y + \frac{\mu^2}{\lambda^2}y^2\right] - \cdots \tag{A.2}$$

expands as a power series convergent for $y \in \overline{D}_{|\lambda|}(0)$. (We are using the assumption that $p \neq 2$.) Furthermore, by Lemma A.0.7, and because λ^{-1} and μ are integers in K, the power series of equation (A.2) has K-integer coefficients and converges on

$\overline{D}_{|\lambda|}(0)$. Let ψ_1 be the function where the $\pm$ is a $+$ and ψ_2 the function where the $\pm$ is a $-$. Then $\psi_1(0) = 0$ and $\psi_2(0) = -\lambda$. Thus, for $|y| \leq |\lambda|$, we have $|\psi_1(y)| < |\lambda|$ and $|\psi_2(y) + \lambda| < |\lambda|$. □

Let $s_1 = \text{rad}(D_1)$ and $s_2 = \text{rad}(D_2)$. Note that $s_i \in p^{\mathbb{Q}}$, since D_i is the image of a rational closed disk under a power series. Let

$$\rho = \max\left\{\frac{s_1}{|\lambda|}, \frac{s_2}{|\lambda|}\right\} < 1.$$

By Lemma 8.2.6, if $\overline{D}_r(a) \subset D_i$, then

$$\text{rad}\left(\psi_i\left(\overline{D}_r(a)\right)\right) \leq \rho r.$$

Let $Y_0 = \overline{D}_{|\lambda|}(0)$. We define Y_i for $i \geq 1$ inductively by $Y_i = \phi^{-1}(Y_{i-1})$. Note that Y_i is closed, since ϕ is continuous; furthermore, $\mathcal{J} \subset Y_i$, because $\mathcal{J} \subset Y_0$. Also note that

$$Y_1 = \overline{D}_{s_1}(-\lambda) \cup \overline{D}_{s_2}(0).$$

Claim A.1.2. *$Y_i \subset Y_{i-1}$ for all $i \geq 1$.*

Proof. Given the description of Y_1 above, it is clear that $Y_1 \subset Y_0$. And given some fixed i for which $Y_i \subset Y_{i-1}$, we have

$$Y_{i+1} = \phi^{-1}(Y_i) \subset \phi^{-1}(Y_{i-1}) = Y_i.$$

□

In light of Claim A.1.2, let $Y = \bigcap_{i=0}^{\infty} Y_i$. Thus, Y is closed and $\mathcal{J} \subset Y$. We will describe Y and show that $\mathcal{J} = Y \subset K$. We need the following technical fact.

Claim A.1.3. *Fix $i \geq 0$. Then Y_i is a disjoint union of 2^i disks,*

$$Y_i = \bigcup_{j=1}^{2^i} D_{i,j},$$

where $D_{i,j}$ is a rational closed disk of radius less than or equal to $\rho^i|\lambda|$ and containing a point of K.

Proof. We proceed by induction on i. By definition, Y_0 is a rational closed disk of radius $|\lambda| = \rho^0|\lambda|$ centered at $0 \in K$. Now suppose we knew that Y_{i-1} was a union of 2^{i-1} disks as described in the statement of the claim. We know that $Y_{i-1} \subset Y_0$, that ϕ has degree 2, and that ψ_1 and ψ_2 are different power series inverses to ϕ on Y_0. Thus,

$$Y_i = \phi^{-1}(Y_{i-1}) = \psi_1(Y_{i-1}) \cup \psi_2(Y_{i-1}),$$

and this union is disjoint. Now consider $D_{i-1,j}$, one of the 2^i disks making up Y_{i-1}. Since it is a closed disk, its image under each of ψ_1 and ψ_2 is closed; furthermore, because it contains a point of k and each ψ_i is defined over K, its images also contain points of K. In addition, the images of ψ_1 and ψ_2 are disjoint; thus, each of the 2^{i-1} disks of Y_{i-1} maps to two disks, and none of the images intersect each other. Therefore, Y_i is a disjoint union of 2^i rational closed disks, each containing a point of K.

The only statement left to prove is the bound on the radii. Given that all disks of Y_{i-1} have radius bounded by $\rho^{i-1}|\lambda|$, we consider a disk $D_{i,j}$ in Y_i. From the construction above, it is the image of some disk D_{i-1,j_1} from Y_{i-1} under either ψ_1 or ψ_2. Thus,

$$\mathrm{rad}\,(D_{i,j}) \leq \rho\,\mathrm{rad}\,(D_{i-1,j_1}) \leq \rho^i|\lambda|.$$

□

Claim A.1.4. $Y = \mathfrak{J}$.

Proof. We know $\mathfrak{J} \subseteq Y$, so it suffices to show that $Y \subseteq \mathfrak{J}$. Pick $y \in Y$ and $\varepsilon > 0$. We will consider the family $\{\phi^n\}$ on $D_\varepsilon(y)$. Pick $i \geq 1$ such that $\rho^i|\lambda| < \varepsilon$, and note that $D_\varepsilon(y) \not\subset Y_i$. However, $Y_i = \phi^{-i}(Y_0)$; thus, there is some $x \in D_\varepsilon(y)$ with $|\phi^i(x)| > \lambda$. Therefore, $\phi^n(x) \to \infty$, but $|\phi^n(y)| \leq |\lambda|$ for all $n \geq 1$. It follows that $\{\phi^n\}$ is not equicontinuous on $D_\varepsilon(y)$ for any $\varepsilon > 0$. By definition, $y \in \mathfrak{J}$. □

Note that the fixed point at $x_0 = (1-\lambda)/(1-\mu)$ is repelling; thus, $x_0 \in \mathfrak{J}$, and so the Julia set is nonempty. By Claims A.1.3 and A.1.4, it is a compact Cantor set contained in

$$D_{|\lambda|}(0) \cup D_{|\lambda|}(-\lambda).$$

Furthermore, Y was defined to be the set of all points whose iterates remain bounded; hence, the entire Fatou set is attracted to ∞.

It is interesting to note that if $\lambda, \mu \in \overline{\mathbb{Q}}_p$, then K is a finite extension of $\mathbb{Q}_p$, and $\mathfrak{J} \subset K$. In particular, all periodic points of ϕ are defined over K.

A.2 Case 2: $p = 2$, and $\phi(z) = z^2 + c$

Throughout this section, $|\cdot|$ will denote the 2-adic absolute value, and $v(\cdot)$ will denote the 2-adic valuation. We will study the function $\phi(z) = z^2 + c$. Let $\mathfrak{F}$ and $\mathfrak{J}$ denote the Fatou and Julia sets of ϕ.

If $|c| \leq 4$, then let x be a solution of

$$x^2 - x + c = 0;$$

in other words, x is a fixed point of ϕ. Moving this fixed point to the origin, we see that ϕ is conjugate to the map

$$\tilde{\phi}(z) = \phi(z + x) - x = z^2 + 2xz + x^2 - x + c = z^2 + 2xz.$$

By Theorem 2.2.1, $|x| \leq 2$; thus, $|2x| \leq 1$, and $\tilde{\phi}$ has good reduction. The Julia set of ϕ is therefore empty.

Now let us consider the case $|c| > 4$. Write $c = -b^2$, so $|b| > 2$. Solving the equation

$$z^2 - z + c = 0,$$

we see that there are two finite fixed points, x_1 and x_2, with $|x_i| = |b|$. Note that $x_1 + x_2 = 1$, and

$$|x_1 - x_2| = \left|\sqrt{1-4c}\right| = |2b|.$$

It is easy to verify that x_1 and x_2 are both repelling.

Our goal for the rest of this section is to determine the set of points attracted to ∞ and to show that the complement of this set is $\mathcal{J}$. Note that if $|z| > |b|$, then $|\phi(z)| = |z^2|$, so $\phi^n(z) \to \infty$. Let $X_{|b|} = \overline{D}_{|b|}(0)$, so $\mathcal{J} \subset X_{|b|}$. For any $r \geq |2b|$, let

$$X_r = \overline{D}_r(x_1) = \overline{D}_r(x_2).$$

Note that for $r = |b|$, the new definition of $X_{|b|}$ coincides with the first definition.

The following claims show how the sets X_r map to each other under ϕ. It should be recalled that $|b| > 2$, so $|4b^2| > |2b|$.

Claim A.2.1. *For $r \geq |4b^2|$, $\phi^{-1}(X_r) = X_{\sqrt{r}}$.*

Proof. Because x_1 is a fixed point, note that for any $z \in \mathbb{C}_p$,

$$\phi(z) - x_1 = z^2 - b^2 - x_1 = z^2 - x_1^2.$$

Suppose that $z \in \mathbb{C}_p$ with $|z - x_1| \leq \sqrt{r}$. Then

$$|z + x_1| = |(z - x_1) + 2x_1| \leq \max\{\sqrt{r}, |2b|\} = \sqrt{r},$$

so

$$|\phi(z) - x_1| = |z - x_1||z + x_1| \leq \left(\sqrt{r}\right)^2 = r,$$

proving the $\supseteq$ inclusion.

On the other hand, suppose $y \in \overline{D}_r(x_1)$ and $\phi(z) = y$ with $z \notin \overline{D}_{\sqrt{r}}(x_1)$. Thus, $|z - x_1| > \sqrt{r} \geq |2b|$, and we have

$$|z + x_1| = |(z - x_1) + 2x_1| = |z - x_1| > \sqrt{r}.$$

Therefore,

$$|y - x_1| = |\phi(z) - x_1| = |z - x_1||z + x_1| > \left(\sqrt{r}\right)^2 = r,$$

contradicting our supposition and proving the $\subseteq$ inclusion. □

Claim A.2.2. *There is some radius r_0 with $|2b| \leq r_0 < |4b^2|$ and a nonnegative integer $n \geq 0$ such that for any $z \in \mathbb{P}^1(\mathbb{C}_p) \setminus X_{r_0}$,*

$$\phi^n(z) \in \mathbb{P}^1(\mathbb{C}_p) \setminus X_{|b|}.$$

Proof. If $s \geq |4b^2|$, then by Claim A.2.1 and the fact that $|x_1 - x_2| = |2b|$,

$$\phi^{-1}(X_s) = X_{\sqrt{s}}.$$

Note that $\sqrt{s} \geq |2b|$. Thus, starting with $X_{|b|}$, we can take inverse images until we get some X_{r_0} with $|2b| \leq r_0 < |4b^2|$. Let n be the number of inverse images we needed to take. The claim then follows immediately. □

As a consequence of Claim A.2.2, $\mathcal{J} \subset X_{r_0}$, and all points outside X_{r_0} are attracted to ∞ and therefore are Fatou. Let $Y_0 = X_{r_0}$. As in Section A.1, we can define inverses to ϕ on Y_0. In fact, we will define them on

$$D_{|4b^2|}(x_1) = D_{|4b^2|}(x_2) \supset Y_0.$$

Claim A.2.3. *There are power series*

$$\psi_1 : D_{|4b^2|}(x_1) \to D_{|2b|}(x_1) \quad \text{and} \quad \psi_2 : D_{|4b^2|}(x_1) \to D_{|2b|}(x_2)$$

which converge on $D_{|4b^2|}(x_1) = D_{|4b^2|}(x_2)$ and which are local inverses of ϕ. Furthermore, ψ_1 and ψ_2 have integer coefficients in the field $\mathbb{Q}_p(x_1) = \mathbb{Q}_p(x_2)$.

Proof. If $y \in D_{|4b^2|}(x_1)$ and $\phi(z) = y$ for some $z \in \mathbb{C}_p$, then

$$z^2 = b^2 + y = b^2 + x_1 + (y - x_1) = x_1^2 + (y - x_1).$$

Therefore,

$$z = \pm x_1 \sqrt{1 + \frac{y - x_1}{x_1^2}}. \tag{A.3}$$

However, $|y - x_1| \leq |4b^2| = |4x_1^2|$; therefore, by Lemma A.0.7, equation (A.3) expands as a power series in $(y - x_i)$ converging for $y \in D_{|4b^2|}(x_1)$. Let ψ_1 be the series where the $\pm$ is $+$, and let ψ_2 be the series where the $\pm$ is $-$. The image of ψ_1 is $D_{|2b|}(x_1)$, and the image of ψ_2 is $D_{|2b|}(-x_1) = D_{|2b|}(x_2)$. □

The rest of the analysis is exactly analogous to that in Section A.1. We define $Y_i = \phi^{-1}(Y_{i-1})$. By applying ψ_1 and ψ_2, we can show that for any $i \geq 0$, Y_i is a disjoint union of 2^i disks, each with radius $r/|2b|^i$. Thus, $Y = \bigcap Y_i$ is a Cantor set, and every point of the complement of Y is attracted to ∞. Since points of Y remain bounded, it follows that $Y = \mathcal{J}$. As in Section A.1, if $c \in \overline{\mathbb{Q}}_p$, then $K = \mathbb{Q}_p(x_1)$ is a finite extension of $\mathbb{Q}_p$, and all points of Y are in K. Thus, all periodic points of ϕ are defined over K.

Appendix B

List of Notation

$\mathbb{Z}_p$	p-adic rational integers, 5
$\mathbb{Q}_p$	p-adic rational numbers, 5
$\overline{\mathbb{Q}}_p$	algebraic closure of $\mathbb{Q}_p$, 5
$\mathbb{C}_p$	completion of $\overline{\mathbb{Q}}_p$, 5
$v_p(\cdot) = v(\cdot)$	p-adic valuation on $\mathbb{C}_p$, 5
$\lvert \cdot \rvert_p = \lvert \cdot \rvert$	p-adic absolute value on $\mathbb{C}_p$, 5
$\mathcal{O}$	ring of integers of $\mathbb{C}_p$, 5
$\mathbb{P}^1(\mathbb{C}_p)$	projective line over $\mathbb{C}_p$, 5
$D_r(a)$	open disk of radius r about a, 5
$\overline{D}_r(a)$	closed disk of radius r about a, 6
ϕ^n	n-fold composition of ϕ, 6
$\mathrm{dist}(\cdot,\cdot)$	distance between two subsets of $\mathbb{C}_p$, 6
$\mathrm{rad}(\cdot)$	radius of a disk in $\mathbb{C}_p$, 6
$\mathcal{F} = \mathcal{F}_\phi$	Fatou set of ϕ, 12
$\mathcal{J} = \mathcal{J}_\phi$	Julia set of ϕ, 12
D^n	n-dimensional polydisk, 15
ν	the multi-index $(i_1, \ldots, i_n)$, 15
z^ν	$z_1^{i_1} \cdots z_n^{i_n}$, 15
$\nu \geq 0$	all indices nonnegative, 15
$\lVert \nu \rVert$	norm of ν, 15
T_n	ring of restricted power series in n variables, 15
$\lVert f \rVert$	norm of the function f, 15
$\mathrm{Max}\, A$	maximal ideal space of the Tate algebra A, 16
$X\left(\dfrac{f}{g}\right)$	rational domain defined by $\{f_i\}$ and g, 17
$A\left(\dfrac{f}{g}\right)$	Tate algebra of $X\left(\dfrac{f}{g}\right)$, 17
$\left(\dfrac{f}{g}\right)^\nu$	$\dfrac{f_1^{i_1} \cdots f_n^{i_n}}{g^{i_1+\cdots+i_n}}$, 18
$\deg \phi^{-1}(x)$	number of points in $\phi^{-1}(x)$, counting multiplicity, 19
Φ	action of ϕ on Fatou D-components, 23

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