

ATTAINING POTENTIALLY GOOD REDUCTION IN ARITHMETIC DYNAMICS

ROBERT L. BENEDETTO

ABSTRACT. Let K be a non-archimedean field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$. If ϕ has potentially good reduction, we give an upper bound, depending only on d , for the minimal degree of an extension L/K such that ϕ is conjugate over L to a map of good reduction. In particular, if $d = 2$ or d is less than the residue characteristic of K , the bound is $d + 1$. If K is discretely valued, we give examples to show that our bound is sharp.

Fix the following notation throughout this paper.

K :	a field
\overline{K} :	an algebraic closure of K
$ \cdot $:	a non-archimedean absolute value on \overline{K}
\mathcal{O}_K :	the ring of integers $\{x \in K : x \leq 1\}$ of K
\mathcal{M}_K :	the maximal ideal $\{x \in K : x < 1\}$ of \mathcal{O}_K
k :	the residue field $\mathcal{O}_K/\mathcal{M}_K$ of K

Let $\phi(z) \in K(z)$ be a rational function. We define the degree of $\phi = f/g$ to be $\deg \phi := \max\{\deg f, \deg g\}$, where $f, g \in K[z]$ have no common factors. We will view ϕ as a dynamical system acting on $\mathbb{P}^1(\overline{K}) = \overline{K} \cup \{\infty\}$. For a thorough treatment of dynamics over such non-archimedean fields, see Chapter 10 of [1], or [2].

The notion of *good reduction* of ϕ first appeared in [6]; see Definition 1.1. We say that ϕ has *potentially good reduction* if ϕ is conjugate over L to a map of good reduction, for some finite extension L/K . In [3], we gave a necessary and sufficient condition for whether or not ϕ has potentially good reduction. In this paper, we turn to a related question: if ϕ has potentially good reduction, how large an extension L/K is required to attain good reduction? In Theorem 3.6 of [9], Rumely showed that when $\deg \phi = d \geq 2$ and ϕ has potentially good reduction, there is an extension L/K of degree at most $(d + 1)^2$ such that ϕ is

Date: December 16, 2013; Revised July 8, 2014.

1991 *Mathematics Subject Classification.* Primary: 37P05 Secondary: 37P20, 11S82.

Key words and phrases. arithmetic dynamics, good reduction, periodic points.

conjugate over L to a map of good reduction. In this paper, we improve Rumely's bound, as follows.

Theorem A. *Let K be a field with non-archimedean absolute value $|\cdot|$ and residue characteristic $p \geq 0$, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$. Define*

$$B = B(p, d) := \begin{cases} p^e(d-1) & \text{if } d \geq 3 \text{ and } d = mp^e \\ & \text{for integers } e, m \geq 1 \text{ with } p \nmid m, \\ dp^e & \text{if } d \geq 3 \text{ and } d = 1 + mp^e \\ & \text{for integers } e, m \geq 1 \text{ with } p \nmid m, \\ d+1 & \text{otherwise.} \end{cases}$$

If ϕ has potentially good reduction, then there is an extension L/K with $[L : K] \leq B$ such that ϕ is conjugate over L to a map of good reduction.

For $d \geq 3$, the bound B of the Theorem A is at most $d(d-1)$. Moreover, if $d = 2$, or if $p > d$, then B is simply $d+1$. In addition, B is a sharp bound, at least for many choices of the field K .

Theorem B. *Let K be a field with non-archimedean absolute value $|\cdot|$ and residue characteristic $p \geq 0$, and let $d \geq 2$ be an integer. Define the integer B as in Theorem A. Suppose K contains an element π such that for any integer $n \geq 2$ with $n|B$, $|\pi|^{1/n} \notin |K|$. Then there is a rational function $\phi \in K(x)$ of degree d and potentially good reduction such that for any extension field L with $[L : K] < B$, ϕ is not conjugate over L to a map of good reduction.*

The existence of the element π in the hypothesis of Theorem B certainly holds if K is discretely valued. After all, in that case, we may choose π to be a uniformizer for K .

The outline of the paper is as follows. We will recall some general facts about arithmetic dynamics and the Berkovich projective line in Section 1. In Section 2, we will state and prove some auxilliary results that we will need. We will then prove Theorem A in Section 3, and Theorem B in Section 4. Finally, in Section 5, we will discuss a conjectural strengthening of Theorem A, that the extension L/K can always be chosen to be totally ramified.

1. BACKGROUND

If $n \in \mathbb{Z}$ is a nonzero integer and $p \geq 2$ is prime, we use the standard notation $v_p(n)$ to indicate the largest integer $e \geq 0$ such that $p^e | n$. For $p = 0$, we will use the following conventions: $v_p(n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$, and $p^0 = 1$.

A map $\phi \in K(z)$ of degree $d = \deg \phi \geq 2$ has exactly $d + 1$ fixed points in $\mathbb{P}^1(\overline{K})$, counted with appropriate multiplicity. If $x \in \overline{K}$ is such a fixed point, its *multiplier* is $\lambda := \phi'(x)$. Given $h \in \text{PGL}(2, \overline{K})$, $h(x)$ is a fixed point of $\psi := h \circ \phi \circ h^{-1}$, and the multiplier of $h(x)$ under ψ is also λ . Thus, we can define the multiplier of a fixed point at ∞ by changing coordinates via such a conjugation.

Note that $\lambda = 1$ if and only if x has multiplicity at least two as a fixed point of ϕ , and that $\lambda = 0$ if and only if x is a critical point of ϕ . We say that x is *attracting* if $|\lambda| < 1$, *repelling* if $|\lambda| > 1$, and *indifferent* if $|\lambda| = 1$.

Given a polynomial $f(z) \in \mathcal{O}_K[z]$, denote by $\overline{f}(z) \in k[z]$ the polynomial formed by reducing all coefficients of f modulo \mathcal{M}_K .

Definition 1.1. Let $\phi(z) \in K(z)$ be a rational function of degree $d \geq 1$. Write $\phi = f/g$ with $f, g \in \mathcal{O}_K[z]$ and with at least one coefficient of f or g having absolute value 1. Let $\overline{\phi} := \overline{f}/\overline{g} \in k(z) \cup \{\infty\}$. We say that ϕ has *good reduction* if $\deg \overline{\phi} = \deg \phi$. Otherwise, we say that ϕ has *bad reduction*.

We say that ϕ has *potentially good reduction* if there is a finite extension L/K in \overline{K} and a map $h \in \text{PGL}(2, L)$ such that $h \circ \phi \circ h^{-1} \in L(z)$ has good reduction.

It is easy to see that for any $\phi \in K(z)$, polynomials $f, g \in \mathcal{O}_K[z]$ exist with $\phi = f/g$ and with at least one coefficient of f or g having absolute value 1. Moreover, the reduction type (good or bad) of ϕ is independent of the choice of the pair f, g . If $\phi \in K[z]$ is a polynomial, then it has good reduction if and only if $\phi \in \mathcal{O}_K[z]$ and the lead coefficient of ϕ has absolute value 1.

Given $a \in \overline{K}$ and $r > 0$,

$$D(a, r) := \{x \in \overline{K} : |x - a| < r\}$$

and

$$\overline{D}(a, r) := \{x \in \overline{K} : |x - a| \leq r\}$$

will denote the open and closed disks, respectively, in \overline{K} containing a and of radius r . If $r \in |\overline{K}^\times|$, we say that the above disks are *rational*.

If $\phi \in \overline{K}(z)$ is a nonconstant rational function, and if $D(a, r) \subseteq \overline{K}$ is a rational open disk containing no poles of ϕ , then we may write

$$\phi(z) = \sum_{n \geq 0} c_n (z - a)^n$$

as a power series converging on $D(a, r)$. In that case, $\{|c_n|r^n : n \geq 0\}$ is bounded and attains its maximum. Moreover, this power series has

an associated *Weierstrass degree*, which is the smallest integer $\ell \geq 0$ such that

$$|c_\ell|r^\ell = \max\{|c_n|r^n : n \geq 0\}.$$

It is a consequence of the Weierstrass Preparation Theorem (and more specifically, of the theory of Newton polygons) that in that case, the image $\phi(D(a, r))$ is also a rational open disk. Moreover, if that image disk $D(b, s)$ contains the point 0, then the Weierstrass degree ℓ of ϕ on $D(a, r)$ is at least 1, and the mapping

$$\phi : D(a, r) \rightarrow D(b, s) \quad \text{is everywhere } \ell\text{-to-1,}$$

meaning that every point $y \in D(b, s)$ has exactly ℓ preimages in $D(a, r)$, counting multiplicity. In particular, we must have $1 \leq \ell \leq \deg \phi$.

The Berkovich projective line $\mathbb{P}_{\text{Ber}}^1$ over K is a certain compact Hausdorff topological space containing $\mathbb{P}^1(\overline{K})$ as a subspace. The precise definition, which uses multiplicative seminorms on \overline{K} -algebras, is rather involved; the interested reader may consult Berkovich's original monograph [4], the detailed exposition in [1], or the summary in [2], for example.¹ We only state some basic properties here, without proofs.

The space $\mathbb{P}_{\text{Ber}}^1$ is uniquely path-connected: given any two distinct points $\xi_0, \xi_1 \in \mathbb{P}_{\text{Ber}}^1$, there is a *unique* arc between them. That is, there is a unique subspace $X \subseteq \mathbb{P}_{\text{Ber}}^1$ homeomorphic to the interval $[0, 1] \subseteq \mathbb{R}$, where $\xi_0, \xi_1 \in X$, and the homeomorphism takes ξ_0 to 0 and ξ_1 to 1.

For each closed disk $\overline{D}(a, r) \subseteq \overline{K}$, the space $\mathbb{P}_{\text{Ber}}^1$ contains a unique associated point, which we shall denote $\zeta(a, r)$. If $r \in |\overline{K}^\times|$, then $\zeta(a, r)$ is said to be a *type II point*. For completeness, we note that the points $\zeta(a, r)$ with $r > 0$ but $r \notin |\overline{K}^\times|$ are of type III. The type I points are simply points of $\mathbb{P}^1(\mathbb{C}_K)$, where \mathbb{C}_K is the completion of \overline{K} . There are also type IV points, corresponding to decreasing chains of disks in \overline{K} with empty intersection. However, in this paper we will only be concerned with the type II points.

For any type II point $\zeta(a, r) \in \mathbb{P}_{\text{Ber}}^1$, the complement $\mathbb{P}_{\text{Ber}}^1 \setminus \{\zeta(a, r)\}$ is no longer connected, but instead consists of infinitely many connected components. When intersected with $\mathbb{P}^1(\overline{K})$, one of these components is $\mathbb{P}^1(\overline{K}) \setminus \overline{D}(a, r)$, while the rest are the infinitely many rational open disks $D(b, r) \subseteq \mathbb{P}^1(\overline{K})$ with $b \in \overline{D}(a, r)$.

Finally, any rational function $\phi \in \overline{K}(z)$ extends uniquely to a continuous function $\phi : \mathbb{P}_{\text{Ber}}^1 \rightarrow \mathbb{P}_{\text{Ber}}^1$. In particular, if $D(a, r) \subseteq \overline{K}$ is a

¹Readers familiar with $\mathbb{P}_{\text{Ber}}^1$ may note that the construction is usually done over the completion \mathbb{C}_K of \overline{K} . Because our absolute value $|\cdot|$ is already defined on all of \overline{K} , however, the multiplicative seminorms over \overline{K} coincide with those over \mathbb{C}_K .

rational open disk containing no poles of ϕ , then $\phi(\zeta(a, r)) = \zeta(b, s)$, where $D(b, s) = \phi(D(a, r))$ is the image in \overline{K} of $D(a, r)$ under ϕ .

2. SOME LEMMAS

We say a point $\xi \in \mathbb{P}_{\text{Ber}}^1$ is *totally invariant* under $\phi \in K(z)$ if $\phi^{-1}(\xi) = \{\xi\}$.

Lemma 2.1. *Let K be a non-archimedean field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$. Then*

- (a) *ϕ has good reduction if and only if the Gauss point $\zeta(0, 1)$ is totally invariant under ϕ .*
- (b) *ϕ has potentially good reduction if and only if there is some type II point $\xi \in \mathbb{P}_{\text{Ber}}^1$ that is totally invariant under ϕ . In that case, ξ is the only totally invariant type II point in $\mathbb{P}_{\text{Ber}}^1$.*

Proof. This is Théorème 5 of [8]. See also Corollary 9.27(C) and Proposition 10.45 of [1]. \square

Lemma 2.2. *Let $a_1, a_2, b_1, b_2, c_1, c_2 \in \overline{K}$, and let $\xi_1, \xi_2 \in \mathbb{P}_{\text{Ber}}^1$ be type II points. Suppose that a_1, b_1 , and c_1 all lie in different components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi_1\}$, and that a_2, b_2 , and c_2 all lie in different components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi_2\}$. Let $h \in \text{PGL}(2, \overline{K})$ be the unique linear fractional map with $h(a_1) = a_2$, $h(b_1) = b_2$, and $h(c_1) = c_2$. Then $h(\xi_1) = \xi_2$.*

Proof. Suppose $h(\xi_1) \neq \xi_2$. Let U be the component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{h(\xi_1)\}$ containing ξ_2 , and let V be the component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi_2\}$ containing $h(\xi_1)$. By the continuity of the map h^{-1} , the points a_2, b_2, c_2 must lie in separate components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{h(\xi_1)\}$. By hypothesis, they also lie in separate components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi_2\}$. Thus, U and V can each contain at most one of these three points; without loss, $c_2 \in \mathbb{P}_{\text{Ber}}^1 \setminus (U \cup V)$.

Since $c_2 \notin V$, the unique arc $\gamma \subseteq \mathbb{P}_{\text{Ber}}^1$ from c_2 to $h(\xi_1)$ must pass through ξ_2 . But then the unique arc $\gamma' \subseteq \mathbb{P}_{\text{Ber}}^1$ from c_2 to ξ_2 does *not* pass through $h(\xi_1)$, and hence $c_2 \in U$, a contradiction. \square

The next Lemma concerns the dynamics of ϕ on the components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ when ξ is totally invariant. It is merely a weak version of a special case of Rivera-Letelier's far more general Classification Theorem [7] in the context of potentially good reduction. However, its proof is much simpler than that of the full Classification Theorem, and the statement below will suffice for our purposes.

Lemma 2.3. *Let K be a non-archimedean field, let $\phi \in K(z)$ be a nonconstant rational function, and suppose that the type II point $\xi \in \mathbb{P}_{\text{Ber}}^1$ is totally invariant under ϕ . Let U be a component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ such that $\phi(U) \subseteq U$. Then $\phi(U) = U$, and either*

- (a) $\phi : U \rightarrow U$ is one-to-one, or
 (b) $\phi : U \rightarrow U$ is ℓ -to-1, for some integer $\ell \geq 2$. Moreover, there is an attracting fixed point $a \in U \cap \mathbb{P}^1(\overline{K})$; and $\lim_{n \rightarrow \infty} \phi^n(x) = a$ for all $x \in U \cap \mathbb{P}^1(\overline{K})$.

Proof. After a \overline{K} -rational change of coordinates, we may assume that $\xi = \zeta(0, 1)$, and that $U = D(0, 1)$. Expand $\phi(z) = \sum_{n \geq 0} c_n z^n \in K[[z]]$ as a power series converging on $D(0, 1)$. Since $\phi(U) \subseteq U$ but also $\phi(\zeta(0, 1)) = \zeta(0, 1)$, we must have $\phi(D(0, 1)) = D(0, 1)$. In particular, $|c_n| \leq 1$ for all $n \geq 0$, with equality for at least one n . Let ℓ be the Weierstrass degree of ϕ on $D(0, 1)$, so that $1 \leq \ell \leq d$. If $\ell = 1$, then we are in case (a), and we are done.

Otherwise, $\ell \geq 2$. Thus,

$$|c_n| < 1 \text{ for } n < \ell, \quad |c_\ell| = 1, \quad \text{and} \quad |c_n| \leq 1 \text{ for } n > \ell.$$

Therefore, a glance at the Newton polygon of $\phi(z) - z$ shows that ϕ has exactly one fixed point a with $|a| < 1$, although *a priori*, a is defined over the completion \mathbb{C}_K of \overline{K} . However, since $\phi \in \overline{K}(z)$, all the fixed points of ϕ , including a , are defined over \overline{K} .

After a \overline{K} -rational translation, we may assume that $a = 0$, so that $c_0 = 0$. Thus, for any $x \in D(0, 1)$,

$$|\phi(x)| \leq \max\{|c_n x^n| : 1 \leq n \leq \ell\}.$$

It follows that $\phi^n(x) \rightarrow 0$ as $n \rightarrow \infty$. □

Lemma 2.3 motivates the following definition.

Definition 2.4. Let K be a non-archimedean field, let $\phi \in K(z)$ be a nonconstant rational function, and suppose that the type II point $\xi \in \mathbb{P}_{\text{Ber}}^1$ is totally invariant under ϕ . Let U be a component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$. If $\phi(U) \subseteq U$, we say U is *fixed* by ϕ .

In that case, if $\phi : U \rightarrow U$ is one-to-one, then we say U is an *indifferent* component; otherwise, we say U is an *attracting* component.

If K is complete and U is an attracting fixed component containing a K -rational point x , then the attracting fixed point $a \in U$ of Lemma 2.3.b must also be K -rational, as it is a limit of iterates of x . Thus, each of the remaining fixed points is defined over an extension of K of degree at most d , improving the *a priori* bound of $d+1$. Although the same conclusion does not necessarily hold when K is not complete, it almost does, as the next lemma makes precise.

Lemma 2.5. *Let K be a non-archimedean field, let $\phi \in K(z)$ be a rational function of degree $d \geq 2$, and let $\xi \in \mathbb{P}_{\text{Ber}}^1$ be a type II point that is totally invariant under ϕ . Suppose that $a \in \mathbb{P}^1(K)$ lies in an*

attracting fixed component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$. Then there is a polynomial $f \in K[z]$ such that

- (a) $\deg f = d$, and
- (b) none of the roots of f in $\mathbb{P}^1(\overline{K})$ lie in the same component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ as a .

Proof. After a K -rational change of coordinates if necessary, we may assume that ∞ is *not* one of the $d + 1$ fixed points of ϕ . Let U be the component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ containing a .

Write $\phi = g/h$ with $g, h \in K[z]$ in lowest terms. Then the fixed points of ϕ in $\mathbb{P}^1(\overline{K})$ are precisely the roots of the polynomial

$$F(z) = h(z)(\phi(z) - z) = g(z) - zh(z) \in K[z].$$

Because ∞ is not fixed, we have $\deg F = d + 1$. By Lemma 2.3.b, there is an attracting fixed point $b \in U \cap \overline{K}$ such that $\lim_{n \rightarrow \infty} \phi^n(a) = b$. Denote the roots of F by $b = b_0, b_1, \dots, b_d \in \overline{K}$.

Choose $\varepsilon > 0$ small enough that for each $j = 0, \dots, d$, the disk $D(b_j, \varepsilon)$ is contained in a single component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$. For each $n \geq 0$, define

$$G_n(z) := F(z) - F(\phi^n(a)) \in K[z],$$

which is a slight perturbation of F . By Proposition 3.4.1.1 of [5], for n large enough, and hence for $\phi^n(a)$ close enough to b , G_n has exactly one root, namely $\phi^n(a)$, in $D(b_0, \varepsilon)$, and its other d roots in $\bigcup_{j=1}^d D(b_j, \varepsilon)$. Thus, the polynomial

$$f(z) := \frac{G_n(z)}{z - \phi^n(a)} \in K[z]$$

has degree d , with none of its roots lying in U . □

Lemma 2.6. *Let K be a non-archimedean field of residue characteristic $p \geq 0$ and algebraic closure \overline{K} , and let $f \in K[z]$ be a polynomial of degree $n \geq 1$. Let $V \subseteq \overline{K}$ be a closed disk that contains all the roots of f , and let $e = v_p(n) \geq 0$. Then there is a point $\alpha \in V$ such that $[K(\alpha) : K] \leq p^e$.*

When $p = 0$, recall our convention that $v_p(n) = 0$ and $p^0 = 1$.

Proof. Without loss, $f(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ is monic. Let $q = p^e$, and let $g(t) \in K[t]$ be the coefficient of z^{n-q} in $f(z+t) \in K[t, z]$, that is,

$$g(t) = \binom{n}{q} t^q + \sum_{j=0}^{q-1} a_{n-q+j} \binom{n-q+j}{n-q} t^j.$$

It is well known that $p \nmid \binom{n}{q}$ (this fact follows easily from the formula for $\text{ord}_p(n!)$ found in, for example, Section V.3.1 of [10]), and hence that $|\binom{n}{q}| = 1$ in K .

Let $\alpha \in \overline{K}$ be a root of g , so that $[K(\alpha) : K] \leq \deg g = q$. Let $h(z) := f(z + \alpha) \in K(\alpha)[z]$, and let $W := V - \alpha \subseteq \overline{K}$, which is a closed disk containing all the roots of h . It suffices to show that $0 \in W$.

We proceed by contradiction. If $0 \notin W$, then denoting the roots of h by $\beta_1, \dots, \beta_n \in \overline{K}$, we must have $|\beta_j - \beta_1| < |\beta_1|$ for all $j = 1, \dots, n$. Therefore, for any $1 \leq j_1 < \dots < j_q \leq n$,

$$(2.1) \quad |\beta_{j_1} \cdots \beta_{j_q} - \beta_1^q| < |\beta_1|^q.$$

By construction, the z^{n-q} -coefficient of $h(z)$ is 0. Thus, examining the same coefficient when writing $h(z) = \prod_{j=1}^n (z - \beta_j)$, we have

$$(2.2) \quad 0 = (-1)^q \sum_{1 \leq j_1 < \dots < j_q \leq n} \beta_{j_1} \cdots \beta_{j_q} = (-1)^q \binom{n}{q} \beta_1^q + \varepsilon,$$

where $|\varepsilon| < |\beta_1|^q$, by inequality (2.1). Since $|\binom{n}{q}| = 1$, then, the right side of equation (2.2) cannot be zero, giving a contradiction. \square

Lemma 2.7. *Let K be a non-archimedean field with algebraic closure \overline{K} , let $\phi \in K(z)$, and let $r > 0$. Suppose that ϕ maps $D(0, r)$ ℓ -to-1 onto itself, for some integer $\ell \geq 2$. Then $r^{\ell-1} \in |K^\times|$.*

Proof. Expand $\phi(z) = \sum_{n \geq 0} c_n z^n \in K[[z]]$ as a power series converging on $D(0, r)$. By hypothesis, the Weierstrass degree of ϕ on $D(0, r)$ is ℓ , and $\phi(D(0, r)) = D(0, r)$. In particular,

$$|c_\ell| r^\ell = r.$$

As a result, c_ℓ cannot be zero, and $r^{\ell-1} = |c_\ell|^{-1} \in |K^\times|$. \square

3. PROOF OF THEOREM A

Theorem A rests on the following four results on the existence of field extensions L/K large enough to define new components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$, where ξ is the totally invariant type II point for a given map ϕ of potentially good reduction. The basic idea underlying these results is twofold. First, an indifferent fixed component U may contain multiple fixed points, but for any x in U , only one of the d preimages of x lies in U . Second, an attracting fixed component U may contain many, or even all, of the preimages of a given point $x \in U$, but only one of the $d + 1$ fixed points.

Theorem 3.1. *Let K be a non-archimedean field of residue characteristic $p \geq 0$, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$ with a totally invariant type II point $\xi \in \mathbb{P}_{\text{Ber}}^1$. Let $e = v_p(d-1) \geq 0$, and let U be an indifferent fixed component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ such that $U \cap \mathbb{P}^1(K) \neq \emptyset$. Then there is a point $\alpha \in \mathbb{P}^1(\overline{K}) \setminus U$ such that $[K(\alpha) : K] \leq p^e$.*

As before, when $p = 0$, recall our convention that $v_p(n) = 0$ and $p^0 = 1$.

Proof. After a K -rational change of coordinates, we may assume that $\infty \in U$. Thus, $V := \mathbb{P}^1(\overline{K}) \setminus U$ is a closed disk contained in \overline{K} .

Counting multiplicity, there are exactly d preimages of $\phi(\infty)$ in $\mathbb{P}^1(\overline{K})$. However, because ∞ is one of them, and because $\phi : U \rightarrow U$ is one-to-one by Lemma 2.3, there must be $d-1$ such preimages in V , and they must be the roots of some polynomial $f(z) \in K[z]$ of degree $d-1$. By Lemma 2.6, then, there exists a point $\alpha \in V$ such that $[K(\alpha) : K] \leq p^e$. \square

Theorem 3.2. *Let K be a non-archimedean field of residue characteristic $p \geq 0$, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$ with a totally invariant type II point $\xi \in \mathbb{P}_{\text{Ber}}^1$. Let $e = v_p(d) \geq 0$, and let U be an attracting fixed component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ such that $U \cap \mathbb{P}^1(K) \neq \emptyset$. Then there is a point $\alpha \in \mathbb{P}^1(\overline{K}) \setminus U$ such that $[K(\alpha) : K] \leq p^e$.*

Proof. After a K -rational change of coordinates, we may assume that $\infty \in U$. Thus, $V := \mathbb{P}^1(\overline{K}) \setminus U$ is a closed disk contained in \overline{K} .

According to Lemma 2.5, with $a = \infty$, there is a polynomial $f \in K[z]$ of degree d and with all its roots in the closed disk V . By Lemma 2.6, then, there exists a point $\alpha \in V$ such that $[K(\alpha) : K] \leq p^e$. \square

Theorem 3.3. *Let K be a non-archimedean field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$ with a totally invariant type II point $\xi \in \mathbb{P}_{\text{Ber}}^1$. Let U and V be distinct components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ that both contain points of $\mathbb{P}^1(K)$ and such that U is fixed, while $\phi(V) \subseteq U$. Then there is a point $\alpha \in \mathbb{P}^1(\overline{K}) \setminus (U \cup V)$ such that $[K(\alpha) : K] \leq d$.*

Proof. After a K -rational change of coordinates, we may assume that $0 \in U$ and $\infty \in V$. There are d preimages of ∞ in $\mathbb{P}^1(\overline{K})$, none of which can be in $U \cup V$, since $\phi(U \cup V) \subseteq U$, whereas $\infty \notin U$. Thus, there is a polynomial $g(z) \in K[z]$ of degree d with all of its roots in $\mathbb{P}^1(\overline{K}) \setminus (U \cup V)$; g is, of course, simply the denominator of ϕ . Choosing α to be one of these roots, we have $[K(\alpha) : K] \leq \deg g = d$. \square

Theorem 3.4. *Let K be a non-archimedean field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$ with a totally invariant type II*

point $\xi \in \mathbb{P}_{\text{Ber}}^1$. Let U and V be distinct components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ that both contain points of $\mathbb{P}^1(K)$. If either or both of the following two conditions hold:

- (a) U and V are both fixed, or
- (b) U is attracting fixed,

then there is a point $\alpha \in \mathbb{P}^1(\overline{K}) \setminus (U \cup V)$ such that $[K(\alpha) : K] \leq d - 1$.

Proof. Under either condition, U is fixed. After a K -rational change of coordinates, we may assume that $0 \in U$ and $\infty \in V$. We consider two cases.

Case 1: U is attracting. Since $0 \in U$ and $\infty \in V$, it must be that $U \cap \mathbb{P}^1(\overline{K}) = D(0, r)$ and $V \cap \mathbb{P}^1(\overline{K}) = \mathbb{P}^1(\overline{K}) \setminus \overline{D}(0, r)$, for some $r > 0$. By Lemma 2.3, there is an integer $\ell \geq 2$ such that $\phi : U \rightarrow U$ is ℓ -to-1. Because $\deg \phi = d$, we must have $\ell \leq d$. By Lemma 2.7, there is some $c \in K^\times$ such that $r^{\ell-1} = |c|$. Let $\alpha \in \overline{K}$ be a root of the polynomial $z^{\ell-1} - c = 0$, so that $|\alpha| = r$. Hence, $\alpha \in \mathbb{P}^1(\overline{K}) \setminus (U \cup V)$, and $[K(\alpha) : K] \leq \ell - 1 \leq d - 1$.

Case 2: U is not attracting. Then by hypothesis, U and V are both fixed. In addition, by Lemma 2.3, U is indifferent, and hence $\phi : U \rightarrow U$ is one-to-one. The equation $\phi(z) = \phi(0)$ has exactly d solutions in $\mathbb{P}^1(\overline{K})$, none of which can be in V (since V is fixed, and $\phi(0) \in U$). Thus, this equation becomes a polynomial equation $g(z) = 0$, where $g \in K[z]$ has degree exactly d . Clearly 0 is a root of g , and hence $f(z) := g(z)/z \in K[z]$ is a polynomial of degree $d - 1$.

Let $\alpha \in \overline{K}$ be a root of f , so that $[K(\alpha) : K] \leq \deg f = d - 1$. Note that α cannot lie in U , since $\phi : U \rightarrow U$ is injective, and the root $z = 0$ of $\phi(z) = \phi(0)$ is already in U . It also cannot lie in V , as noted above. Thus, $\alpha \in \mathbb{P}^1(\overline{K}) \setminus (U \cup V)$, as desired. \square

Proof of Theorem A. Assume that ϕ has potentially good reduction. By Lemma 2.1, there is a type II point ξ that is totally invariant under ψ . We consider several cases.

Case 1: $\mathbb{P}^1(K)$ intersects at least three different components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$. Choose $a, b, c \in K$ in separate components, let $h \in \text{PGL}(2, K)$ be the unique linear fractional map with $h(a) = 0$, $h(b) = \infty$, and $h(c) = 1$, and let $\psi := h \circ \phi \circ h^{-1} \in K(z)$. By Lemma 2.2, $h(\xi) = \zeta(0, 1)$. Thus, $\zeta(0, 1)$ must be totally invariant under ψ ; by Lemma 2.1, ψ has good reduction. Since h is defined over K , and $[K : K] = 1 < B$, we are done.

Case 2: $\mathbb{P}^1(K)$ intersects exactly two components, U and V , of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$, where U is indifferent fixed but V is not fixed. By continuity, $\phi(V)$ must be contained in some component, and since $V \cap \mathbb{P}^1(K) \neq \emptyset$,

that component must also contain K -rational points. However, by assumption, $\phi(V) \not\subseteq V$. Thus, $\phi(V) \subseteq U$.

By Theorem 3.3, there exists an extension L/K in \overline{K} such that $[L : K] \leq d$, and $\mathbb{P}^1(L)$ intersects at least three different components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$. By Case 1, then, ϕ is conjugate over L to a map of good reduction. Since $d < B$, we are done.

Case 3: $\mathbb{P}^1(K)$ intersects exactly two components, U and V , of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$, and *either* U and V are both fixed, *or* U is attracting fixed. By Theorem 3.4, there is an extension L/K in \overline{K} such that $[L : K] \leq d-1$ and $\mathbb{P}^1(L)$ intersects at least three different components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$. By Case 1, then, ϕ is conjugate over L to a map of good reduction. Since $d-1 < B$, we are done.

Case 4: $\mathbb{P}^1(K)$ intersects exactly two components, U and V , of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$, but neither is fixed. In particular, neither can contain any fixed points in $\mathbb{P}^1(\overline{K})$. The fixed points, however, are roots of a polynomial $f \in K[z]$ of degree $d+1$. (After all, we may make a K -rational change of coordinates if necessary to guarantee that ∞ is not fixed.) Let α be one of the fixed points, and let $L := K(\alpha)$. Then $[L : K] \leq \deg f = d+1$, and $\mathbb{P}^1(L)$ intersects at least three different components of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$, namely U , V , and the component containing α . By Case 1, then, ϕ is conjugate over L to a map of good reduction. Since $d+1 \leq B$, we are done.

Case 5: $\mathbb{P}^1(K)$ intersects exactly one component U of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$, and U is an indifferent fixed component. Let $e := v_p(d-1)$. By Theorem 3.1, there exists $\alpha \in \mathbb{P}^1(\overline{K}) \setminus U$ such that $[K(\alpha) : K] \leq p^e$. Let V be the component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ containing α . Then with respect to the field $K(\alpha)$, we are now in either Case 2 or Case 3 above, and hence there is an extension $L/K(\alpha)$ with $[L : K(\alpha)] \leq d$ and such that ϕ is conjugate over L to a map of good reduction. Thus, $[L : K] \leq dp^e \leq B$, and we are done.

Case 6: $\mathbb{P}^1(K)$ intersects exactly one component U of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$, but U is not indifferent fixed. Of course, U must still be fixed, since the image $\phi(U)$ must be contained in a component but also contain K -rational points, and hence $\phi(U) \subseteq U$. By Lemma 2.3, then, U must be attracting fixed.

Let $e := v_p(d)$. By Theorem 3.2, there is a point $\alpha \in \mathbb{P}^1(\overline{K}) \setminus U$ such that $[K(\alpha) : K] \leq p^e$. Let V be the component of $\mathbb{P}_{\text{Ber}}^1 \setminus \{\xi\}$ containing α . Then with respect to the field $K(\alpha)$, we are now in Case 3 above, and hence there is an extension $L/K(\alpha)$ with $[L : K(\alpha)] \leq d-1$ and such that ϕ is conjugate over L to a map of good reduction. Thus, $[L : K] \leq p^e(d-1) \leq B$, and we are done. \square

Remark 3.5. In this Remark, we assume the reader is familiar with some of the standard machinery of non-archimedean dynamics.

The Julia set $\mathcal{J} \subseteq \mathbb{P}_{\text{Ber}}^1$ of a given rational function $\phi \in K(z)$ consists of precisely one point if and only if ϕ has potentially good reduction; see Théorème 4 of [8] and Theorem 10.105 of [1]. In that case, the one Julia point is the totally invariant type II point of Lemma 2.1. The field L in our Theorem A therefore has the property that the (one-point) Julia set of ϕ has a point in the convex hull of $\mathbb{P}^1(L)$. The referee posed to us the following related question: for an arbitrary rational function $\phi \in K(z)$, can we bound the degree of an extension L/K such that the Julia set of ϕ has a point in the convex hull of $\mathbb{P}^1(L)$?

The answer is yes, with a bound of $d = \deg \phi$, if K is complete, or $d + 1$ otherwise. The fact that this is smaller than our earlier bound B does not contradict Theorem B, as the convex hull condition does not force the diameter of the type II Julia point to lie in $|L|$. Here is the proof.

If $\mathbb{P}^1(K)$ intersects the Julia set \mathcal{J} of ϕ , then we are done by setting $L = K$. Thus, we may assume that $\mathbb{P}^1(K)$ is contained in the Fatou set \mathcal{F} of ϕ . If $\mathbb{P}^1(K)$ intersects more than one component of \mathcal{F} , then choosing $a, b \in \mathbb{P}^1(K)$ in different components, the unique arc in $\mathbb{P}_{\text{Ber}}^1$ from a to b must intersect \mathcal{J} , and we are again done by setting $L = K$. We may therefore assume that $\mathbb{P}^1(K)$ is contained in a single connected component U of \mathcal{F} . Since $\phi(0) \in \mathbb{P}^1(K)$, this component must be fixed.

According to the Classification Theorem of [7], the component U is either attracting or indifferent. If it is attracting and K is complete, then the component U contains a unique attracting fixed point $a = \lim_{n \rightarrow \infty} \phi^n(0)$, which lies in $\mathbb{P}^1(K)$ by our completeness assumption. Meanwhile, ϕ must have d other fixed points (counting multiplicity), all of which must be distinct from a , since a is attracting and therefore does not have multiplier 1. Let α be one of these other fixed points, and let $L = K(\alpha)$, so that $[L : K] \leq d$. Since α cannot lie in U , the unique arc in $\mathbb{P}_{\text{Ber}}^1$ from α to 0 must intersect \mathcal{J} , and we are done.

Similarly, if U is attracting and K is not complete, then choosing the same point fixed point α and $L = K(\alpha)$, we have $[L : K] \leq d + 1$.

Finally, if the fixed component U containing $\mathbb{P}^1(K)$ is indifferent, then $\phi : U \rightarrow U$ is injective. Let $a = \phi(0)$. Then $\phi^{-1}(a) \setminus \{0\}$ is nonempty (since $0 \in U$ cannot be a critical point), and so we may choose $\alpha \neq 0$ such that $\phi(\alpha) = a$. Let $L = K(\alpha)$, so that $[L : K] \leq d$. Since α cannot lie in U (as $\phi : U \rightarrow U$ is injective), the unique arc in $\mathbb{P}_{\text{Ber}}^1$ from α to 0 must intersect \mathcal{J} , and we are done.

4. PROOF OF THEOREM B

We will need the following result on the field of definition of a type II point.

Theorem 4.1. *Let K be a non-archimedean field, and let $a, b \in \overline{K}$ with $b \neq 0$. Let $j, m \geq 1$ be the smallest positive integers such that $|a|^j \in |K|$ and $|b|^m \in |K|$. Let L/K be an extension in \overline{K} , let $h \in \mathrm{PGL}(2, L)$, and suppose that $h(\zeta(a, |b|)) = \zeta(0, 1)$.*

- (a) *If $|a| \leq |b|$, then the ramification index of L/K , and hence also $[L : K]$, must be divisible by m .*
- (b) *If $|a| > |b|$, then the ramification index of L/K , and hence also $[L : K]$, must be divisible by $\mathrm{lcm}(j, m)$.*

Proof. Let $c_1 = h^{-1}(0)$, $c_2 = h^{-1}(\infty)$, and $c_3 = h^{-1}(1)$, all of which lie in $\mathbb{P}^1(L)$. Since $h : \mathbb{P}_{\mathrm{Ber}}^1 \rightarrow \mathbb{P}_{\mathrm{Ber}}^1$ is a homeomorphism mapping $\zeta(a, |b|)$ to $\zeta(0, 1)$, and since $0, \infty$, and 1 all lie in separate components of $\mathbb{P}_{\mathrm{Ber}}^1 \setminus \{\zeta(0, 1)\}$, the three points c_1, c_2 , and c_3 must lie in separate components of $\mathbb{P}_{\mathrm{Ber}}^1 \setminus \{\zeta(a, |b|)\}$. In particular, at most one can lie outside $\overline{D}(a, |b|)$. Without loss, then, $c_1, c_2 \in \overline{D}(a, |b|)$, and since both lie in separate components of $\mathbb{P}_{\mathrm{Ber}}^1 \setminus \{\zeta(a, |b|)\}$, we must have $|b| = |c_2 - c_1| \in |L^\times|$. Thus, the ramification index $[|L^\times| : |K^\times|]$ of L/K is divisible by m .

If $|a| > |b|$, then $|a| = |c_1| \in |L^\times|$ as well. Therefore, $[|L^\times| : |K^\times|]$ is also divisible by j , and hence by $\mathrm{lcm}(j, m)$. \square

Proof of Theorem B. Let K, B , and π be as in the statement of the Theorem. We may assume that $|p| \leq |\pi| < 1$. After all, if K is discretely valued, then we may choose π to be a uniformizer for K . And if K is not discretely valued, then $|K^\times|^B$ is dense in $(0, \infty)$, and hence there is some $a \in K^\times$ such that $|p| \leq |a|^B < 1$. Thus, replacing π by $a^{Bn}\pi$ for an appropriate $n \in \mathbb{Z}$, we have $|p| \leq |\pi| < 1$, as desired.

Let $\gamma \in \overline{K}$ be a B -th root of π , and let $L = K(\gamma)$. By the defining property of π , the extension L/K is totally ramified, with $[L : K] = B$. We now consider three cases.

Case 1: $d \geq 3$ and $p|d$, so that $B = p^e(d-1)$, where $d = mp^e$ with $e, m \geq 1$ and $p \nmid m$. Define $q := p^e$, and

$$\phi(z) := z + (\pi^{-1}z^q - 1)^m \in K[z].$$

Clearly $\deg \phi = mq = d$. Let $\alpha := \gamma^{d-1}$ and $\beta := \gamma^q$, so that $\alpha^q = \beta^{d-1} = \pi$. We will show that ϕ is conjugate over L to a map of good reduction, but not over any field \tilde{L}/K with $[\tilde{L} : K] < B = q(d-1)$.

Define $h(z) = \beta^{-m}(z - \alpha)$. (This conjugating map is chosen to move a fixed point of ϕ to 0 and then to scale to make the polynomial monic.)

Then $h(\zeta(\alpha, |\beta|^m)) = \zeta(0, 1)$, and

$$(4.1) \quad \begin{aligned} h(\phi(h^{-1}(z))) &= \beta^{-m} \left[\beta^m z + \alpha + \left(\alpha^{-q} \sum_{j=1}^q \binom{q}{j} \alpha^{q-j} \beta^{mj} z^j \right)^m - \alpha \right] \\ &= z + \left(\sum_{j=1}^q \binom{q}{j} \alpha^{-j} \beta^{mj-1} z^j \right)^m. \end{aligned}$$

Note that the lead coefficient of the polynomial (4.1) is $(\alpha^{-q} \beta^{d-1})^m = 1^m = 1$. Note also that $|\alpha^{-j} \beta^{mj-1}| = |\pi|^{E_j}$, where

$$E_j = -\frac{j}{q} + \frac{mj-1}{d-1} = \frac{j-q}{q(d-1)} > \frac{-1}{d-1} > -1.$$

Therefore, for $1 \leq j \leq q-1$, the coefficient of z^j inside the m -th power in (4.1) has absolute value

$$\left| \binom{q}{j} \alpha^{-j} \beta^{mj-1} \right| \leq |p| \cdot |\pi|^{E_j} < |p| \cdot |\pi|^{-1} \leq 1.$$

Thus, (4.1) is a monic polynomial in $\mathcal{O}_L[z]$, and hence it has good reduction.

Finally, by the uniqueness statement of Lemma 2.1.b, given an extension \tilde{L}/K in \overline{K} and a map $\tilde{h}(z) \in \text{PGL}(2, \overline{K})$ for which $\tilde{h} \circ \phi \circ \tilde{h}^{-1}$ has good reduction, we must have $\tilde{h}(\zeta(\alpha, |\beta|^m)) = \zeta(0, 1)$. However,

$$|\beta|^m = |\pi|^{m/(d-1)} < |\pi|^{m/d} = |\pi|^{1/q} = |\alpha|.$$

By Theorem 4.1.b, then, $[\tilde{L} : K] \geq \text{lcm}(q, d-1) = q(d-1)$.

Case 2: $d \geq 3$ and $p|(d-1)$, so that $B = p^e d$, where $d = 1 + mp^e$ with $e, m \geq 1$ and $p \nmid m$. Define $q := p^e$, and

$$\phi(z) := z + \frac{\pi^{d-1}}{(z^q - \pi^{q-1})^m} \in K(z).$$

Writing the above expression as a single fraction shows that $\deg \phi = 1 + qm = d$. Let $\alpha := \gamma^d$ and $\beta = \gamma^q$, so that $\alpha^q = \beta^d = \pi$. We will show that ϕ is conjugate over L to a map of good reduction, but not over any field \tilde{L}/K with $[\tilde{L} : K] < B = qd$.

Define $h(z) = \beta^{-(d-1)}(z - \alpha^{q-1})$. (This conjugating map is chosen to move a pole of ϕ to 0 and then to scale to change the numerator of the second term of ϕ from π^{d-1} to 1.) Then $h(\zeta(\alpha^{q-1}, |\beta|^{d-1})) = \zeta(0, 1)$,

and

$$\begin{aligned}
 h(\phi(h^{-1}(z))) &= \beta^{-(d-1)} \left[\beta^{d-1} z + \frac{\beta^{d(d-1)}}{\left(\sum_{j=1}^q \binom{q}{j} \alpha^{(q-j)(q-1)} \beta^{j(d-1)} z^j \right)^m} \right] \\
 (4.2) \qquad &= z + \frac{1}{\left(\sum_{j=1}^q \binom{q}{j} \alpha^{(q-j)(q-1)} \beta^{-(q-j)(d-1)} z^j \right)^m}.
 \end{aligned}$$

Consider the coefficient of z^j inside the m -power in the denominator of (4.2). This coefficient is 1 for $j = q$, and for $1 \leq j \leq q - 1$ has absolute value

$$\left| \binom{q}{j} \right| |\alpha^{q-1} \beta^{-(d-1)}|^{q-j} = \left| \binom{q}{j} \right| |\pi|^{E_j} \leq |p| |\pi|^{E_j},$$

where

$$E_j = (q-j) \left(\frac{q-1}{q} - \frac{d-1}{d} \right) = -\frac{(q-j)(d-q)}{dq} > -\frac{(d-q)}{d} > -1.$$

Thus, still for $1 \leq j \leq q - 1$, the z^j -coefficient in question has absolute value strictly less than $|p| \cdot |\pi|^{-1} \leq 1$. Hence, the denominator of (4.2) is a monic polynomial $g(z) \in \mathcal{O}_L[z]$ with reduction $\bar{g}(z) = z^{d-1}$. Thus, the rational function in (4.2) is $(zg(z) + 1)/g(z)$, which has reduction $(z^d + 1)/z^{d-1}$, exhibiting the desired good reduction.

Once again, given an extension \tilde{L}/K in \bar{K} and a linear fractional map $\tilde{h}(z) \in \text{PGL}(2, \bar{K})$ for which $\tilde{h} \circ \phi \circ \tilde{h}^{-1}$ has good reduction, we must have $\tilde{h}(\zeta(\alpha^{q-1}, |\beta|^{d-1})) = \zeta(0, 1)$, by the uniqueness statement of Lemma 2.1.b. However,

$$|\beta^{d-1}| = |\pi|^{(d-1)/d} < |\pi|^{(q-1)/q} = |\alpha^{q-1}|.$$

By Theorem 4.1.b, then, $[\tilde{L} : K] \geq \text{lcm}(d, q) = dq$.

Case 3: In the remaining case, we have $B = d + 1$. (In fact, the example we are about to give will work in Cases 1 and 2 as well if $\pi \in K$ can be chosen to have the desired property for $d + 1$ rather than B .) Define

$$\phi(z) := \frac{\pi}{z^d},$$

let $\beta = \gamma \in \bar{K}$ be a $(d + 1)$ -st root of π , and let $L := K(\beta)$, so that $[L : K] = d + 1$. Let $h(z) = \beta^{-1}z$. Then $h(\zeta(0, |\beta|)) = \zeta(0, 1)$, and

$$h \circ \phi \circ h^{-1}(z) = \frac{\pi}{\beta^{d+1} z^d} = \frac{1}{z^d},$$

which has good reduction.

As before, given an extension \tilde{L}/K in \overline{K} and a map $\tilde{h}(z) \in \mathrm{PGL}(2, \overline{K})$ for which $\tilde{h} \circ \phi \circ \tilde{h}^{-1}$ has good reduction, we must have $\tilde{h}(\zeta(0, |\beta|)) = \zeta(0, 1)$. By Theorem 4.1.a, then, $[\tilde{L} : K] \geq d + 1$. \square

5. THEOREM A AND RAMIFICATION

The examples in our proof of Theorem B attain the sharpness of the bound B by forcing the extension L/K to have ramification degree at least B . This observation motivates the following conjecture.

Conjecture 1. *Let K , ϕ , and B be as in Theorem A. Assume that the residue field k of K is perfect. If ϕ has potentially good reduction, then there is a totally ramified extension L/K with $[L : K] \leq B$ such that ϕ is conjugate over L to a map of good reduction.*

Conjecture 1 would be false without the hypothesis that k is perfect. For example, let $K = k((t))$, where $k = \mathbb{F}_p(s)$, and define $\phi(z) = t^{1-p}(z^p - s) + z$. It is easy to check that $h^{-1} \circ \phi \circ h(z) = z^p + z$, where $h(z) = tz + \sqrt[p]{s}$, which moves $\zeta(0, 1)$ to $\zeta(\sqrt[p]{s}, |t|)$. Any other coordinate change \tilde{h} also moving $\zeta(0, 1)$ to $\zeta(\sqrt[p]{s}, |t|)$ would have to be defined over a field L with residue field containing $\sqrt[p]{s}$. Since $[k(\sqrt[p]{s}) : k] = p$, L/K cannot be totally ramified, as the residue field extension degree must be at least p .

A proof of Conjecture 1 could perhaps proceed along the same lines as our proof of Theorem A, provided we could prove totally ramified versions of the four theorems in Section 3.

This is definitely possible for Theorems 3.3 and 3.4. In the proof of each, after the change of coordinates to move the two K -rational points to 0 and ∞ , the point α could be replaced by some β with $|\beta| = |\alpha|$ so that $K(\beta)/K$ is totally ramified and $[K(\beta) : K] \leq [K(\alpha) : K]$. Such $\beta \in \overline{K}$ certainly exists; if $n \geq 1$ is the smallest positive integer such that $|\alpha|^n \in |K|$, then pick $b \in K$ such that $|b| = |\alpha|^n$, and let β be an n -th root of b . By the minimality of n , the extension $K(\beta)/K$ must be totally ramified.

The obstacle, then, is to prove that we can choose the field extensions from Theorems 3.1 and 3.2 to be totally ramified. Thus, Conjecture 1 reduces to proving the following extension of Lemma 2.6.

Conjecture 2. *Let K , f , V , p , n , and e be as in Lemma 2.6. Assume that the residue field k of K is perfect. Then there is a point α in the disk V such that $K(\alpha)/K$ is totally ramified, and $[K(\alpha) : K] \leq p^e$.*

Acknowledgements. The author thanks Keith Conrad, Xander Faber, and the referee for their helpful discussions and suggestions. The

author also gratefully acknowledges the support of NSF grant DMS-1201341.

REFERENCES

- [1] Matthew H. Baker and Robert S. Rumely, *Potential Theory and Dynamics on the Berkovich Projective Line*, Amer. Math. Soc., Providence, 2010.
- [2] Robert L. Benedetto, Non-archimedean dynamics in dimension one: Lecture notes, Arizona Winter School 2010. Available at <http://swc.math.arizona.edu/aws/2010/2010BenedettoNotes-09Mar.pdf>.
- [3] Robert L. Benedetto, A criterion for potentially good reduction in non-archimedean dynamics, preprint. Available at [arXiv:1311.6695](https://arxiv.org/abs/1311.6695).
- [4] Vladimir G. Berkovich, *Spectral Theory and Analytic Geometry over Non-archimedean Fields*, Amer. Math. Soc., Providence, 1990.
- [5] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert, *Non-Archimedean Analysis: A systematic approach to rigid analytic geometry*, Springer-Verlag Berlin, Heidelberg 1984.
- [6] Patrick Morton and Joseph H. Silverman, Rational periodic points of rational functions, *Inter. Math. Res. Notices* **2** (1994), 97–110.
- [7] Juan Rivera-Letelier, Dynamique des fonctions rationnelles sur des corps locaux, *Astérisque* **287** (2003), 147–230.
- [8] Juan Rivera-Letelier, Points périodiques des fonctions rationnelles dans l'espace hyperbolique p -adique, *Comment. Math. Helv.* **80** (2005), 593-629.
- [9] Robert S. Rumely, The minimal resultant locus, preprint. Available at [arXiv:1304.1201](https://arxiv.org/abs/1304.1201).
- [10] Alain M. Robert, *A Course in p -adic Analysis*, Springer-Verlag, New York, 2000.

AMHERST COLLEGE, AMHERST, MA 01002, USA
E-mail address: rlbenedetto@amherst.edu