DISCRETENESS OF POSTCRITICALLY FINITE MAPS IN p-ADIC MODULI SPACE

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Abstract. Let $p \geq 2$ be a prime number and let $\mathbb{C}_p$ be the completion of an algebraic closure of the p-adic rational field $\mathbb{Q}_p$. Let $f_c(z)$ be a one-parameter family of rational functions of degree $d \geq 2$, where the coefficients are meromorphic functions defined at all parameters c in some open disk $D \subseteq \mathbb{C}_p$. Assuming an appropriate stability condition, we prove that the parameters c for which f_c is postcritically finite (PCF) are isolated from one another in the p-adic disk D, except in certain trivial cases. In particular, all PCF parameters of the family $f_c(z) = z^d + c$ are p-adically isolated.

1. Introduction

Let K be a field with algebraic closure $\overline{K}$, and let $f(z) \in K(z)$ be a separable rational function. The degree of f is defined to be the maximum of the degrees of the numerator and denominator of f. Define $f^n(z)$ to be the n-th iterate of f under composition; that is, $f^0(z) = z$, and for each $n \geq 0$, $f^{n+1}(z) = f \circ f^n(z)$. The *forward orbit* of a point $x \in \mathbb{P}^1(\overline{K})$ is the set $\{f^n(x) : n \geq 0\}$; the *strict forward orbit* of x is $\{f^n(x) : n \geq 1\}$. We say x is *periodic* under f if $f^n(x) = x$ for some $n \geq 1$; equivalently, x belongs to its own strict forward orbit. Similarly, we say that x is *preperiodic* under f if there are integers $n > m \geq 0$ such that $f^n(x) = f^m(x)$; equivalently, the forward orbit of x is finite. We say x is *strictly preperiodic* if it is preperiodic but not periodic. We say that f is *postcritically finite*, or PCF, if every critical point $c \in \mathbb{P}^1(\overline{K})$ of f is preperiodic under f.

Let $f_c(z)$ be a one-parameter analytic family of rational functions of degree d. That is, each coefficient of f_c is an analytic function of $c \in D$, where D is some set of parameters; and for each specific parameter $c \in D$, the function $f_c(z) \in K(z)$ is a rational function of degree d. In this paper, we consider the set of parameters $c \in D$ for which f_c is PCF; we call such points *PCF parameters*. In particular, our focus is on the case that $K = \mathbb{C}_p$, the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic rationals, where $p \geq 2$ is a prime number.

There are a number of cases when it is easy to see that there are many PCF parameters. First, suppose that $f_c = h_c^{-1} \circ g \circ h_c$, where $g(z) \in K(z)$ is a (fixed) rational function of degree $d \geq 2$, and where $h_c(z)$ is a one-parameter family of linear fractional transformations. This is the *isotrivial* case, and if $g(z)$ itself is PCF, then so is f_c for every parameter c. Second, suppose that E_c is a one-parameter family of elliptic curves, $m \geq 2$ is an integer, and f_c is the *flexible Lattès map* associated with the multiplication-by-m map $[m] : E_c \to E_c$. (That is, if we write E_c in Weierstrass form, then $f_c \circ x_c = x_c \circ [m]$,

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where $x_c : E_c \to \mathbb{P}^1$ is the x-coordinate map. See Definition 4.1 for a slightly more general definition of flexible Lattès maps.) It is well known, and easy to check, that all Lattès maps are PCF, and hence every c is a PCF parameter for such a family.

As a third, more complicated case, let $K = \mathbb{C}$, and consider the quadratic polynomial family $f_c(z) = z^2 + c \in \mathbb{C}[c, z]$. Since the critical point at $z = \infty$ is fixed, a given map f_c is PCF if and only if the critical point at $z = 0$ is preperiodic. The parameters c for which $z = 0$ is strictly preperiodic are called *Misiurewicz parameters*, and they are dense in the boundary $\partial\mathbf{M}$ of the Mandelbrot set $\mathbf{M}$. (See, for example, [7, Theorem VIII.1.5ff] or [11, Proposition 2.1]. The parameters c for which $z = 0$ is periodic lie in the interior of $\mathbf{M}$ but also accumulate on $\partial\mathbf{M}$.) Hence, the complex quadratic polynomial family features PCF parameters accumulating at many points, including one another. (This phenomenon is not unique to the complex case $K = \mathbb{C}$. Indeed, Rivera-Letelier described similar Misiurewicz bifurcations in the p-adic setting in [14]; see Example 7.3.)

By contrast, our main result, Theorem 1.1 below, states that if $K = \mathbb{C}_p$ and the family f_c is stable in an appropriate sense, then such accumulation of PCF parameters is impossible. The underlying flavor of this statement is similar to that of Scanlon's theorem [15, 16] on the Tate-Voloch conjecture [19], although our methods are different. See Section 2.1 for the notation $D(b, R)$ and $\overline{D}(b, R)$ for open and closed disks in $\mathbb{C}_p$, and Section 2.2 for the field $\mathbb{M}_p(D)$ of p-adic meromorphic functions on a disk $D \subseteq \mathbb{C}_p$; in addition, $\mathbb{P}^1(\mathbb{M}_p(D))$ of course denotes $\mathbb{M}_p(D) \cup \{\infty\}$.

Theorem 1.1. *Let $p \geq 2$ be a prime number, let $S > 0$, let $\Phi(c, z) \in \mathbb{M}_p(D(0, S))(z)$, and suppose that for each $c \in D(0, S)$,*

$$f_c(z) := \Phi(c, z)$$

is a rational function in $\mathbb{C}_p(z)$ of degree $d \geq 2$. Let $\alpha_1, \ldots, \alpha_{2d-2} \in \mathbb{P}^1\big(\mathbb{M}_p(D(0, S))\big)$ be meromorphic functions on $D(0, S)$ such that for each $c \in D(0, S)$, the critical points of f_c, repeated according to multiplicity, are $\alpha_1(c), \ldots, \alpha_{2d-2}(c)$.

For each $i = 1, \ldots, 2d - 2$, suppose that there are integers $N_i > M_i \geq 0$ and rational open disks $U_{i,0}, \ldots, U_{i,N_i} \subseteq \mathbb{P}^1(\mathbb{C}_p)$ such that for all $c \in D(0, S)$,

- $\alpha_i(c) \in U_{i,0}$,
- $f_c(U_{i,j}) \subseteq U_{i,j+1}$ *for all* $j = 0, \ldots, N_i - 1$, *and*
- $U_{i,N_i} \subseteq U_{i,M_i}$.

Then either

(a) *for any $0 < s < S$, there are only finitely many $c \in \overline{D}(0, s)$ such that f_c is postcritically finite,*
(b) *for every $c \in D(0, S)$, f_c is a flexible Lattès map, or*
(c) *for every $b, c \in D(0, S)$, f_b is conjugate to f_c.*

The stability conditions of Theorem 1.1, i.e., the bullet points concerning the disks $U_{i,j}$, happen to be satisfied by certain families of good reduction, which we now define. Recall that a non-archimedean field K_v with absolute value $|\cdot|_v$ has ring of integers $\mathcal{O}_v := \{x \in K_v : |x|_v \leq 1\}$, maximal ideal $\mathcal{M}_v := \{x \in K_v : |x|_v < 1\}$, and residue field $k_v := \mathcal{O}_v/\mathcal{M}_v$. (For $K_v = \mathbb{C}_p$, the residue field is isomorphic to $\overline{\mathbb{F}}_p$, the algebraic closure of the field $\mathbb{F}_p$ of p elements.)

Definition 1.2. Let K_v be a non-archimedean field with absolute value $|\cdot|_v$, ring of integers $\mathcal{O}_v$, and residue field k_v, and let $f(z) \in K_v(z)$ be a rational function of degree

$d \geq 1$. Write $f = g/h$, where $g, h \in \mathcal{O}_v[z]$, with at least one coefficient a of g or h satisfying $|a|_v = 1$. Define $\overline{g}, \overline{h} \in k_v[z]$ by reducing each coefficient of g, h modulo v. We say f has (*explicit*) *good reduction* if its reduction, the rational function $\overline{f} := \overline{g}/\overline{h} \in k_v(z)$, satisfies $\deg(\overline{f}) = d$.

The reduction map from $\mathcal{O}_v \to k_v$ induces a map $\mathbb{P}^1(K_v) \to \mathbb{P}^1(k_v)$, which we also denote $x \mapsto \overline{x}$. If $f \in K_v(z)$ has good reduction, then $\overline{f(x)} = \overline{f}(\overline{x})$ for every $x \in \mathbb{P}^1(K_v)$.

Theorem 1.3. *Let $p \geq 2$ be a prime number, let $S > 0$, let $\Phi(c, z) \in \mathbb{M}_p(D(0, S))(z)$, and let $\alpha_1, \ldots, \alpha_{2d-2} \in \mathbb{P}^1\big(\mathbb{M}_p(D(0, S))\big)$. Suppose that for each $c \in D(0, S)$,*

- *$f_c(z) := \Phi(c, z)$ is a rational function in $\mathbb{C}_p(z)$ of degree $d \geq 2$ and of p-adic explicit good reduction;*
- *the reduction $\overline{f}_c \in \overline{\mathbb{F}}_p(z)$ satisfies $\overline{f}_c = \overline{f}_0$;*
- *the critical points of f_c, repeated according to multiplicity, are $\alpha_1(c), \ldots, \alpha_{2d-2}(c)$; and*
- *for each $i = 1, \ldots, 2d-2$, the reduction $\overline{\alpha_i(c)} \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$ satisfies $\overline{\alpha_i(c)} = \overline{\alpha_i(0)}$.*

Then either

(a) *for any $0 < s < S$, there are only finitely many $c \in \overline{D}(0, s)$ such that f_c is postcritically finite,*
(b) *for every $c \in D(0, S)$, f_c is a flexible Lattès map, or*
(c) *for every $b, c \in D(0, S)$, f_b is conjugate to f_c.*

As a special case of Theorem 1.3, we can show that the unicritical family $z^d + c$ over $\mathbb{C}_p$ has all PCF parameters isolated. Here is a more precise statement.

Theorem 1.4. *Let $p \geq 2$ be a prime number, and let $d \geq 2$ be an integer. Define*

$$f_c(z) := z^d + c.$$

If $c \in \mathbb{C}_p$ is a parameter for which f_c is postcritically finite, then $|c|_p \leq 1$. Moreover, for any $a \in \mathbb{C}_p$ with $|a|_p \leq 1$ and for any radius $0 < s < 1$, there are only finitely many $c \in \overline{D}(a, s)$ such that f_c is postcritically finite.

As we will see in Examples 7.1 and 7.2, Theorem 1.4 cannot be strengthened to conclude that there are only finitely many PCF parameters in an open disk $D(a, 1)$ of radius 1. An application of Theorem 1.4, towards a result on the integrality of PCF parameters in the unicritical family, will appear in [3].

The outline of the paper is as follows. In Section 2, we set notation and recall the essential background we need, especially on non-archimedean analysis. In Sections 3 and 4, we prove technical lemmas needed to avoid degenerations of isotrivial and Lattès families. Section 5 is devoted to the statement and proof of Theorem 5.1, which is our central tool, using certain p-adic dynamical results to study the orbit of a single marked point in a one-parameter p-adic family. In Section 6, we use Theorem 5.1 and Thurston Rigidity (stated here as Theorem 6.1) to prove Theorems 1.1, 1.3, and 1.4. Finally, in Section 7, we present various examples showing that the conclusions of our theorems cannot be significantly strengthened, and that the stability conditions cannot be dropped from their hypotheses.

2. Notation and background

In this section, we recall some basic facts from non-archimedean analysis, following [2, Chapters 2–3]. Throughout the rest of the paper, for ease of notation, we fix a prime number $p \geq 2$, and we denote the absolute value on $\mathbb{C}_p$ simply as $|\cdot|$ rather than $|\cdot|_p$.

2.1. **Disks and power series.** Fix $b \in \mathbb{C}_p$ and $R > 0$. We write

$$D(b,R) := \{x \in \mathbb{C}_p \,:\, |x-b| < R\} \quad \text{and} \quad \overline{D}(b,R) := \{x \in \mathbb{C}_p \,:\, |x-b| \leq R\}$$

for the open and closed disks, respectively, centered at b of radius R. If $R \in |\mathbb{C}_p^\times|$, then we call $D(b,R)$ a *rational open disk*, and $\overline{D}(b,R)$ a *rational closed disk*, which satisfy $D(b,R) \subsetneq \overline{D}(b,R)$. If $R \notin |\mathbb{C}_p^\times|$, then we call $D(b,R) = \overline{D}(b,R)$ an *irrational disk*. In spite of the names, all disks in $\mathbb{C}_p$ are open *and* closed topologically. In addition, for any $b' \in D(b,R)$, we have $D(b',R) = D(b,R)$. Similarly, for any $b' \in \overline{D}(b,R)$, we have $\overline{D}(b',R) = \overline{D}(b,R)$.

We also define a disk D in $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ to be either a disk in $\mathbb{C}_p$ or the complement $\mathbb{P}^1(\mathbb{C}_p) \smallsetminus D'$ of a disk $D' \subseteq \mathbb{C}_p$. In the latter case, if D' is rational closed (resp., rational open, irrational), we say D is rational open (resp., rational closed, irrational).

Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z-b)^n \in \mathbb{C}_p[[z-b]]$$

be a power series with coefficients in $\mathbb{C}_p$. Then f converges on the open disk $D(b,R)$ if and only if the sequence $\{|a_n|r^n\}_{n\geq 0}$ is bounded for every $0 < r < R$; equivalently, if and only if $\lim_{n\to\infty} |a_n|r^n = 0$ for every $0 < r < R$. In that case, the *Weierstrass degree* of f on $D(b,R)$ is the *smallest* integer $m \geq 0$ such that

$$|a_m|R^m = \max\big\{|a_n|R^n \,:\, n \geq 0\big\}, \tag{2.1}$$

or ∞ if the maximum is not attained. If the Weierstrass degree of $f - a_0$ is an integer $m \geq 1$, then f maps $D(b,R)$ onto $D(a_0,S)$, where $S := |a_m|R^m$; moreover, every point of $D(a_0,S)$ has exactly m preimages in $D(b,R)$, counted with multiplicity.

If $R \in |\mathbb{C}_p^\times|$, then f converges on the rational closed disk $\overline{D}(b,R)$ if and only if $\lim_{n\to\infty} |a_n|R^n = 0$. In that case, the Weierstrass degree of f on $\overline{D}(0,R)$ is the *greatest* integer $m \geq 0$ satisfying equation (2.1). As with open disks, f maps $\overline{D}(b,R)$ onto $\overline{D}(a_0,S)$, where $S = |a_m|R^m$ and m is the Weierstrass degree of $f - a_0$ on $\overline{D}(b,R)$; moreover, every point of $\overline{D}(a_0,S)$ has exactly m preimages in $\overline{D}(b,R)$, counted with multiplicity. In particular, a convergent power series f can have only finitely many zeros on a rational closed disk *unless* f is the trivial power series, i.e., is identically zero. Since any open disk is a union of countably many nested rational closed disks, then, a nontrivial convergent power series can have only countably many zeros on an open disk.

We will also occasionally need to consider power series $F(z,w) \in \mathbb{C}_p[[z,w]]$ in two variables. The main fact we will need is the following. Writing such a series as $F(x,y) = \sum_{i,j\geq 0} A_{ij}x^i y^j$, suppose that F converges for all (x,y) in the bidisk $D(0,r) \times D(0,s)$, with image contained in the disk $D(0,t)$, with $r,s,t > 0$. Then we have

$$|A_{ij}|r^i s^j \leq t \quad \text{for all } i,j \geq 0, \qquad \text{with } |A_{00}| < t.$$

2.2. **Meromorphic functions.** If $\phi \in \mathbb{C}_p(z)$ is a rational function of degree $d \geq 1$ with no poles in a given disk $D \subseteq \mathbb{C}_p$ — either open or closed — then ϕ has a power series expansion that converges on D. Moreover, the Weierstrass degree of this power series is at most d. (The existence of such a power series holds for the exact same reasons as over $\mathbb{C}$. One can add, subtract, and multiply convergent power series, and one can compute reciprocals of power series with no zeros using the identity $1/(1-z) = \sum z^n$ for $|z| < 1$.) More generally, we make the following definition.

Definition 2.1. Let $D \subseteq \mathbb{C}_p$ be a disk, either open or closed. A function $f : D \to \mathbb{P}^1(\mathbb{C}_p)$ is a (*rigid*) *meromorphic function* on D if there exist power series g, h converging on D such that $f = g/h$. We denote the set of all meromorphic functions on D by $\mathbb{M}_p(D)$.

Just as for $\mathbb{C}$, the set $\mathbb{M}_p(D)$ is a field under addition and multiplication of functions, and we may speak of zeros and poles of meromorphic functions. We will also occasionally abuse terminology and refer to the constant function ∞ as being meromorphic; that is, we consider every element of the projective line $\mathbb{P}^1(\mathbb{M}_p(D)) = \mathbb{M}_p(D) \cup \{\infty\}$ to be a meromorphic function on D. A meromorphic function on D that is not identically 0 or ∞ has only finitely many zeros and poles in any proper subdisk of D; in particular, it has only countably many zeros and poles in D, all of which are isolated. A meromorphic function with no poles in D is in fact given by a power series converging on D. If two meromorphic functions $f, g \in \mathbb{M}_p(D)$ agree on an uncountable subset of D, then $f = g$.

Many of the above facts (in both Sections 2.1 and 2.2) are consequences of the Weierstrass Preparation Theorem (see, for example, [2, Theorem 14.2]). This result states that a nontrivial power series $f \in \mathbb{C}_p[[z-b]]$ which converges on a rational closed disk $D = \overline{D}(b, R)$ can be factored uniquely as $f = gh$, where $g \in \mathbb{C}_p[z]$ is a monic polynomial g with all of its roots in D, and where $h \in \mathbb{C}_p[[z-b]]$ converges on D with $|h(z)|$ a nonzero constant on D. (Moreover, the degree of g is the Weierstrass degree of f on D.) We also have the following fact.

Lemma 2.2. *Let $f(z)$ be a nontrivial power series converging on $D(0, S)$, let $0 < s < S$, and let $y_1, \ldots, y_m$ be the zeros of f in $\overline{D}(0, s)$, where $0 \leq m < \infty$. Then for any $r > 0$, there exists $\varepsilon > 0$ so that*

$$|f(x)| \geq \varepsilon \quad \textit{for all } x \in \overline{D}(0, s) \smallsetminus \bigcup_{i=1}^{m} D(y_i, r).$$

Proof. Increasing s slightly if necessary, we may assume without loss that $s \in |\mathbb{C}_p^\times|$. The conclusions are then immediate from the Weierstrass Preparation Theorem. □

2.3. **Norm and distortion.** Let $h(z) = \sum_{n \geq 0} a_n z^n \in \mathbb{C}_p[[z]]$ be a power series converging on $D(0, R)$. Then for any radius $0 < r < R$, we define $\|h\|_{\zeta(0,r)}$ to be the sup-norm of h on $\overline{D}(0, r)$. That is,

$$\|h\|_{\zeta(0,r)} := \sup\left\{|h(x)| \,:\, x \in \overline{D}(0, r)\right\} = |a_d| r^d, \tag{2.2}$$

where d is the Weierstrass degree of h on $\overline{D}(0, r)$. (The notation $\|\cdot\|_{\zeta(0,r)}$ is meant to evoke the fact that this norm is a point $\zeta(0, r)$ on the Berkovich disk $D_{\text{an}}(0, R)$ corresponding to the disk $\overline{D}(0, r)$.) If we define a function $L_h : (-\infty, \log R) \to \mathbb{R}$ by

$$L_h(\log r) := \log \|h\|_{\zeta(0,r)},$$

then L_h is continuous and piecewise linear, with nonnegative integer slopes. Specifically, the slope of L_h just to the right of $\log r$ is the Weierstrass degree of h on $\overline{D}(0,r)$. (The graph of L_h is essentially the Newton copolygon of h.)

As in [1, Section 4], we also define a quantity $\delta(h,\zeta(0,r))$, which we call the *distortion* of the power series h on the disk $\overline{D}(0,r)$, by

$$\delta(h,\zeta(0,r)) := \log r + \log \|h'\|_{\zeta(0,r)} - \log \|h\|_{\zeta(0,r)}. \tag{2.3}$$

Then $\log r \mapsto \delta(h,\zeta(0,r))$ is again a continuous, real-valued, piecewise linear function on $(-\infty, \log R)$ with (possibly negative) integer slopes, since both h and h' are power series converging on $D(0,R)$. Indeed, if h' and h have Weierstrass degrees ℓ and m, respectively, on $\overline{D}(0,r)$, then the slope of this function just to the right of $\log r$ is $1+\ell-m$.

3. Isotrivial families

As noted in the introduction, a family f_c of rational maps is said to be *isotrivial* if it is conjugate to a constant map g. That is, $g \in \mathbb{C}_p(z)$ is a rational function of degree d, and $f_c = h_c^{-1} \circ g \circ h_c$, where h_c is a one-parameter family of linear fractional transformations. We will need the following lemma to identify isotrivial families when they arise in proving our main results.

Lemma 3.1. *Let $S>0$, let $g(z) \in \mathbb{C}_p(z)$ be a rational function of degree $d \geq 2$, and let $f_c(z) \in \mathbb{M}_p(D(0,S))(z)$. Let $\beta_1,\beta_2,\beta_3 \in \mathbb{P}^1\big(\mathbb{M}_p(D(0,S))\big)$ be three distinct meromorphic functions on $D(0,S)$, and let $\gamma_1,\gamma_2,\gamma_3$ be three distinct points in $\mathbb{P}^1(\mathbb{C}_p)$. Suppose that for every $c \in D(0,S)$, f_c is a rational function in $\mathbb{C}_p(z)$ of degree d. Suppose also that for some uncountable subset $W \subseteq D(0,S)$ and for every $c \in W$, there is a linear fractional transformation $h_c \in \mathrm{PGL}(2,\mathbb{C}_p)$ such that*

$$f_c = h_c^{-1} \circ g \circ h_c \qquad \text{and} \quad h_c(\beta_i(c)) = \gamma_i \text{ for } i=1,2,3.$$

Then f_c is conjugate to g for all *$c \in D(0,S)$.*

Proof. **Step 1**. After a change of coordinates, we may assume without loss that $\gamma_1 = 0$, $\gamma_2 = \infty$, and $\gamma_3 = 1$. We may similarly assume that none of β_1,β_2,β_3 is identically ∞.

Let Y be the set of parameters $c \in D(0,S)$ for which at least two of $\beta_1(c)$, $\beta_2(c)$, $\beta_3(c)$ have the same value. Define

$$\tilde{h}_c(z) := \frac{\big(z-\beta_1(c)\big)\big(\beta_3(c)-\beta_2(c)\big)}{\big(z-\beta_2(c)\big)\big(\beta_3(c)-\beta_1(c)\big)} \in \mathrm{PGL}\left(2, \mathbb{M}_p(D(0,S))\right), \tag{3.1}$$

and let $\tilde{f}_c := \tilde{h}_c^{-1} \circ g \circ \tilde{h}_c \in \mathbb{M}_p(D(0,S))(z)$. Then for each $c \in D(0,S) \smallsetminus Y$, we have $\tilde{h}_c \in \mathrm{PGL}(2,\mathbb{C}_p)$. Moreover, for all c in the uncountable set $W \subseteq D(0,S) \smallsetminus Y$, we have $\tilde{h}_c = h_c$, and hence $\tilde{f}_c = f_c$ for such c. Since the coefficients of both $\tilde{f}_c$ and f_c are meromorphic functions on $D(0,S)$ that agree for all $c \in W$, we in fact have $\tilde{f}_c = f_c$ as elements of $\mathbb{M}_p(D(0,S))(z)$, and hence $\tilde{f}_c = f_c$ for all $c \in D(0,S)$. Thus, it suffices to show that Y is empty.

Step 2. Suppose, towards a contradiction, that there is some $c_0 \in Y$. Since the meromorphic functions $\beta_i - \beta_j$ are nontrivial for all $i \neq j$, all points of Y are isolated. Translating by c_0 in the c-variable, we may assume that $c_0 = 0$.

Let $\mathbb{L} := \mathbb{C}_p((c))$, the field of formal Laurent series over $\mathbb{C}_p$, which is a complete non-archimedean field with respect to the valuation ord_0, where $\mathrm{ord}_0(\alpha)$ denotes the order of vanishing of $\alpha \in \mathbb{L}$ at $c = 0$. Let $\mathcal{O}_{\mathbb{L}} := \mathbb{C}_p[[c]]$ be the ring of integers in $\mathbb{L}$, and let $\widehat{\mathbb{L}}$ be the completion of an algebraic closure of $\mathbb{L}$. Then $f_c(z)$ is a rational function in $\mathbb{L}(z)$ of degree d, and by hypothesis, $f_0(z)$ is a rational function in $\mathbb{C}_p(z)$ of the same degree d. However, $f_0(z)$ is precisely the reduction of f_c with respect to the valuation ord_0; thus, $f_c \in \mathbb{L}(z)$ has explicit good reduction with respect to ord_0, in the sense of Definition 1.2.

On the other hand, $g(z) \in \mathbb{L}(z)$ also has explicit good reduction with respect to ord_0 (since all its coefficients are constants in $\mathbb{C}_p$), and $f_c, g \in \mathbb{L}(z)$ are conjugate via $\tilde{h}_c \in \mathrm{PGL}(2, \mathbb{L})$. By Propositions 8.12 and 8.13 of [2], then, we must have $\tilde{h}_c \in \mathrm{PGL}(2, \mathcal{O}_{\mathbb{L}})$. (See also [13, Théorème 4]. In the language of Berkovich spaces, the explicit good reduction of f_c is equivalent to saying that the Gauss point $\zeta(0,1)$ in the Berkovich projective line over $\widehat{\mathbb{L}}$ is invariant under f_c, by [2, Proposition 8.12]. However, the fact that $g = \tilde{h}_c \circ f_c \circ \tilde{h}_c^{-1}$ has explicit good reduction says that $\tilde{h}_c^{-1}(\zeta(0,1))$ is also invariant under f_c; but there can only be one such invariant point, by [2, Proposition 8.13]. Thus, $\tilde{h}_c(\zeta(0,1)) = \zeta(0,1)$, which is equivalent to saying $\tilde{h}_c \in \mathrm{PGL}(2, \mathcal{O}_{\mathbb{L}})$, by [2, Proposition 4.10(b)].)

The fact that $\tilde{h}_c \in \mathrm{PGL}(2, \mathcal{O}_{\mathbb{L}})$, and hence that $\tilde{h}_c^{-1} \in \mathrm{PGL}(2, \mathcal{O}_{\mathbb{L}})$, implies that $\beta_1(0) = \tilde{h}_0^{-1}(0)$, $\beta_2(0) = \tilde{h}_0^{-1}(\infty)$, and $\beta_3(0) = \tilde{h}_0^{-1}(1)$ are all distinct. Thus, $0 \notin Y$, as desired. □

Observe that the hypothesis that $\deg(f_c) = d$ for all $c \in D(0, S)$ is essential. For example, consider the family $f_c(z) = cz^2 + z$ for $c \in D(0,1)$. Then for every $c \in D(0,1) \smallsetminus \{0\}$, we have $f_c(z) = c^{-1}g(cz)$, where $g(z) = z^2 + z$. However, $f_0(z) = z$, which is *not* conjugate to $g(z)$, but which also has strictly smaller degree.

4. Lattès maps

There are a number of equivalent definitions of (flexible) Lattès maps in the literature. Our working definition, which is equivalent to the others by [17, Proposition 6.51] and [18, Proposition III.1.7(a)], is as follows.

Definition 4.1. Let $f \in \mathbb{C}_p(z)$ be a rational function of degree $d \geq 2$. We say that f is a *Lattès map* if there exist

- an elliptic curve E defined over $\mathbb{C}_p$,
- a morphism $\psi : E \to E$, and
- a finite, separable morphism $q : E \to \mathbb{P}^1$

such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ {\scriptstyle q}\downarrow & & {\scriptstyle q}\downarrow \\ \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

Furthermore, if we write E in Legendre form $y^2 = x(x-1)(x-\lambda)$ with $\lambda \in \mathbb{C}_p \smallsetminus \{0,1\}$, and if

- ψ is given by $\psi(P) := [m]P + T$ for some integer $m \geq 2$ and some 2-torsion point $T \in E[2]$, and
- $q = h \circ x$ for some linear fractional transformation $h \in \mathrm{PGL}(2, \mathbb{C}_p)$, where $x : E \to \mathbb{P}^1$ is the x-coordinate map,

then we say f is a *flexible* Lattès map.

In particular, the degree d of a flexible Lattès map f is precisely $d = m^2$. For more on such maps, see [10] or [17, Sections 6.4–6.5].

In [10, Corollary 4.8], Milnor proved the following characterization of Lattès maps; he stated it over $\mathbb{C}$, but because $\mathbb{C}$ and $\mathbb{C}_p$ are isomorphic as abstract fields, it also holds in our context. First, any Lattès map is PCF, and its strictly postcritical set consists of either three or four points. (The strictly postcritical set is, of course, the union of the strict forward orbits of the critical points.) Second, a PCF map with exactly four points in its strictly postcritical set is Lattès if and only if all of its critical points are simple — that is, they map to their images with multipliclity 2 — and none are strictly postcritical.

In particular, flexible Lattès maps have this postcritical structure, with exactly four postcritical points. Specifically, the strictly postcritical set is the four-element set $h(x(E[2]))$, where h is the linear fractional transformation of Definition 4.1. Indeed, after changing coordinates by h so that the commutative diagram of the definition becomes simply $x \circ \psi = f \circ x$, the critical points of f are precisely the $2d - 2$ points of $x(E[2m] \smallsetminus E[2])$, that is, the x-coordinates of the $2m$-torsion points that are not 2-torsion. The image of $E[2m]$ under ψ is $E[2]$, and hence the image of $x(E[2m])$ under f is $x(E[2])$.

To prove Theorem 1.1, we will need the following criterion about families of maps having exactly four strictly postcritical points and satisfying Milnor's criterion.

Lemma 4.2. *Let $S > 0$, let $f_c(z) \in \mathbb{M}_p(D(0,S))(z)$, and let $\alpha_1, \ldots, \alpha_{2d-2} \in \mathbb{P}^1\big(\mathbb{M}_p(D(0,S))\big)$. Suppose, for each $c \in D(0,S)$, that f_c is a rational function of degree $d \geq 2$ with critical points $\alpha_1(c), \ldots, \alpha_{2d-2}(c)$, repeated according to multiplicity. Suppose also that there are uncountably many parameters $c \in D(0,S)$ for which the critical points are all distinct, and for which the strictly postcritical set consists of exactly four points, none of which are critical.*

Then either

(a) *for all $c \in D(0,S)$, f_c is is a flexible Lattès map, or*
(b) *for every $b, c \in D(0,S)$, f_b is conjugate to f_c.*

Proof. **Step 1**. Let $W \subseteq D(0,S)$ be the uncountable set of parameters c satisfying the given restrictions on the orbits of the critical points. By Milnor's result in [10, Corollary 4.8], the map f_c is Lattès for every $c \in W$. On the other hand, there are only countably many conjugacy classes of rigid (i.e., non-flexible) Lattès maps in $\mathbb{C}_p(z)$. Indeed, by the results of [17, Section 6.5], any such map arises from an elliptic curve E with CM (of which there are only countably many isomorphism classes), an endomorphism of E (of which there are only countably many), and a finite subgroup of $\mathrm{Aut}(E)$ to quotient by (of which there are only finitely many).

Suppose there are an uncountable subset $W' \subseteq W$ and a rigid Lattès map $g(z)$ such that for each $c \in W'$, there exists $h_c(z) \in \mathrm{PGL}(2, \mathbb{C}_p)$ satisfying $f_c = h_c^{-1} \circ g \circ h_c$. Let $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{P}^1(\mathbb{C}_p)$ be three distinct postcritical points of g. For each $i = 1, 2, 3$, let $\beta_i(c) := h_c^{-1}(\gamma_i)$, each of which must be of the form

$$\beta_i(c) = f_c^{m_i}\big(\alpha_{j_i}(c)\big). \tag{4.1}$$

In particular, $\beta_i \in \mathbb{P}^1\big(\mathbb{M}_p(D(0,S))\big)$ for each $i=1,2,3$. (There may be different possible choices for m_i and j_i for different $c \in W'$, but only finitely many. Thus, shrinking W' if necessary, we may assume equations (4.1) hold for all $c \in W'$ and all $i=1,2,3$, with W' still uncountable.) The hypotheses of Lemma 3.1 therefore hold, and hence f_c is conjugate to g for every $c \in D(0,S)$, yielding conclusion (b).

Step 2. We may assume for the remainder of the proof that f_c is rigid Lattès for only countably many c; thus, shrinking W if necessary, we may assume that f_c is flexible Lattès for all c in the uncountable set W. For all such c, the strictly postcritical set of f_c consists of *four* distinct points of the form $\beta_i(c)$ as in equation (4.1), for $i=1,2,3,4$. After a change of coordinates, we may assume that no β_i is identically ∞.

For each $c \in W$, let $E_c : y^2 = x(x-1)(x-\lambda(c))$ be the associated elliptic curve in Legendre form, T_c the associated 2-torsion point, $m \geq 2$ the associated integer, and $h_c \in \mathrm{PGL}(2,\mathbb{C}_p)$ the associated linear fractional transformation, as in Definition 4.1. Since $h_c \circ x(E_c[2]) = \{\beta_1(c),\beta_2(c),\beta_3(c),\beta_4(c)\}$, and $x(E_c([2])) = \{\infty,0,1,\lambda(c)\}$, we may assume (by reindexing if necessary) that

$$h_c(\lambda(c)) = \beta_1(c),\ h_c(0) = \beta_2(c),\ h_c(1) = \beta_3(c), \text{ and } h_c(\infty) = \beta_4(c) \tag{4.2}$$

for uncountably many such $c \in W$. Shrink the uncountable set W if necessary so that equation (4.2) holds for *all* $c \in W$. For such c, then, we must have

$$h_c^{-1}(z) := \frac{(z-\beta_2(c))(\beta_3(c)-\beta_4(c))}{(z-\beta_4(c))(\beta_3(c)-\beta_2(c))}, \tag{4.3}$$

and hence

$$\lambda(c) = \frac{(\beta_1(c)-\beta_2(c))(\beta_3(c)-\beta_4(c))}{(\beta_1(c)-\beta_4(c))(\beta_3(c)-\beta_2(c))} \in \mathbb{M}_p(D(0,S)). \tag{4.4}$$

Note that λ cannot be identically equal to any of $0,1,\infty$, since E_c is an elliptic curve for all $c \in W$.

Step 3. Let $Y \subseteq D(0,S) \smallsetminus W$ be the set of parameters $c \in D(0,S)$ for which at least two of the values $\beta_1(c),\beta_2(c),\beta_3(c),\beta_4(c)$ coincide. Since the meromorphic functions $\beta_i - \beta_j$ are nontrivial for all $i \neq j$, all points of Y are isolated. We claim that Y is empty.

Suppose, towards a contradiction, that there is some $c_0 \in Y$. Translating in the c-variable, we may assume $c_0 = 0$. Reindexing $\beta_1,\ldots,\beta_4$ again if necessary, with the associated changes to E_c, T_c, and h_c, we may assume that $\lambda(0) \neq 1,\infty$. Thus, $\mathrm{ord}_0(\lambda) \geq 0 = \mathrm{ord}_0(\lambda - 1)$, where ord_0 denotes the order of vanishing at $c=0$. To prove our claim, it suffices to show that $\lambda(0) \neq 0$, i.e., that $\mathrm{ord}_0(\lambda) = 0$.

As in the proof of Lemma 3.1, let $\mathbb{L} := \mathbb{C}_p((c))$ with valuation ord_0, and with ring of integers $\mathcal{O}_{\mathbb{L}} := \mathbb{C}_p[[c]]$. Also as in that proof, the map $f_c \in \mathbb{L}(z)$ has explicit good reduction with respect to ord_0.

If $\mathrm{ord}_0(\lambda) > 0$, then E_0 is a singular curve. Thus, if we consider E_c as an elliptic curve over the discretely valued field $\mathbb{L}$, then E_c has multiplicative reduction. (See, for example, [18, Proposition VII.5.1(b)].) In light of Exercise 10.19(a) and Proposition 8.12 of [2], then, the map f_c does *not* have explicit good reduction. (More precisely, the Julia set of f_c in the Berkovich projective line over $\widehat{\mathbb{L}}$, the completion of an algebraic closure of $\mathbb{L}$, is a line segment, containing infinitely many points. However, by [2, Proposition 8.12], the Julia set of a map of explicit good reduction consists of only one point.) This contradiction to the previous paragraph implies that $\mathrm{ord}_0(\lambda) = 0$, proving our claim.

Step 4. By the claim of Step 3, the points $\beta_1(c), \beta_2(c), \beta_3(c), \beta_4(c)$ are distinct for each $c \in D(0, S)$. Hence, for each $c \in D(0, S)$, the map h_c described by equations (4.2) and (4.3) lies in $\mathrm{PGL}(2, \mathbb{C}_p)$, and $\lambda(c) \neq 0, 1, \infty$, where λ is the meromorphic function of equation (4.4).

In addition, for each $c \in W$, the 2-torsion point T_c of Step 2 must be one of the four 2-torsion points of E_c, namely $(\lambda(c), 0)$, $(0, 0)$, $(1, 0)$, or O, the point at infinity. Shrinking the uncountable set W if necessary, we may assume that T_c is always the same one of these four points, for every $c \in W$. For *every* $c \in D(0, S)$, define $\tilde{T}_c$ to be this torsion point on E_c.

For every $c \in D(0, S)$, define $\tilde{f}_c$ to be the flexible Lattès map associated to the curve $E_c : y^2 = x(x - 1)(x - \lambda(c))$, morphism $\psi : P \mapsto [m]P + \tilde{T}_c$, and linear fractional transformation $h_c \in \mathrm{PGL}(2, \mathbb{C}_p)$. Then $f_c, \tilde{f}_c \in \mathbb{M}_p(D(0, S))(z)$, and $f_c = \tilde{f}_c$ for all $c \in W$. Since W is uncountable, we have $f_c = \tilde{f}_c$ for all $c \in D(0, S)$. □

5. The orbit of a marked point

The heart of our proof of Theorem 1.1 is the following result on the orbit of a single marked point in a family $f_c(z)$.

Theorem 5.1. *Let $S > 0$, let $f_c(z) := \Phi(c, z) \in \mathbb{M}_p(D(0, S))(z)$, and let $\alpha \in \mathbb{P}^1\big(\mathbb{M}_p(D(0, S))\big)$. Suppose that for each $c \in D(0, S)$, $f_c(z)$ is a rational function in $\mathbb{C}_p(z)$ of degree $d \geq 2$. Let $N > M \geq 0$ be integers, and let $U_0, \ldots, U_N \subseteq \mathbb{P}^1(\mathbb{C}_p)$ be rational open disks such that for all $c \in D(0, S)$,*

- *$\alpha(c) \in U_0$,*
- *$f_c(U_j) \subseteq U_{j+1}$ for all $j = 0, \ldots, N - 1$, and*
- *$U_N \subseteq U_M$.*

Then either

(a) *for any $s \in (0, S)$, there are are only finitely many $c \in \overline{D}(0, s)$ such that $\alpha(c)$ and all critical points of f_c in $U_M \cup \cdots \cup U_{N-1}$ are preperiodic under f_c, or*
(b) *there are integers $n > m \geq 0$ such that $f_c^n(\alpha(c)) = f_c^m(\alpha(c))$ for all $c \in D(0, S)$.*

To prove Theorem 5.1, we will need the following lemma about dynamics in a disk-shaped attracting basin. It is a variant of [4, Theorem 4.1].

Lemma 5.2. *Let $R > 0$, let $\gamma \in D(0, R)$, and let*

$$h(z) = \sum_{i \geq \ell} A_i z^i \in \mathbb{C}_p[[z]] \quad \textit{with } \ell \geq 1 \textit{ and } A_\ell \neq 0$$

be a nontrivial power series fixing the point 0, and converging on $D(0, R)$ with Weierstrass degree $d \geq 1$. Suppose that $h(D(0, R)) \subseteq D(0, R)$, and that

$$|A_\ell| R^{\ell - 1} < \min\big\{|m|^m \, : \, 1 \leq m \leq d\big\}. \tag{5.1}$$

Then at least one of the following three conclusions holds:

(a) *$h(\gamma) = 0$.*
(b) *γ has infinite forward orbit under h.*
(c) *h has a critical point in $D(0, R)$ that has infinite forward orbit under h.*

Proof. Let $r > 0$ be the smallest radius such that h has a zero in the punctured disk $\overline{D}(0,r) \smallsetminus \{0\}$, or $r = R$ if h has no zeros in $D(0,R) \smallsetminus \{0\}$. Then

$$|A_i|r^i \le |A_\ell|r^\ell \quad \text{for all } i \ge \ell,$$

as otherwise h would have a root in $D(0,r) \smallsetminus \{0\}$. Therefore,

$$|h(x)| = |A_\ell|\,|x|^\ell \quad \text{for all } x \in D(0,r). \tag{5.2}$$

Combined with condition (5.1), it follows that if $0 < |x| < r$, then $0 < |h(x)| < |x|$. In particular, the only preperiodic point of h in $D(0,r)$ is the fixed point $x = 0$.

If $h(\gamma) \in D(0,r)$, then either conclusion (a) holds, or else $h(\gamma)$ is a nonzero point in $D(0,r)$ and hence is not preperiodic, yielding conclusion (b). Similarly, if h' has any zeros in $D(0,r) \smallsetminus \{0\}$, then h has a wandering critical point, yielding conclusion (c). Hence, we assume for the remainder of the proof that $|h(\gamma)| \ge r$, and that h' has no roots in $D(0,r) \smallsetminus \{0\}$. These two assumptions imply that

$$h\big(\overline{D}(0,|\gamma|)\big) \supseteq \overline{D}(0,r)$$

(because $h(\overline{D}(0,|\gamma|)$ is a disk containing both $h(\gamma)$ and 0), and

$$|h'(x)| = |\ell A_\ell|\,|x|^{\ell-1} \quad \text{for all } x \in D(0,r). \tag{5.3}$$

Since the image of any disk $\overline{D}(0,t)$ is a disk $\overline{D}(0,u)$, where u is a continuous and increasing function of t, there is a unique radius $S \in [r,|\gamma|]$ such that h maps $\overline{D}(0,S)$ onto $\overline{D}(0,r)$. Let m be the Weierstrass degree of h on $\overline{D}(0,S)$. Because 0 has ℓ preimages at 0 and at least one other of absolute value r, we have $m \ge \ell + 1$. Moreover, since $\overline{D}(0,S) \subseteq D(0,R)$, we also have $m \le d$.

Define a function $L : (-\infty, \log R) \to \mathbb{R}$ by

$$\begin{aligned} L(\log t) &= m\log t + m\log\|h'\|_{\zeta(0,t)} - (m-1)\log\|h\|_{\zeta(0,t)} \\ &= m\delta(h,\zeta(0,t)) + \log\|h\|_{\zeta(0,t)}, \end{aligned}$$

where the norm $\|\cdot\|_{\zeta(0,t)}$ and distortion δ are as defined in equations (2.2) and (2.3). Then L is a continuous, piecewise linear function (with all slopes integers). Equations (5.2) and (5.3) together imply that

$$\begin{aligned} L(\log r) &= m\log r + m\log\left(\big|\ell A_\ell\big|r^{\ell-1}\right) - (m-1)\log\left(\big|A_\ell\big|r^\ell\right) \\ &= \log\left(\big|A_\ell\big|r^\ell\right) + m\log|\ell|. \end{aligned}$$

On the other hand, the distortion δ satisfies $\delta(h,\zeta(0,S)) \ge |m|$, for example by [1, Lemma 4.2] or [4, Lemma 3.3]. Since $h(\zeta(0,S)) = \zeta(0,r)$, then

$$\begin{aligned} L(\log S) &= m\delta(h,\zeta(0,S)) + \log\|h\|_{\zeta(0,S)} \ge m\log|m| + \log r \\ &> \log\left(|A_\ell|R^{\ell-1}\right) + \log r \ge \log\left(|A_\ell|r^\ell\right) \\ &\ge \log\left(\big|A_\ell\big|r^\ell\right) + m\log|\ell| = L(\log r), \end{aligned}$$

where the second inequality is by hypothesis (5.1). Thus, there must be some $\rho \in (r, S)$ such that L has positive slope at $\log \rho$. Define

$$\begin{aligned} M &= \text{number of roots of } h' \text{ in } \overline{D}(0,\rho), \text{ counted with multiplicity,} \\ a &= \text{number of roots } y \text{ of } h' \text{ in } \overline{D}(0,\rho) \text{ with } h(y) \neq 0, \text{ counted with multiplicity, and} \\ b &= \text{number of distinct roots of } h \text{ in } \overline{D}(0,\rho). \end{aligned}$$

Then the slope of L at $\log \rho$ is $m(M+1)-(m-1)m$, because $\log \|F\|_{\zeta(0,t)}$ increases with respect to $\log t$ with slope equal to the number of zeros of F in $\overline{D}(0,t)$, and because h has m roots in $\overline{D}(0,\rho)$. In addition, since a root of h of multiplicity $n \geq 2$ is also a root of h' of multiplicity $n-1$, we have $M - a = m - b$. Furthermore, we have $b \geq 2$ by our choice of r, because h has at least two distinct roots in $\overline{D}(0,\rho)$, at 0 and at some x with $|x| = r$. Thus, since the slope of L at $\log \rho$ is positive, we have

$$0 < m(M+1) - (m-1)m = m(m+a-b+1) - m^2 + m = m(a-b+2) \leq ma,$$

and hence $a > 0$. That is, there is some critical point β of h in $\overline{D}(0,\rho)$ with $h(\beta) \neq 0$. Thus, $h(\beta) \in D(0,r) \smallsetminus \{0\}$, whence β is wandering. □

Proof of Theorem 5.1. After a change of coordinates, we may assume that the open disk U_M is $U_M = D(0,R)$ for some $R > 0$. Thus, $f_c^M(\alpha(c)) \in U_M = D(0,R)$ for every $c \in D(0,S)$, and $f_c^{N-M}(z) \in U_N \subseteq D(0,R)$ for every $(c,z) \in D(0,S) \times D(0,R)$.

Define $\Psi : D(0,S) \times D(0,R) \to D(0,R)$ by

$$\Psi_c(z) = \Psi(c,z) := f_c^{N-M}\Big(z + f_c^M\big(\alpha(c)\big)\Big) - f_c^M\big(\alpha(c)\big) \in \mathbb{C}_p[[c,z]].$$

Expanded as a power series, we have

$$\Psi_c(z) = \Psi(c,z) = \sum_{i,j \geq 0} A_{i,j} c^i z^j$$

with $A_{i,j} \in \mathbb{C}_p$ satisfying

$$|A_{i,j}| S^i R^j \leq R \quad \text{for all } i,j \geq 0, \tag{5.4}$$

and with $|A_{0,0}| < R$.

For $i = 0$ and $j = 1$, we have $|A_{0,1}| \leq 1$. We consider two cases: that $|A_{0,1}| < 1$, and that $|A_{0,1}| = 1$.

Case 1: Attracting. Suppose that $|A_{0,1}| < 1$. We will show that there is a power series $\beta(c) \in \mathbb{C}_p[[c]]$ such that $\beta(c)$ is an attracting fixed point of Ψ_c for all $c \in D(0,S)$. (That is, $\Psi_c(\beta(c)) = \beta(c)$, and $|\Psi_c'(\beta(c))| < 1$.) Then, for each $s \in (0,S)$, we will construct a finite set $\mathcal{E}_s \subseteq \overline{D}(0,s)$ such that either

- there exists $n \geq 1$ such that for all $c \in \overline{D}(0,s)$, we have $\Psi_c^n(0) = \beta(c)$, or
- for all $c \in \overline{D}(0,s) \smallsetminus \mathcal{E}_s$, either 0 or a critical point of Ψ_c in $D(0,R)$ has infinite forward orbit under Ψ_c,

from which the theorem will follow.

Step 1. For any radius $s \in [0,S)$, define

$$t_s := \max\left\{ \frac{s}{S}, \frac{|A_{0,0}|}{R}, |A_{0,1}| \right\} < 1. \tag{5.5}$$

Claim 5.2.1. For all $c \in \overline{D}(0,s)$ and all $x, y \in \overline{D}(0, t_s R)$, we have

$$\Psi_c(x) \in \overline{D}(0, t_s R) \quad \text{and} \quad \big|\Psi_c(x) - \Psi_c(y)\big| \le t_s |x-y|.$$

To prove Claim 5.2.1, observe that for any such c, x, y, and for any $i, j \ge 0$, we have

$$\big|A_{i,j} c^i x^j\big| \le \left(\frac{s}{S}\right)^i t_s^j \big(|A_{i,j}| S^i R^j\big) \le t_s^{i+j} R,$$

using inequality (5.4). Thus, $|A_{i,j}c^i x^j| \le t_s R$ if $i + j \ge 1$; moreover, $|A_{0,0}| \le t_s R$ by definition of t_s. The first conclusion of the claim is then immediate. For the second,

$$\begin{aligned} \big|\Psi_c(x) - \Psi_c(y)\big| &= \left| \sum_{i,j\ge 0} A_{i,j} c^i (x^j - y^j) \right| \\ &\le |x-y| \max_{i\ge 0, j\ge 1} \left\{ \big|A_{i,j} c^i (x^{j-1} + x^{j-2} y + \cdots + y^{j-1})\big| \right\} \\ &\le |x-y| \max_{i\ge 0, j \ge 1} \left\{ \left(\frac{s}{S}\right)^i t_s^{j-1} \big(|A_{i,j}| S^i R^{j-1}\big) \right\} \le t_s |x-y|, \end{aligned}$$

where the final inequality is because for $i + j \ge 2$, inequality (5.4) yields

$$\left(\frac{s}{S}\right)^i t_s^{j-1} \big(|A_{i,j}| S^i R^{j-1}\big) \le t_s^{i+j-1} \le t_s,$$

and because $|A_{0,1}| \le t_s$ for $i = 0$ and $j = 1$. Thus, we have proven Claim 5.2.1.

Step 2. To prove the existence of the fixed point $\beta(c)$, for each $n \ge 0$, define

$$g_n(c) \in \mathbb{C}_p[[c]] \quad \text{by} \quad g_n(c) := \Psi_c^n(0).$$

Then each g_n is a convergent power series on $D(0,S)$, with image in $D(0,R)$. Moreover, for each $c \in D(0,S)$, by repeated application of Claim 5.2.1, we see that $\{g_n(c)\}_{n\ge 0}$ is a Cauchy sequence of points in $D(0,R)$, and hence that it converges to some point $\beta(c)$; furthermore, by Claim 5.2.1 again, this convergence is uniform in $c \in \overline{D}(0,s)$. Because each g_n is a power series converging on $D(0,S)$ with image in $D(0,R)$, the limit function $\beta(c)$ is also such a power series. By construction, we have $\Psi_c(\beta(c)) = \beta(c)$, as desired.

By Claim 5.2.1 yet again, for any $s \in (0,S)$ and any $c \in \overline{D}(0,s)$, we have

$$\beta(c) \in \overline{D}(0, t_s R), \quad \text{and} \quad \big|\Psi_c'\big(\beta(c)\big)\big| \le t_s < 1. \tag{5.6}$$

In particular, since its multiplier is $\Psi_c'\big(\beta(c)\big)$, we see that $\beta(c)$ is an *attracting* fixed point of Ψ_c.

Step 3. Changing coordinates to move $\beta(c)$ to the origin, define

$$H_c(z) = H(c,z) := \Psi\big(c, z + \beta(c)\big) - \beta(c) \in \mathbb{C}_p[[c,z]],$$

which converges on $D(0,S) \times D(0,R)$ with image in $D(0,R)$ and with $H(c,0) = 0$. Write

$$H_c(z) = H(c,z) = \sum_{j\ge \ell} \sum_{i \ge 0} B_{i,j} c^i z^j,$$

where $\ell \ge 0$ is the smallest integer such that some coefficient $B_{i,\ell}$ is nonzero. (Such ℓ exists since the original function $f_c(z)$ is a nonconstant rational function of z.) In fact, we have $\ell \ge 1$, since $H(c,0) = 0$. Furthermore,

$$|B_{i,j}| S^i R^j \le R \quad \text{for all } i \ge 0 \text{ and } j \ge \ell, \tag{5.7}$$

since the image of H is contained in $D(0,R)$. Note also that for any $c \in D(0,S)$, the Weierstrass degree of H_c on $D(0,R)$ is at most $\deg(f_c^{N-M}) = d^{N-M}$.

For each radius $s \in (0,S)$, define

$$\mathcal{E}_s := \Big\{ y \in \overline{D}(0,s) \, : \, \sum_{i\geq 0} B_{i,\ell} y^i = 0 \Big\}.$$

Note that $\mathcal{E}_s$ is finite, because it is the set of zeros of a nontrivial power series on a disk $\overline{D}(0,s)$ which is strictly contained in a larger disk $D(0,S)$ on which the series converges. For each $y \in \mathcal{E}_s$, we may choose a radius $\sigma(y) > 0$ such that

$$0 < \Big| \sum_{i\geq 0} B_{i,\ell} c^i \Big| < R^{1-\ell} \min\big\{ |m|^m \, : \, 1 \leq m \leq d^{N-M} \big\} \quad \text{for all } c \in \overline{D}\big(y, \sigma(y)\big) \smallsetminus \{y\}.$$

Hence, for any $c \in \overline{D}(y,\sigma(y)) \smallsetminus \{y\}$, if we set $h(z) = H_c(z)$ and $\gamma := -\beta(c)$, then one of the three conclusions of Lemma 5.2 holds. That is, either

- $H_c(-\beta(c)) = 0$, i.e., $\Psi_c(0) = \beta(c)$, or
- $z = 0$ has infinite forward orbit under Ψ_c, or
- Ψ_c has a critical point in $D(0,R)$ with infinite forward orbit under Ψ_c.

Step 4. It remains to consider parameters c in the set

$$W := \overline{D}(0,s) \smallsetminus \Big(\bigcup_{y\in\mathcal{E}_s} \overline{D}(y,\sigma(y)) \Big).$$

By Lemma 2.2 and the definition of $\mathcal{E}_s$, there is some $\varepsilon > 0$ such

$$C_c \geq \varepsilon \quad \text{for all } c \in W, \quad \text{where } C_c := \Big| \sum_{i\geq 0} B_{i,\ell} c^i \Big|.$$

Let $r := t_s R$, and recall from equation (5.6) that the power series β maps $\overline{D}(0,s)$ into $\overline{D}(0,r) \subsetneq D(0,R)$. Then for all $(c,z) \in \overline{D}(0,s) \times \overline{D}(0,r)$, we have

$$\big| H_c(z) \big| = \big| \Psi_c\big(z + \beta(c)\big) - \Psi_c\big(\beta(c)\big) \big| \leq t_s |z|,$$

by Claim 5.2.1 and the fact that $\Psi_c(\beta(c)) = \beta(c)$. Pick $n \geq 1$ large enough that $t_s^n < \varepsilon r^{\ell-1}$. Then $H_c^n(-\beta(c)) \in D(0, \varepsilon r^\ell)$ for any $c \in W$.

Furthermore, we claim that

$$|H_c(z)| = C_c |z|^\ell \quad \text{for any } (c,z) \in W \times D(0, \varepsilon r^\ell). \tag{5.8}$$

Indeed, for any such (c,z) with $z \neq 0$, and for any $i \geq 0$ and $j \geq \ell+1$, we have

$$\big| B_{i,j} c^i z^j \big| \leq \big| B_{i,j} \big| S^i R^{j-1} \left(\frac{|z|}{R} \right)^{j-1} |z| \leq \left(\frac{|z|}{R} \right)^{\ell} |z| \leq \frac{|z|}{r^\ell} |z|^\ell < \varepsilon |z|^\ell \leq C_c |z|^\ell,$$

using inequality (5.7). Thus, the z^ℓ term in the power series for $H_c(z)$ has strictly larger absolute value than all the other terms, yielding equation (5.8), as claimed.

In addition, we have $0 < C_c|z|^\ell < |z|$ for any such (c,z), because $C_c R^\ell \leq R$, by inequality (5.7) again. That is, by equation (5.8), we have $0 < |H_c(z)| < |z|$. Thus, for any $c \in W$, since we have $H_c^n(-\beta(c)) \in D(0, \varepsilon r^\ell)$, it follows either that $H_c^n(-\beta(c)) = 0$, or else that $-\beta(c)$ has infinite forward orbit under H_c. Hence, either

- $\Psi_c^n(0) = \beta(c)$, or

- $z = 0$ has infinite forward orbit under Ψ_c.

Step 5. For any $s \in (0, S)$, let $\mathcal{E}_s$ be the finite set from Step 3, and let $n \geq 1$ be the integer chosen in Step 4 (so that $t_s^n < \varepsilon r^\ell$). By the bullet lists at the end of Steps 3 and 4, for any $c \in \overline{D}(0, s) \smallsetminus \mathcal{E}_s$, we have either

(a) $\Psi_c^n(0) = \beta(c)$, or
(b) $z = 0$ has infinite forward orbit under Ψ_c, or
(c) Ψ_c has a critical point in $D(0, R)$ with infinite forward orbit under Ψ_c.

If statement (a) occurs for infinitely many parameters $c \in \overline{D}(0, s)$, then the power series $\Psi_c^n(0) - \beta(c) \in \mathbb{C}_p[[c]]$ must be trivial, since it converges on the strictly larger disk $D(0, S)$. Thus, we would have $\Psi_c^n(0) = \Psi_c^{n+1}(0) = \beta(c)$ for all $c \in D(0, S)$, and hence

$$f_c^{(n+1)N-nM}(\alpha(c)) = f_c^{nN-(n-1)M}(\alpha(c)) \quad \text{for all } c \in D(0, S),$$

yielding conclusion (b) of Theorem 5.1.

Otherwise, the set $\mathcal{E}'_s := \{c \in \overline{D}(0, s) : \Psi_c^n(0) = \beta(c)\}$ is finite. For any $c \in \overline{D}(0, s)$ outside the finite set $\mathcal{E}_s \cup \mathcal{E}'_s$, either statement (b) or statement (c) above occurs. For a given such c, statement (b) implies that $\alpha(c)$ has infinite forward orbit under f_c, and statement (c) implies that f_c has a critical point in $U_M \cup \cdots \cup U_{N-1}$ with infinite forward orbit. Hence, conclusion (a) of Theorem 5.1 holds, completing Case 1.

Case 2: Indifferent. Assume for the remainder of the proof that $|A_{0,1}| = 1$. Then for each $c \in D(0, S)$, the power series $\Psi_c(z) \in \mathbb{C}_p[[z]]$ has Weierstrass degree 1 and maps $D(0, R)$ bijectively onto itself.

Step 1. Because the residue field of $\mathbb{C}_p$ is isomorphic to $\overline{\mathbb{F}}_p$, there is an integer $e \geq 1$ such that $|A_{0,1}^e - 1| < 1$. Replacing N by $e(N - M) + M$ in the statement of Theorem 5.1 and in the definition of $\Psi(c, z) = \Psi_c(z)$ at the start of this proof, we may assume that $e = 1$, and hence that $|A_{0,1} - 1| < 1$.

Given a fixed radius $s \in (0, S)$, choose $\tilde{s} \in (s, S)$, and pick a radius r with

$$\max\left\{|A_{0,0}|, \frac{R\tilde{s}}{S}\right\} < r < R.$$

Then $|A_{0,0}| < r$, $|A_{0,1} - 1|r < r$, and for all $j \geq 2$,

$$\big|A_{0,j}\big|r^j = \big|A_{0,j}\big|R^j\left(\frac{r}{R}\right)^j \leq R\left(\frac{r}{R}\right)^2 = \left(\frac{r}{R}\right)r,$$

via inequality (5.4). Pick a real number t with

$$\max\left\{\frac{|A_{0,0}|}{r}, |A_{0,1} - 1|, \frac{r}{R}, \frac{R\tilde{s}}{rS}\right\} < t < 1.$$

Then by the above bounds, we have

$$|A_{0,j}|r^j < tr \quad \text{for all } j \geq 2 \text{ and } j = 0, \quad \text{with } |A_{0,1} - 1|r < tr. \tag{5.9}$$

In addition, for all $i \geq 1$ and all $j \geq 0$, inequality (5.4) yields

$$|A_{i,j}|\tilde{s}^i r^j \leq |A_{i,j}|S^i R^j\left(\frac{\tilde{s}}{S}\right)^i \leq R\left(\frac{\tilde{s}}{S}\right) < tr. \tag{5.10}$$

Combining inequalities (5.9) and (5.10), then, we have

$$\big|\Psi_c(z) - z\big| < tr \quad \text{for all } (c, z) \in \overline{D}(0, \tilde{s}) \times \overline{D}(0, r). \tag{5.11}$$

Step 2. The *iterative logarithm* of Ψ_c, defined in [12, Définition 3.7], is the function

$$\Lambda(c,z) := \lim_{n\to\infty} p^{-n}\big(\Psi_c^{p^n}(z) - z\big). \tag{5.12}$$

According to [12, Lemme 3.11(iii)], the bound (5.11) yields that

$$\big|\Lambda(c,z) - p^{-n}\big(\Psi_c^{p^n}(z) - z\big)\big| \le C_n r|p|^n \quad \text{for all } (c,z)\in \overline{D}(0,\tilde{s})\times\overline{D}(0,r) \text{ and } n\ge 1,$$

where, as shown in the proof of [12, Lemme 3.11(iii)],

$$C_n := \max_{k\ge 1} |k|^{-1}\Big(t|p|^{-1/(p^n(p-1))}\Big)^k. \tag{5.13}$$

Note that for all n large enough, the expression inside the maximum on the right side of equation (5.13) approaches 0 as $k\to\infty$, since $0<t<1$; thus, C_n is defined and finite for all such n. Moreover, the sequence $\{C_n\}$ is decreasing. Hence, the limit of equation (5.12) in fact converges uniformly on $(c,z)\in\overline{D}(0,\tilde{s})\times\overline{D}(0,r)$. Therefore, $\Lambda(c,z)\in\mathbb{C}_p[[c,z]]$ is a power series converging on $\overline{D}(0,\tilde{s})\times\overline{D}(0,r)$.

In addition, by [12, Proposition 3.16], a point $(c,z)\in\overline{D}(0,\tilde{s})\times\overline{D}(0,r)$ is a zero of Λ if and only if z is periodic under Ψ_c. Since $\Psi_c:\overline{D}(0,r)\to\overline{D}(0,r)$ is bijective, this condition is equivalent to saying that z is preperiodic under Ψ_c.

Step 3. Define $F(c) := \Lambda(c,0)\in\mathbb{C}_p[[c]]$, which is a power series converging on $\overline{D}(0,\tilde{s})$, with zeros precisely at the parameters c for which $z=0$ is preperiodic under Ψ_c, i.e., for which $\alpha(c)$ is preperiodic under f_c. If F is not identically zero, then F has only finitely many zeros in the strictly smaller disk $\overline{D}(0,s)$. That is, there are only finitely many $c\in\overline{D}(0,s)$ for which $\alpha(c)$ is preperiodic under f_c, implying conclusion (a) of Theorem 5.1.

Otherwise, we have $F=0$, and hence $\alpha(c)$ is preperiodic under f_c for all $c\in\overline{D}(0,\tilde{s})$. There are only countably many choices of integers $n>m\ge 0$, and hence there must be some such choice of m and n such that uncountably many points $c\in D(0,S)$ are zeros of the power series $f_c^n(\alpha(c)) - f_c^m(\alpha(c))$. A power series with uncountably many zeros must be identically zero, and hence $f_c^n(\alpha(c)) = f_c^m(\alpha(c))$ for all $c\in D(0,S)$, yielding conclusion (b). ☐

6. Proofs of main theorems

Our main results are all consequences of Theorem 5.1, using Thurston's powerful rigidity theorem, which we state below as Theorem 6.1, in the language of critical orbit relations. Fix an integer $d\ge 2$, and consider a separable rational function $f(z)$ of degree d with marked critical points $\alpha_1,\ldots,\alpha_{2d-2}$. A *set of critical orbit relations* for f is a set of equations of the form $f^m(\alpha_i) = f^n(\alpha_j)$ for integers $m,n\ge 0$ and indices $i,j\in\{1,\ldots,2d-2\}$, including at least one equation of the form $f^{n_i}(\alpha_i) = f^{m_i}(\alpha_i)$ for each α_i, with $n_i>m_i\ge 0$. Thus, a set of critical orbit relations specifies a finite forward orbit for each critical point, and it may also include other restrictions, such as that some of the critical points coincide, or that, for example, $f^3(\alpha_1) = f^4(\alpha_2)$.

Theorem 6.1 (Thurston Rigidity)**.** *Fix $d\ge 2$, and let $\mathcal{C}$ be a set of critical orbit relations. Assume that $\mathcal{C}$ is not consistent with a flexible Lattès map. Then, up to conjugacy by linear fractional transformations, there are only finitely many rational functions $f(z)\in\mathbb{C}(z)$ of degree d satisfying the relations of $\mathcal{C}$.*

The original form [20] of Thurston's result is different; see also [8, Theorem 1] for the statement and proof. An explanation of why the original implies Theorem 6.1 appears in [5, Corollary 3.7]. As with Milnor's criterion, the fact that $\mathbb{C}$ and $\mathbb{C}_p$ are isomorphic as abstract fields shows that Theorem 6.1 also holds for $\mathbb{C}_p$ in place of $\mathbb{C}$.

Proof of Theorem 1.1. For each $i = 1, \ldots, 2d-2$, one of the conclusions of Theorem 5.1 holds for $\alpha_i(c)$. If there is some such i for which conclusion (a) of Theorem 5.1 holds, then conclusion (a) of Theorem 1.1 holds, and we are done.

Thus, we may assume for the remainder of the proof that conclusion (b) of Theorem 5.1 holds for every $\alpha_i(c)$. That is, for each $i = 1, \ldots, 2d-2$, there exist integers $n_i > m_i \geq 0$ such that

$$f_c^{n_i}\big(\alpha_i(c)\big) = f_c^{m_i}\big(\alpha_i(c)\big) \tag{6.1}$$

for all $c \in D(0, S)$. In addition, for each $i \neq j$, there may be integers $e_{i,j}, e_{j,i} \geq 0$ so that

$$f_c^{e_{i,j}}\big(\alpha_i(c)\big) = f_c^{e_{j,i}}\big(\alpha_j(c)\big) \tag{6.2}$$

for uncountably many $c \in D(0, S)$. Without loss, for each i, the integers $n_i > m_i \geq 0$ are the smallest for which equation (6.1) holds for uncountably many c. (That is, for any other such $n_i' > m_i' \geq 0$, we have $n_i' \geq n_i$ and $m_i' \geq m_i$.) Similarly, for each relevant $i \neq j$, we may assume that the integers $e_{i,j}, e_{j,i} \geq 0$ are the smallest for which equation (6.2) holds for uncountably many c.

Since the meromorphic functions $f_c^{n_i}(\alpha_i(c)) - f_c^{m_i}(\alpha_i(c))$ and $f_c^{e_{i,j}}(\alpha_i(c)) - f_c^{e_{j,i}}(\alpha_j(c))$ have uncountably many zeros, they are trivial on $D(0, S)$. Thus,

$$\boxed{\text{the relations of equations (6.1) and (6.2) hold for } \textit{all } c \in D(0, S).} \tag{6.3}$$

However, for each i, and for each of the finitely many smaller choices of integers $\ell > k \geq 0$, there could be (at most) countably many parameters $c \in D(0, S)$ for which $f_c^k(\alpha_i(c)) = f_c^\ell(\alpha_i(c))$. Similarly, for each $i \neq j$ and each of the finitely many smaller choices of $k, \ell \geq 0$, there could be (at most) countably many parameters $c \in D(0, S)$ for which $f_c^k(\alpha_i(c)) = f_c^\ell(\alpha_j(c))$.

Suppose that the relations of equations (6.1) and (6.2) are consistent with a flexible Lattès map. That is, for uncountably many parameters $c \in D(0, S)$, all $2d-2$ critical points of f_c are distinct, and the strictly postcritical set consists of four points, none of which are critical. By Lemma 4.2, either conclusion (b) or conclusion (c) of Theorem 1.1 holds, and we are done.

Otherwise, by Thurston Rigidity (Theorem 6.1), there are only finitely many rational functions, up to conjugacy, that satisfy the critical orbit relations (6.1) and (6.2). Thus, there exist some $g(z) \in \mathbb{C}_p(z)$ and an uncountable subset $W \subseteq D(0, S)$ such that f_c is conjugate to g for all $c \in W$. That is, for any $c \in W$, there exists $h_c \in \mathrm{PGL}(2, \mathbb{C}_p)$ such that $f_c = h_c^{-1} \circ g \circ h_c$. We consider two cases.

Case 1. If the forward orbits of the critical points of g together contain at least three distinct points $\gamma_1, \gamma_2, \gamma_3$, then there are corresponding distinct meromorphic functions

$$\beta_i(c) := f_c^{e_i}\big(\alpha_{j_i}(c)\big) \quad \text{for } i = 1, 2, 3,$$

for some indices $j_i \in \{1, \ldots, 2d-2\}$ and some integers $e_i \geq 0$, such that $h_c(\beta_i(c)) = \gamma_i$ for each $c \in W$ and each $i = 1, 2, 3$. Thus, by Lemma 3.1, conclusion (c) of Theorem 1.1 holds; again, we are done.

Case 2. Finally, suppose that the union of the forward orbits of the critical points of g consists of at most two points. Since g must have at least two distinct critical points, this means that it has exactly two, either with each of them fixed or with each of them mapping to the other. Thus, after a change of coordinates, we may assume that $g(z) = z^d$ or that $g(z) = 1/z^d$, with critical points at $z = 0, \infty$.

Since f_c is conjugate to g for uncountably many c, and since equations (6.1) and (6.2) are optimal for such c, then possibly after reindexing, those equations say that

$$\alpha_1(c) = \alpha_2(c) = \cdots = \alpha_{d-1}(c) \quad \text{and} \quad \alpha_d(c) = \alpha_{d+1}(c) = \cdots = \alpha_{2d-2}(c) \tag{6.4}$$

with either

$$f_c\big(\alpha_1(c)\big) = \alpha_1(c) \quad \text{and} \quad f_c\big(\alpha_d(c)\big) = \alpha_d(c) \tag{6.5}$$

if $g(z) = z^d$, or

$$f_c\big(\alpha_1(c)\big) = \alpha_d(c) \quad \text{and} \quad f_c\big(\alpha_d(c)\big) = \alpha_1(c) \tag{6.6}$$

if $g(z) = 1/z^d$. By comment (6.3), equations (6.4) hold for every $c \in D(0, S)$, as do either equations (6.5) if $g(z) = z^d$, or equations (6.6) if $g(z) = 1/z^d$. Moreover, we must have $\alpha_1(c) \neq \alpha_d(c)$ for all $c \in D(0, S)$; indeed, if some parameter c_0 had $\alpha_1(c_0) = \alpha_d(c_0)$, then $\alpha_1(c_0)$ would have more than d preimages under f_{c_0}, counted with multiplicity.

Hence, for any $c \in D(0, S)$, we may change coordinates in the z-variable to move $\alpha_1(c)$ to 0, and to move $\alpha_d(c)$ to ∞. Therefore, if $g(z) = z^d$, equations (6.4) and (6.5) together imply that for every $c \in D(0, S)$, the rational function f_c is conjugate to $z \mapsto z^d$. Similarly, if $g(z) = 1/z^d$, equations (6.4) and (6.6) together imply that for every $c \in D(0, S)$, the rational function f_c is conjugate to $z \mapsto 1/z^d$. Either way, f_c is conjugate to g for every $c \in D(0, S)$, yielding conclusion (c) of Theorem 1.1. □

Proof of Theorem 1.3. For each point $a \in \mathbb{P}^1(\mathbb{C}_p)$, let $W(\overline{a})$ be the inverse image of the point $\overline{a} \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$ under the reduction map $\mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\overline{\mathbb{F}}_p)$. That is, if $|a| \leq 1$, then $W(\overline{a}) = D(a, 1)$, and if $|a| > 1$, then $W(\overline{a}) = \mathbb{P}^1(\mathbb{C}_p) \smallsetminus \overline{D}(0, 1)$.

For each $i = 1, \ldots, 2d-2$ and for every $j \geq 0$, let $U_{i,j} := W\big(\overline{f_0^j(\alpha_i(0))}\big)$. By the fourth bullet point of the hypotheses, we have $\alpha_i(c) \in U_{i,0}$ for every $c \in D(0, S)$. Furthermore, by the first and second bullet points, for every $j \geq 0$ and every $c \in D(0, S)$, we have

$$\begin{aligned} f_c(U_{i,j}) &= f_c\Big(W\Big(\overline{f_0^j(\alpha_i(0))}\Big)\Big) \subseteq W\Big(\overline{f}_c\Big(\overline{f_0^j(\alpha_i(0))}\Big)\Big) \\ &= W\Big(\overline{f}_0\Big(\overline{f_0^j(\alpha_i(0))}\Big)\Big) = W\Big(\overline{f_0^{j+1}(\alpha_i(0))}\Big) = U_{i,j+1}. \end{aligned}$$

In addition, because $\overline{f}_0 \in \overline{\mathbb{F}}_p(z)$ and $\overline{\alpha_i(0)} \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$ must both be defined over some finite subfield of $\overline{\mathbb{F}}_p$, there exist some $N_i > M_i \geq 0$ such that $\overline{f}_0^{N_i}(\overline{\alpha_i(0)}) = \overline{f}_0^{M_i}(\overline{\alpha_i(0)})$. Hence, $U_{i,N_i} = U_{i,M_i}$.

We have verified that the family f_c satisfies all the hypotheses of Theorem 1.1 holds. Therefore, one of the three desired conclusions follows. □

Proof of Theorem 1.4. Counting with multiplicity, the map $f_c(z) = z^d + c$ has $d - 1$ critical points at $z = 0$, and the other $d - 1$ at the (fixed) point at ∞.

For any $c \in \mathbb{C}_p$ with $|c| > 1$, and for any $x \in \mathbb{C}_p$ with $|x| > |c|^{1/d}$, we have $|f_c(x)| = |x|^d > |x|$. Since $|f_c(0)| = |c| > |c|^{1/d}$ for such c, it follows that

$$0 < |f_c(0)| < |f_c^2(0)| < |f_c^3(0)| < \cdots,$$

and hence the critical point at $z = 0$ is not preperiodic. Thus, f_c is not PCF for $|c| > 1$, proving the first claim of Theorem 1.4.

Given any $a \in \mathbb{C}_p$ with $|a| \leq 1$, then for any $c \in D(0,1)$, define $g_c(z) := z^d + a + c$, and also define $\alpha_i(c) := 0$ for each $i = 1, \ldots, d-1$ and $\alpha_i(c) := \infty$ for $i = d, \ldots, 2d-2$. Then for each $c \in D(0,1)$, the map g_c has explicit good reduction, with $\overline{g}_c(z) = z^d + \overline{a} = \overline{g}_0(z)$, and clearly with $\overline{\alpha_i(c)} = \overline{\alpha_i(0)}$. Hence, by Theorem 1.3, either every g_c is flexible Lattès, or the family is isotrivial, or our desired conclusion holds.

However, g_c is not flexible Lattès. Indeed, for $d \geq 3$, there are repeated critical points, violating Milnor's criterion; and for $d = 2$, the degree of g_c is not a square. That is, the second conclusion of Theorem 1.3 cannot hold.

In addition, we claim that there are only finitely many $c \in D(0,1)$ for which g_c is conjugate to g_0. Indeed, if $a = 0$, then no other g_c is conjugate to $g_0(z) = z^d$, since z^d is the only map in the family with all critical points fixed. If $a \neq 0$, then both g_0 and g_c have fixed critical points at $z = \infty$ and non-fixed critical points at $z = 0$. Thus, if $g_c = h^{-1} \circ g_0 \circ h$ for some $h \in \mathrm{PGL}(2, \mathbb{C}_p)$, then we must have $h(0) = 0$ and $h(\infty) = \infty$, so that $h(z) = bz$ for some $b \in \mathbb{C}_p^\times$. Therefore,

$$z^d + a + c = g_c(z) = b^{-1} g_0(bz) = b^{d-1} z^d + b^{-1}(a + c),$$

so that b must be a $(d-1)$-st root of unity. Hence there are at most $d - 1$ choices of c with g_c conjugate to g_0, proving our claim.

Therefore, the family g_c is neither Lattès nor isotrivial. By Theorem 1.3, then, our desired conclusion holds. □

7. Examples

In this section, we present examples to justify our earlier claims that our main theorems cannot be significantly strengthened. In particular, Examples 7.1 and 7.2 illustrate that the conclusions of Theorem 1.4 cannot be significantly strengthened even for the quadratic family $f_c(z) = z^2 + c$. Moreover, Example 7.3 illustrates the necessity of the stability hypotheses of Theorems 1.1 and 1.3.

Example 7.1. Consider the family $f_c(z) = z^2 + c$ of Theorem 1.4, with $d = 2$ and $p = 2$. Then both $c = 0$ and $c = -2$ are PCF parameters of f_c lying in the 2-adic disk $D(0,1)$, and all three of $c = -1$ and $c = \pm i$ are PCF parameters in the 2-adic disk $D(1,1)$. More generally, the roots of the polynomial

$$g_n(c) := f_c^n(0) + f_c^{n-1}(0)$$

are parameters c for which the orbit of the critical point $z = 0$ satisfies $f_c^{n+1}(0) = f_c^n(0)$ but no simpler critical orbit relation. However, it is easy to check that $g_n(c) \equiv c^{2^{n-1}}$ (mod 2), and therefore all roots of g^n lie in the 2-adic disk $D(0,1)$. For any such root c, the map f_c has an attracting fixed point $\beta \in D(0,1)$, and the critical point $z = 0$ lands on β after exactly n iterations. In particular, there are infinitely many PCF parameters in the disk $D(0,1)$.

Example 7.2. Consider the family $f_c(z) = z^2 + c$ of Theorem 1.4, with $d = 2$, and this time with $p = 3$. For any parameter c in the 3-adic disk $D(1,1) \subseteq \mathbb{C}_3$, we have $f_c(0) \equiv 1 \pmod 3$ and $f_c^n(0) \equiv -1 \pmod 3$ for all $n \geq 2$. Changing coordinates via $c = 1 + b$ and $z = -1 + w$, consider the family

$$F_b(w) := f_{1+b}(w-1) + 1 = w^2 - 2w + (b+3) \in \mathcal{O}_3[[b,w]],$$

where $\mathcal{O}_3$ is the ring of integers in $\mathbb{C}_3$.

For each $n \geq 1$, let $I_n := \langle 3, w^{n+1} \rangle + b\langle b, w\rangle^{n-1}$, which is an ideal of the ring $R := \mathcal{O}_3[[b,w]]$. The reduction of F_b modulo the ideal $I_1 := \langle 3, b, w^2 \rangle$ is simply $w \mapsto w$. Inspired by a similar idea in [9, Proposition 4.1], it is a simple exercise to check that if $G, H \in R$ are of the form $G(w) \equiv w + A \bmod I_{n+1}$ and $H(w) \equiv w + B \bmod I_{n+1}$ with $A, B \in I_n$, then $G \circ H(w) \equiv w + A + B \bmod I_{n+1}$. Therefore, $G^3(w) \equiv w \bmod I_{n+1}$. By induction, it follows that $F_b^{3^n}(w) \equiv w \bmod I_{n+1}$ for every $n \geq 0$, and hence that $F_b^{3^n}(-b) \equiv -b \bmod \langle 3, b^{n+1} \rangle$. On the other hand, viewed as a polynomial in the two variables b, w, the highest total-degree term of $F_b^{3^n}(w)$ is w^{D_n}, where $D_n = 2^{3^n}$. Thus,

$$h_n(b) := F_b^{3^n}(-b) + b \in \mathcal{O}_3[b]$$

is a monic polynomial of degree D_n whose reduction modulo 3 has a zero of order at least $n+2$ at $\bar{b} = 0$. In particular, h_n has at least $n+2$ roots in the disk $D(0,1) \subseteq \mathbb{C}_3$.

The critical points of $F_b(w)$ are $w = 1, \infty$, and we have $F_b(1) = b + 2$, and

$$F_b^2(1) = F_b(b+2) = F_b(-b) = b^2 + 3b + 3.$$

Thus, every root b of h_n satisfies $F_b^{3^n}(b+2) = -b$, and hence $F_b^{2+3^n}(1) = F_b^2(1)$. That is, every such b is a PCF parameter of the family F_b. Since h_n has at least $n+2$ roots in $D(0,1)$, and since these roots are distinct by [6, Lemma 4], it follows that the family F_b has infinitely many PCF parameters $b \in D(0,1)$. (More precisely, Buff's result says that the polynomial $f_c^{1+3^n}(0) + f_c(0) \in \mathbb{Z}[c]$ has no repeated roots; after our change of coordinates, the same is true of h_n.) Therefore, the original family f_c has infinitely many PCF parameters $c \in D(1,1)$. For any such c, the map f_c fixes the disk $D(-1,1)$, which contains periodic points of period 3^n for every $n \geq 0$; and the critical point $z = 0$ lands on one such periodic point after 2 iterations.

Our final example illustrates that the stability conditions of Theorems 1.1 and 1.3 cannot be removed in general.

Example 7.3. Fix a prime number $p \geq 2$, and let $d = p + 1$. Define

$$f_c(z) := \left(c + \frac{p+1}{p}\right)\big(pz^{p+1} - (p+1)z^p + 1\big),$$

which has critical points at $z = 0, 1, \infty$. Writing $\gamma(c) := c + (p+1)/p$, we have

$$\infty \overset{p+1}{\longmapsto} \infty, \qquad 1 \overset{2}{\longmapsto} 0 \overset{p}{\longmapsto} \gamma(c), \qquad \frac{p+1}{p} \longmapsto \gamma(c),$$

where the numbers above the arrows indicate ramification indices. Thus, f_c is PCF if and only if $\gamma(c)$ is preperiodic. In particular, $\gamma(0) = (p+1)/p$ is a fixed point of f_0, which in turn is therefore PCF. In addition, we have $f_c'(z) = p(p+1)\gamma(c)z^{p-1}(z-1)$, and hence

$$f_0'\big(\gamma(0)\big) = p(p+1)\gamma(0) \cdot \gamma(0)^{p-1} \cdot \frac{1}{p} = \frac{(p+1)^{p+1}}{p^p},$$

which has p-adic absolute value greater than 1. That is, $\gamma(0)$ is a repelling fixed point of f_0. Since there are other choices of c for which $\gamma(c)$ is *not* a fixed point of f_c, then in the terminology of [14, Section 5], the family of rational maps f_c with marked critical point $z = 1$ has a Misiurewicz bifurcation at $c = 0$.

By [14, Key Lemma], then, there is an infinite sequence of distinct parameters $c_n \in \mathbb{C}_p$ with $c_n \to 0$ such that $z = 1$ is preperiodic under f_{c_n}, and hence such that f_{c_n} is PCF, for every n. The intuitive reason such a sequence exists is because for nonzero parameters c very close to 0, the point $z = \gamma(c)$ will be very close to a repelling fixed point $\beta(c)$. After many iterations, $f_c^m(\gamma(c))$ will be pushed away from $\beta(c)$, and by a careful choice of such c, we can get this orbit to land wherever we like, for example on some preimage of $\beta(c)$, yielding a new PCF parameter.

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