PREPERIODIC POINTS OF POLYNOMIALS OVER GLOBAL FIELDS

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Abstract. Given a global field K and a polynomial ϕ defined over K of degree at least two, Morton and Silverman conjectured in 1994 that the number of K-rational preperiodic points of ϕ is bounded in terms of only the degree of K and the degree of ϕ. In 1997, for quadratic polynomials over $K = \mathbb{Q}$, Call and Goldstine proved a bound which was exponential in s, the number of primes of bad reduction of ϕ. By careful analysis of the filled Julia sets at each prime, we present an improved bound on the order of $s \log s$. Our bound applies to polynomials of any degree (at least two) over any global field K.

Let K be a field, and let $\phi \in K(z)$ be a rational function. Let ϕ^n denote the n^{th} iterate of ϕ under composition; that is, ϕ^0 is the identity function, and for $n \geq 1$, $\phi^n = \phi \circ \phi^{n-1}$. We will study the dynamics ϕ on the projective line $\mathbb{P}^1(K)$. In particular, we say a point x is *preperiodic* under ϕ if there are integers $n > m \geq 0$ such that $\phi^m(x) = \phi^n(x)$. The point $y = \phi^m(x)$ satisfies $\phi^{n-m}(y) = y$ and is said to be *periodic* (of period $n - m$). Note that $x \in \mathbb{P}^1(K)$ is preperiodic if and only if its orbit $\{\phi^n(x) : n \geq 0\}$ is finite.

For example, let $K = \mathbb{Q}$ and $\phi(z) = z^2 - 29/16$. Then $\{5/4, -1/4, -7/4\}$ forms a periodic cycle (of period 3), and $-5/4$, $1/4$, $7/4$, and $\pm 3/4$ each land on this cycle after one or two iterations. In addition, the point ∞ is of course fixed. These nine $\mathbb{Q}$-rational points are all preperiodic under ϕ. Meanwhile, it is not difficult to see that no other point in $\mathbb{P}^1(\mathbb{Q})$ is preperiodic by showing that the denominator of a rational preperiodic point must be 4, and that the absolute value must be less than 2.

In general, for any global field K, any dimension $N \geq 1$, and any morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ over K of degree at least two, Northcott proved in 1950 that the number of K-rational preperiodic points of ϕ is finite [25]. More precisely, he showed that the preperiodic points form a set of bounded arithmetic height. Years later, by analogy with the Theorems of Mazur [19] and Merel [20] on K-rational torsion of elliptic curves, Morton and Silverman proposed the following Conjecture [23].

Uniform Boundedness Conjecture. (Morton and Silverman, 1994)
Given integers $D, N \geq 1$ and $d \geq 2$, there is a constant $\kappa = \kappa(D, N, d)$ with the following property. Let K be a number field with $[K : \mathbb{Q}] = D$, and let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree d defined over K. Then ϕ has at most κ preperiodic points in $\mathbb{P}^N(K)$.

The analogy between preperiodic points and torsion comes from the fact that the torsion points of an elliptic curve E are precisely the preperiodic points of the multiplication-by-two map $[2] : E \to E$. In fact, taking x-coordinates, the map $[2]$ induces a rational function (known to dynamicists as a Lattès map) $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 whose preperiodic points

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are precisely the x-coordinates of the torsion points of E. Thus, Merel's Theorem would follow as a simple corollary of the Morton and Silverman Conjecture for $N = 1$ and $d = 4$. More generally, Fakhruddin has shown [12] that the full Morton and Silverman Conjecture for $D = 1$ would imply uniform boundedness of torsion for abelian varieties.

The Conjecture seems to be very far from a proof. However, there is growing evidence that it is valid, at least in the simplest case, that $K = \mathbb{Q}$, $N = 1$, and ϕ is a polynomial of degree 2. (The problem then reduces to considering $\phi_c(z) = z^2 + c$, with $c \in \mathbb{Q}$.) In particular, the computations in [22] and [13] show that ϕ_c never has a rational point of period 4 or 5, respectively. Moreover, Poonen showed in 1998 that if ϕ_c never has rational periodic points of period greater than 5, then it never has more than 9 rational preperiodic points [28]. (That bound, if true, would be sharp, in light of the $c = -29/16$ example above.) Those results all considered moduli spaces, for fixed $n > m \geq 0$, of pairs (c, x) such that $\phi_c^n(x) = \phi_c^m(x)$, giving curves analogous to modular curves, but with no known structure to take the place of a Hecke ring. Instead, the theorems were proven by delicate *ad hoc* computations on the particular curves that arose.

Other researchers have found bounds for the longest possible period of a K-rational periodic point by analyzing at a prime of "good reduction" (see Definition 2.1 below); see, for example, [11, 23, 24, 26, 27, 35]. If s is the total number of primes of "bad reduction," then these results lead to bounds on the order of at least $d^{s^{4D}}$ for the number of K-rational periodic points (cf. Corollary B of [23], for example).

A different strategy (for the family $\phi_c(z) = z^2+c$ for $c \in \mathbb{Q}$) appeared in a 1997 theorem of Call and Goldstine [9], who showed that ϕ_c has at most $1+2^{s+2}$ rational preperiodic points, where s is the number of primes of bad reduction. They analyzed the dynamics at the primes v of bad, not good, reduction, by studying the filled Julia set $\mathcal{K}_v$ (see Definition 2.2 below). All preperiodic points in $\mathbb{Q}$ sit inside $\mathcal{K}_v$, which in turn lies in a union of two v-adic disks, each of volume 1. (A slightly different condition holds at $v = 2, \infty$.) For good v, a single such disk suffices. The bound of $O(2^s)$ then follows naturally.

In this paper, we will still work only with polynomials and only in dimension 1, but the degree $d \geq 2$ and global field K of definition will be arbitrary. Like Call and Goldstine, we will study dynamics over the associated complete valued fields $\mathbb{C}_v$. We refer the reader to [4, 10, 21] for expositions on complex dynamics (where v is archimedean and $\mathbb{C}_v \cong \mathbb{C}$), and to [5, 6, 8, 17, 30, 31, 33] for papers exploring various aspects of the newer realm of p-adic and non-archimedean dynamics. By a detailed analysis of the filled Julia sets $\mathcal{K}_v \subseteq \mathbb{C}_v$, we will obtain the following substantial improvement over the results of [9].

Main Theorem. *Let K be a global field, let $\phi \in K[z]$ be a polynomial of degree $d \geq 2$, and let s be the number of bad primes of ϕ in K. Then the number of preperiodic points of ϕ in $\mathbb{P}^1(K)$ is at most $O(s \log s)$.*

A more precise statement appears in Theorem 7.1; the big-O constant is essentially $(d^2 - 2d + 2)/\log d$. The idea of the proof is to consider, for each $v \in M_K$, the product $P_v = \prod_{i \neq j} |x_i - x_j|_v$, where $\{x_1, \ldots, x_N\}$ are finite K-rational preperiodic points of ϕ. (The product P_v is related to transfinite diameters and capacities, as discussed in Section 4.) If r_v is the diameter of the filled Julia set $\mathcal{K}_v$, then $P_v \leq r_v^{N(N-1)}$ naively; but in Lemmas 3.4.a and 4.1, we obtain $P_v \leq r_v^{(d-1)N \log_d N}$, with some correction factors for v archimedean.

The key, however, is our treatment of the prime w with filled Julia set $\mathcal{K}_w$ of the largest diameter. We partition $\mathcal{K}_w$ into two pieces, and we show in Lemmas 3.4.b, 5.1, and 6.3 that the corresponding product P_w on each piece satisfies $P_w \le r_w^{(d-1)N(\log_d N - AN + B)}$ for certain simple constants A and B. The product P of all the P_v's, restricted to preperiodic points in the given piece of $\mathcal{K}_w$, is then bounded by $r_w^{(d-1)NE}$, where $E = s \log_d N - AN + B$. For N slightly larger than $(1/A)s \log_d s$ (see Lemma 3.5), we get $E < 0$, so that $P < 1$, which contradicts the product formula for the global field K. Thus, we get a bound of about $(1/A)s \log_d s$ on each piece; summing the two bounds gives the Theorem.

Of course, the details are complicated. In Sections 1 and 2, we will set terminology and recall fundamental facts concerning local and global fields, bad primes, and filled Julia sets. In Section 3, we will introduce notation for certain expressions that will arise later, and we will bound these expressions in a series of technical but completely elementary Lemmas. In Section 4, we will discuss transfinite diameters and prove our first nontrivial bound for P_v, for general bad primes. In Sections 5 and 6, we will describe the partition of the filled Julia set at a bad prime. Finally, in Section 7, we will state Theorem 7.1 and combine all the results from the preceding sections to prove it.

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1. Global Fields and Local Fields

In this section we present the necessary fundamentals from the theory of local and global fields. We also set some notational conventions for this paper. Although this material is well known to number theorists, we present it for the convenience of dynamicists. See Section B.1 of [16] or Section 4.4 of [29] for more details on global fields and sets of absolute values; see [14, 18] for expositions concerning the local fields $\mathbb{C}_v$.

1.1. **Global fields and absolute values.** Throughout this paper, K will denote a global field. That is, K is either a number field (i.e., a finite extension of $\mathbb{Q}$) or a function field over a finite field (i.e., a finite extension of $\mathbb{F}_p(T)$ for some prime p). We will write M_K for the set of standard absolute values on K. That is, M_K consists of functions $|\cdot|_v : K \to \mathbb{R}$ satisfying $|x|_v \ge 0$ (with equality if and only if $x = 0$), $|xy|_v = |x|_v|y|_v$, and $|x + y|_v \le |x|_v + |y|_v$, for all $x, y \in K$. (We will frequently abuse notation and write $v \in M_K$ when our meaning is clear.) Moreover, the absolute values in M_K are chosen to satisfy a *product formula*, which is to say that for each $v \in M_K$, there is an integer $n_v \ge 1$ such that for all $x \in K^\times$,

$$\prod_{v \in M_K} |x|_v^{n_v} = 1. \tag{1}$$

Implicit in the product formula is the fact that for any $x \in K^\times$, we have $|x|_v = 1$ for all but finitely many $v \in M_K$.

All but finitely many $v \in M_K$ satisfy the ultrametric triangle inequality

$$|x + y|_v \le \max\{|x|_v, |y|_v\}.$$

(Note that $|n|_v \le 1$ for all $n \in \mathbb{Z}$.) Such v are called *non-archimedean* absolute values; the finitely many exceptions are called *archimedean* absolute values. The non-archimedean absolute values in M_K correspond to prime ideals of the ring of integers of K. Hence, we often refer to the absolute values $v \in M_K$ as primes of K, even when v is archimedean.

If K is a function field, then all absolute values are non-archimedean. If K is a number field, then there are archimedean absolute values, each of which, when restricted to $\mathbb{Q}$, is the familiar absolute value $|\cdot|$, commonly denoted $|\cdot|_\infty$; we write $v|\infty$, and we have

$$\sum_{v\in M_K,\, v|\infty} n_v = [K:\mathbb{Q}]. \tag{2}$$

If v is non-archimedean, then $|K^\times|_v$ is a discrete subset of $\mathbb{R}$, and we say that v is a *discrete valuation* on K. In that case, choose $\pi_v \in K$ such that $|\pi_v|_v \in (0,1)$ is the largest absolute value less than 1 attained in $|K^\times|_v$. Then π_v is called a *uniformizer* of K at v, and we have $|K^\times|_v = \{|\pi_v|_v^m : m \in \mathbb{Z}\}$. Moreover, if K is a number field, then $|\pi_v|_v^{-n_v} = p^f$ for some prime number $p \in \mathbb{Z}$ and some positive integer f, and $|\cdot|_v$ restricted to $\mathbb{Q}$ is the usual p-adic absolute value on $\mathbb{Q}$. In this case, we say that v *lies above* p.

1.2. **Local fields.** For each $v \in M_K$, we can form the completion K_v (often called the local field at v) of K with respect to $|\cdot|_v$. We write $\mathbb{C}_v$ for the completion of an algebraic closure $\overline{K}_v$ of K_v. (The absolute value v extends in a unique way to $\overline{K}_v$ and hence to $\mathbb{C}_v$.) The field $\mathbb{C}_v$ is then a complete and algebraically closed field. If v is archimedean, then K_v is isomorphic either to $\mathbb{R}$ (in which case we call v a real prime) or to $\mathbb{C}$ (in which case we call v a complex prime), and $\mathbb{C}_v \cong \mathbb{C}$. We will henceforth avoid the notation K_v, as we will soon introduce the notation $\mathcal{K}_v$ to denote a completely different object in Section 2.

If v is non-archimedean, then $\mathbb{C}_v$ is not locally compact, but it has other convenient properties not shared by $\mathbb{C}$. In particular, the disk $\mathcal{O}_v = \{c \in \mathbb{C}_v : |c|_v \le 1\}$ forms a ring, called the *ring of integers*, which has a unique maximal ideal $\mathcal{M}_v = \{c \in \mathbb{C}_v : |c|_v < 1\}$. The quotient $k_v = \mathcal{O}_v/\mathcal{M}_v$ is called the *residue field* of $\mathbb{C}_v$. The natural reduction map from $\mathcal{O}_v$ to k_v, sending $a \in \mathcal{O}$ to $\overline{a} = a + \mathcal{M}_v \in k_v$, will be used to define good and bad reduction of a polynomial in Definition 2.1 below; but after proving a few simple Lemmas about good and bad reduction, we will not need to refer to $\mathcal{O}_v$, $\mathcal{M}_v$, or k_v again.

1.3. **Disks.** Let $\mathbb{C}_v$ be a complete and algebraically closed field with absolute value v. Given $a \in \mathbb{C}_v$ and $r > 0$, we write

$$\overline{D}(a,r) = \{x \in \mathbb{C}_v : |x-a|_v \le r\} \quad \text{and} \quad D(a,r) = \{x \in \mathbb{C}_v : |x-a|_v < r\}$$

for the closed and open disks, respectively, of radius r centered at a. Note our convention that all disks have positive radius.

If v is non-archimedean and $U \subseteq \mathbb{C}_v$ is a disk, then the radius of U is unique; it is the same as the diameter of the set U viewed as a metric space. However, any point $b \in U$ is a center. That is, if $|b-a|_v \le r$, then $\overline{D}(a,r) = \overline{D}(b,r)$, and similarly for open disks. It follows that two disks intersect if and only if one contains the other.

Still assuming that v is non-archimedean, the set $|\mathbb{C}_v^\times|_v$ of absolute values actually attained by elements of $\mathbb{C}_v^\times$ is usually not all of $(0,\infty)$. As a result, if $r \in (0,\infty) \setminus |\mathbb{C}_v^\times|_v$, then $D(a,r) = \overline{D}(a,r)$ for any $a \in \mathbb{C}_v$. However, if $r \in |\mathbb{C}_v^\times|_v$, then $D(a,r) \subsetneq \overline{D}(a,r)$.

2. Bad Reduction and Filled Julia Sets

The following definition originally appeared in [23]. We have modified it slightly so that "bad reduction" now means not potentially good, as opposed to not good.

Definition 2.1. *Let $\mathbb{C}_v$ be a complete, algebraically closed non-archimedean field with absolute value $|\cdot|_v$, ring of integers $\mathcal{O}_v = \{c \in \mathbb{C}_v : |c|_v \le 1\}$, and residue field k_v. Let $\phi(z) \in \mathbb{C}_v(z)$ be a rational function with homogenous presentation*

$$\phi([x,y]) = [f(x,y), g(x,y)],$$

where $f, g \in \mathcal{O}_v[x,y]$ are relatively prime homogeneous polynomials of degree $d = \deg \phi$, and at least one coefficient of f or g has absolute value 1. We say that ϕ has good reduction *at v if $\overline{f}$ and $\overline{g}$ have no common zeros in $k_v \times k_v$ besides $(x,y) = (0,0)$. We say that ϕ has* potentially good reduction *at v if there is some linear fractional transformation $h \in \mathrm{PGL}(2, \mathbb{C}_v)$ such that $h^{-1} \circ \phi \circ h$ has good reduction. If ϕ does not have potentially good reduction, we say it has* bad reduction *at v.*

Naturally, for $f(x,y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$, the reduction $\overline{f}(x,y)$ in Definition 2.1 means $\sum_{i=0}^{d} \overline{a}_i x^i y^{d-i}$. By convention, if $\mathbb{C}_v \cong \mathbb{C}$ is archimedean, we declare all rational functions in $\mathbb{C}_v(z)$ to have bad reduction.

In this paper, we will consider only polynomial functions ϕ of degree at least 2; that is, $\phi(z) = a_d z^d + \cdots + a_0$, where $d \ge 2$, $a_i \in \mathbb{C}_v$, and $a_d \ne 0$. If $\mathbb{C}_v$ is non-archimedean, then, it is easy to check that ϕ has good reduction if and only if $|a_i|_v \le 1$ for all i and $|a_d|_v = 1$. In particular, by the product formula, if $\phi \in K[z]$ for a global field K, then there can be only finitely many primes $v \in M_K$ at which ϕ has bad reduction.

The main focus of our investigation will be filled Julia sets. The motivating idea is that for a polynomial ϕ, all of the interesting dynamics involves points that do not escape to the attracting fixed point at ∞ under iteration. We rephrase the standard definition from complex dynamics more generally to include the non-archimedean setting, as follows.

Definition 2.2. *Let $\mathbb{C}_v$ be a complete, algebraically closed field with absolute value $|\cdot|_v$, and let $\phi(z) \in \mathbb{C}_v[z]$ be a polynomial of degree $d \ge 2$. The* filled Julia set *of ϕ at v is*

$$\mathcal{K}_v = \{x \in \mathbb{C}_v : \{|\phi^n(x)|_v\}_{n\ge1} \text{ is bounded}\}.$$

We note four fundamental properties of filled Julia sets. First, $\mathcal{K}_v$ is invariant under ϕ; that is, $\phi^{-1}(\mathcal{K}_v) = \phi(\mathcal{K}_v) = \mathcal{K}_v$. Second, all the finite preperiodic points of ϕ (that is, all the preperiodic points in $\mathbb{P}^1(\mathbb{C}_v)$ other than the fixed point at ∞) are contained in $\mathcal{K}_v$. Third, if U_0 is a disk containing $\mathcal{K}_v$, then $\mathcal{K}_v = \bigcap_{n\ge0} \phi^{-n}(U_0)$. Finally, if the polynomial $\phi \in \mathbb{C}_v[z]$ has good reduction, then $\mathcal{K}_v = \overline{D}(0,1)$.

Filled Julia sets have been studied extensively in the archimedean case $\mathbb{C}_v = \mathbb{C}$. If $\phi_d(z) = z^d$, then the (complex) filled Julia $\mathcal{K}$ of ϕ_d is simply the closed unit disk $\overline{D}(0,1)$. Meanwhile, since the degree d Chebyshev polynomial ψ_d satisfies $\psi_d \circ h = h \circ \phi_d$, where $h(z) = z + 1/z$, it follows that the complex filled Julia set of ψ_d is the interval $[-2,2]$ in the real line. These two examples are misleadingly simple, however; most filled Julia sets are complicated fractal sets. For example, for $|c| > 2$, the filled Julia set of $\phi(z) = z^2 + c$ is homeomorphic to the Cantor set. For more complex examples (sometimes of the Julia set, which is the boundary of the filled Julia set), see [4, 10, 21].

On the other hand, while complex filled Julia sets are always compact, their non-archimedean counterparts are not usually compact. Fortunately, this technicality will not be an obstacle for our investigations. For the convenience of the reader, we present a few examples of non-archimedean filled Julia sets here. More examples may be found in [6, 30].

Example 2.3. Given $\mathbb{C}_v$ non-archimedean and $d \geq 2$ with $|d-1|_v = 1$, fix $c \in \mathbb{C}_v$, and consider $\phi(z) = z^d - c^{d-1}z$. If $|c|_v \leq 1$, then ϕ has good reduction, and hence $\mathcal{K}_v = \overline{D}(0,1)$. Thus, we consider $|c|_v > 1$; let $r = |c|_v$ and $U_0 = \overline{D}(0,r)$. Note that for $|x|_v > r$, we have $|\phi(x)|_v = |x|_v^d$, so that $\phi^n(x) \to \infty$. That is, $\mathcal{K}_v \subseteq U_0$.

The set $\phi^{-1}(0)$ consists of d points, all distance r from one another. Using standard non-archimedean mapping properties (see, for example, Section 2 of [7]), it is not hard to show that $\phi^{-1}(U_0)$ consists of d disks of radius r^{2-d}, each centered at one of the points of $\phi^{-1}(0)$. Moreover, each of these smaller disks maps one-to-one and onto U_0, and in fact ϕ multiplies distances by a factor of $r^{d-1} = |\phi'(0)|_v$ on each smaller disk. It follows that $U_n = \phi^{-n}(U_0)$ is a union of d^n disks, each of radius $r^{1-(d-1)n}$. (The sets U_n are nested so that each disk of U_n contains exactly d disks of U_{n+1}, arranged so that any two are the maximal distance $r^{1-(d-1)n}$ apart.) It is then easy to verify that $\mathcal{K}_v = \bigcap U_n$ is homeomorphic to a Cantor set on d intervals (which, incidentally, is homeomorphic to the standard Cantor set).

Example 2.4. Given $\mathbb{C}_v$ non-archimedean and $d \geq 3$ with $|d-1|_v = 1$, fix $a \in \mathbb{C}_v$, and consider $\phi(z) = z^d - az^{d-1}$. If $|a|_v \leq 1$, then ϕ has good reduction, and hence $\mathcal{K}_v = \overline{D}(0,1)$. Once again, then, we consider $|a|_v > 1$ and set $r = |a|_v$, so that $\mathcal{K}_v \subseteq U_0 = \overline{D}(0,r)$.

This time, however, $\phi^{-1}(U_0)$ consists of only two disks. One, $W_1 = \overline{D}(a, r^{-(d-2)})$, is small and maps one-to-one onto U_0; but the other, $W_2 = \overline{D}(0,1)$, is comparatively large, and it maps $(d-1)$-to-1 onto U_0. Because of the fixed critical point at 0, we see that $\phi^{-2}(U_0)$ consists of two disks inside W_1 (one mapping to W_1, and the other to W_2), and d disks inside W_2 ($d-1$ mapping to W_1, and the last mapping $(d-1)$-to-1 to W_2). In general, each $U_n = \phi^{-n}(U_0)$ will be a union of disks. Each disk of U_{n-1} has one preimage in U_n inside W_1 and (with one exception) $d-1$ preimages inside W_2. The exception is the disk D_{n-1} of U_{n-1} containing 0; it has only one preimage D_n inside W_2, mapping $(d-1)$-to-one onto D_{n-1}. Ultimately $\mathcal{K}_v$ consists of the disk $V = \overline{D}(0, r^{-1/(d-2)})$ and all of its preimages together with a vaguely Cantor-like set at which the preimages of V accumulate.

Thus, in contrast with Example 2.3, $\mathcal{K}_v$ is neither a disk nor compact. In general, the filled Julia set of a polynomial of bad reduction over $\mathbb{C}_v$ will look something like this one. However, the dynamics can be even more complicated when there are regions on which ϕ maps n-to-1 for some integer n divisible by p, the characteristic of the residue field k_v.

The preceding comments and examples made frequent reference to disks U_0 containing $\mathcal{K}_v$. The smallest such disk will be of particular importance to us. The following Lemma shows the existence of the smallest disk and gives a partial characterization of it.

Lemma 2.5. *Let $\mathbb{C}_v$ be a complete, algebraically closed field with absolute value $|\cdot|_v$. Let $\phi \in \mathbb{C}_v[z]$ be a polynomial of degree $d \geq 2$ with lead coefficient $a_d \in \mathbb{C}_v$. Denote by $\mathcal{K}_v$ the filled Julia set of ϕ in $\mathbb{C}_v$. Then:*

a. *There is a unique smallest disk $U_0 \subseteq \mathbb{C}_v$ which contains $\mathcal{K}_v$.*
b. *U_0 is a closed disk of some radius $r'_v \in |\mathbb{C}_v^\times|_v$, with $r'_v \geq |a_d|_v^{-1/(d-1)}$.*
c. *If $|\cdot|_v$ is non-archimedean, then ϕ has potentially good reduction if and only if $r'_v = |a_d|_v^{-1/(d-1)}$. In that case, $\mathcal{K}_v = U_0$.*

Proof. Choose $\alpha \in \mathbb{C}_v$ such that $\alpha^{d-1} = a_d$, and let $\psi(z) = \alpha\phi(\alpha^{-1}z)$, which is a monic polynomial with filled Julia set $\alpha\mathcal{K}_v$. Given the scaling factors of $|a_d|_v^{-1/(d-1)} = |\alpha|_v^{-1}$ in parts (b) and (c), we may assume without loss that $a_d = 1$.

If $\mathbb{C}_v$ is archimedean, then $\mathbb{C}_v \cong \mathbb{C}$. In that case, $\mathcal{K}_v \subseteq \mathbb{C}$ is a compact set. (See, for example, Lemma 9.4 of [21].) Since $\mathcal{K}_v$ is bounded, there is a unique smallest disk U_0 containing $\mathcal{K}_v$. (See, for example, Exercise 3 in Appendix I of [34].) Moreover, because $\mathcal{K}_v$ is compact, this disk must be closed.

The filled Julia set of a monic polynomial over $\mathbb{C}$ has capacity 1; see, for example, Theorem 4.1 of [3]. Meanwhile, the capacity of the disk U_0 is exactly its radius r_v. Since $U_0 \supseteq \mathcal{K}_v$, we must have $r_v \geq 1$, proving the Lemma in the archimedean case. (See Remark 2.6 below for an alternate proof not using capacity theory.)

If $\mathbb{C}_v$ is non-archimedean, then let $b \in \mathbb{C}_v$ be a fixed point of ϕ. (Such b exists because $\phi(z) - z$ is a polynomial of degree $d \geq 2$.) Clearly $b \in \mathcal{K}_v$. By the coordinate change $z \mapsto z + b$, we may assume that $b = 0$. Write $\phi(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z$. Let $\tilde{r}_v = \max\{|a_i|_v^{1/(d-i)} : i = 1, \ldots, d-1\}$ and $r_v = \max\{\tilde{r}_v, 1\}$; note that $r_v \in |\mathbb{C}_v^\times|_v$.

If $r_v = 1$, then ϕ is a monic polynomial with coefficients in $\mathcal{O}_v$. Hence, ϕ has good reduction; with $U_0 = \overline{D}(0,1)$, the Lemma follows.

For the remainder of the proof, assume $r_v > 1$. Then the Newton polygon (see Section 6.5 of [14] or Section IV.3 of [18]) for the equation $\phi(z) = 0$ shows that there is some $c \in \mathbb{C}_v$ with $|c|_v = r_v$ and $\phi(c) = 0$. In particular, any disk containing $\mathcal{K}_v$ must contain $\overline{D}(0, r_v)$.

Moreover, if $|z|_v > r_v$, then the z^d term has larger absolute value than any other term of $\phi(z)$, so that $|\phi(z)|_v = |z|_v^d$. By induction, $|\phi^n(z)|_v = |z|_v^{d^n}$ for all $n \geq 1$. It follows that $\mathcal{K}_v \subseteq \overline{D}(0, r_v)$. By the previous paragraph, $\overline{D}(0, r_v)$ is the smallest disk U_0 containing $\mathcal{K}_v$.

On the other hand, for any $x \in \mathbb{C}_v$ with $r_v < |x|_v \leq r_v^d$, all preimages of x lie in U_0. Thus, $\mathcal{K}_v \subsetneq U_0$. However, if ϕ had potentially good reduction, then $\mathcal{K}_v$ would be a disk, since a polynomial of good reduction has filled Julia set equal to $\overline{D}(0,1)$. Since U_0 is the smallest disk containing $\mathcal{K}_v$, ϕ does not have potentially good reduction. ☐

Remark 2.6. The fact that $r_v \geq 1$ in the archimedean case can also be proven directly, without reference to the power of capacity theory. The following alternate argument was suggested to the author by Laura DeMarco.

Suppose that $\mathcal{K}_v \subseteq \overline{D}(a, r_v)$ for some $r_v < 1$; let $s = (1 + r_v)/2$. Since ϕ is a (monic) polynomial of degree at least 2, there is some radius $R > 1$ such that for all $z \in \mathbb{C}$ with $|z - a| > R$, we have $|\phi(z) - a| > R$. Let $A = \{z \in \mathbb{C} : s \leq |z - a| \leq R\}$. Every point of the annulus A is attracted to ∞ under iteration of ϕ. Since A is compact, there is some $n \geq 1$ such that $f(z) = \phi^n(z) - a$ has $|f(z)| > 1$ for all $z \in A$. Note that all d^n zeros of f lie in $D(a, s)$. Let $g(z) = (z - a)^{d^n} - f(z)$, which is a polynomial of degree strictly less than d^n. However, for all $z \in \mathbb{C}$ with $|z - a| = s$, we have

$$|f(z) + g(z)| = |z - a|^{d^n} = s^{d^n} < 1 < |f(z)|.$$

By Rouché's Theorem (noting that $g(z) \neq 0$ for $|z - a| = s$), f and g have the same number of zeros in $D(a, s)$, counting multiplicity. That is a contradiction; thus, $r_v \geq 1$.

The next two Lemmas give slightly more detailed information about the filled Julia set for a polynomial of bad reduction over a non-archimedean field.

Lemma 2.7. *With notation as in Lemma 2.5, suppose that $\mathbb{C}_v$ is non-archimedean and $r'_v > |a_d|_v^{-1/(d-1)}$. Then $\phi^{-1}(U_0)$ is a disjoint union of closed disks $D_1, \ldots, D_\ell \subseteq U_0$, where $2 \leq \ell \leq d$. Moreover, there are positive integers $d_1, \ldots, d_\ell$ with $d_1 + \cdots + d_\ell = d$ such that for each $i = 1, \ldots, \ell$, ϕ maps D_i d_i-to-1 onto U_0. That is, $\phi(D_i) = U_0$, and every point U_0 has exactly d_i pre-images in D_i, counting multiplicity.*

Proof. By the proof of Lemma 2.5 in the $r_v > 1$ case, we have $\phi^{-1}(U_0) \subsetneq U_0$.

We now construct the disks $D_1, \ldots, D_\ell$ inductively. For each $i = 1, 2, \ldots$, suppose we already have $D_1, \ldots, D_{i-1}$, and choose $b_i \in \phi^{-1}(U_0) \setminus (D_1 \cup \cdots \cup D_{i-1})$. (If this is not possible, then skip to the next paragraph.) By Lemmas 2.3 and 2.6 of [7], there is a unique disk D_i containing b_i which maps onto U_0, and this disk must be closed. Since D_j was also unique for each $j < i$, the new disk D_i must be disjoint from D_j. In addition, by Lemma 2.2 of [7], ϕ maps D_i d_i-to-1 onto U_0, for some integer $d_i \geq 1$.

This process must stop with $\ell \leq d$, because for any $a \in U_0$, $\phi^{-1}(a)$ consists of exactly d points, counting multiplicity, and since each $d_i \geq 1$, at least one must be contained in each D_i. Counting elements of $\phi^{-1}(a)$ also shows that $d_1 + \cdots + d_\ell = d$.

Finally, if $\ell = 1$, then $D_1 = \phi^{-1}(U_0) \subsetneq U_0$ is a single disk. Thus, $\mathcal{K}_v = \phi^{-1}(\mathcal{K}_v) \subseteq D_1$; but U_0 was by definition the smallest disk containing $\mathcal{K}_v$. Hence, we must have $\ell \geq 2$. ☐

Lemma 2.8. *Let K be a field with a discrete valuation v, and let $\pi_v \in K$ be a uniformizer at v. Let $\mathbb{C}_v$ be the completion of an algebraic closure of K. Let $\phi(z) = a_d z^d + \cdots + a_0 \in K[z]$ be a polynomial of degree $d \geq 2$. Denote by $\mathcal{K}_v$ the filled Julia set of ϕ in $\mathbb{C}_v$, and let $r'_v > 0$ be the radius of the smallest disk in $\mathbb{C}_v$ containing $\mathcal{K}_v$. Suppose that $r'_v > |a_d|_v^{-1/(d-1)}$. If $\mathcal{K}_v \cap K \neq \emptyset$, then*

$$|a_d|_v^{1/(d-1)} r'_v \geq \begin{cases} |\pi_v|_v^{-1} & \text{if } d = 2, \\ |\pi_v|_v^{-1/[(d-1)(d-2)]} & \text{if } d \geq 3. \end{cases}$$

Proof. Given $b \in \mathcal{K}_v \cap K$, we may replace ϕ by $\phi(z+b) - b \in K[z]$, which is a polynomial with the same degree and lead coefficient as ϕ, but with filled Julia set translated by $-b$. In particular, the radius r'_v is preserved; so we may assume without loss that $0 \in \mathcal{K}_v$.

As in the proof of Lemma 2.5, choose $\alpha \in \mathbb{C}_v$ such that $\alpha^{d-1} = a_d$, and let

$$\psi(z) = \alpha\phi(\alpha^{-1}z) = z^d + \sum_{i=0}^{d-1} \alpha^{1-i} a_i z^i.$$

Then ψ is a monic polynomial with filled Julia set $\mathcal{K}'_v = \alpha\mathcal{K}_v$; hence, the radius r_v of the smallest disk containing $\mathcal{K}'_v$ satisfies $r_v > 1$. However, ψ may not be defined over K.

Let j be the largest index between 0 and $d-1$ that maximizes $\lambda_j = |\alpha^{1-j} a_j|_v^{1/(d-j)}$. Note that $\lambda_j > 1$; for if not, then ψ has good reduction, contradicting Lemma 2.5.c. The Newton polygon for the equation $\psi(z) = 0$ shows that there is some $\beta \in \mathbb{C}_v$ with $\psi(\beta) = 0$ and $|\beta|_v = \lambda_j$. We have $0, \beta \in \mathcal{K}'_v$; hence, $r_v \geq \lambda_j$.

If $j = 0$, then a simple induction shows that $|\psi^n(0)|_v = |\alpha a_0|_v^{d^{n-1}}$ for $n \geq 1$. Since $|\alpha a_0|_v > 1$, this contradicts the hypothesis that $0 \in \mathcal{K}'_v$.

Thus, $1 \leq j \leq d-1$, and we write $|a_d|_v = |\pi_v|_v^{e_1}$ and $|a_j|_v = |\pi_v|_v^{e_2}$; note that $e_1, e_2 \in \mathbb{Z}$. Our assumptions say that

$$r_v \geq \lambda_j = |\alpha^{1-j} a_j|_v^{1/(d-j)} = |\pi_v|_v^f > 1, \qquad \text{where} \qquad f = \frac{1}{d-j}\left(\frac{(1-j)e_1}{d-1} + e_2\right) < 0.$$

If $j = 1$, then $f = e_2/(d-1) \leq -1/(d-1)$, which proves the Lemma for $d = 2$. If $2 \leq j \leq d-1$, then $f \leq -1/[(d-1)(d-j)] \leq -1/[(d-1)(d-2)]$, and we are done. ☐

Remark 2.9. The bounds of Lemma 2.8 are sharp. Indeed, one can check that they are attained by $\phi(z) = z^2 - \pi_v^{-1} z$ for $d = 2$ and by $\phi(z) = \pi_v^d z^d - \pi_v z^2$ for $d \geq 3$.

3. Elementary Computations

We will now define and bound certain integer quantities that will appear as exponents in the rest of the paper. The reader is encouraged to read the statements of Definition 3.1, Lemma 3.4, and Lemma 3.5 but to skip the proofs, which are tedious but completely elementary, until after seeing their use in Theorem 7.1.

We will write $\log_d x$ to denote the logarithm of x to base d.

Definition 3.1. *Let $N \geq 0$ and $d \geq 2$ be integers. We define $E(N,d)$ to be twice the sum of all base-d coefficients of all integers from 0 to $N-1$. That is,*

$$E(N,d) = 2\sum_{j=0}^{N-1} e(j,d), \qquad \text{where} \qquad e\left(\sum_{i=0}^{M} c_i d^i, d\right) = \sum_{i=0}^{M} c_i,$$

for $c_i \in \{0,1,\ldots,d-1\}$.

Moreover, if m is an integer satisfying $1 \leq m \leq d$, we may write $N = c_0 + mk$ for unique integers $c_0 \in \{0,1,\ldots,m-1\}$ and $k \geq 0$. We then define

$$e(N,m,d) = c_0 + e(k,d) - (d-m)k \qquad \text{and} \qquad f(N,m,d) = c_0 + e(k,d),$$

and

$$E(N,m,d) = 2\sum_{j=0}^{N-1} e(j,m,d) \qquad \text{and} \qquad F(N,m,d) = 2\sum_{j=0}^{N-1} f(j,m,d).$$

We declare $E(N,d) = E(N,m,d) = F(N,m,d) = 0$ for $N \leq 1$. Clearly, $E(N,d)$ and $F(N,m,d)$ are always positive for $N \geq 1$; but for N large and $m < d$, $E(N,m,d)$ is negative. Note that $e(N,d,d) = f(N,d,d) = f(N,1,d) = e(N,d)$, and therefore

$$E(N,d,d) = F(N,d,d) = F(N,1,d) = E(N,d). \tag{3}$$

We will need the following two auxiliary Lemmas.

Lemma 3.2. *Let N,m,d be integers satisfying $N \geq 1$, $d \geq 2$, and $1 \leq m \leq d$. Write $N = c + mk$ with $0 \leq c \leq m-1$ and $k \geq 0$. Then:*

a. $F(N,m,d) = (m-c)E(k,d) + cE(k+1,d) + (m-1)N - c(m-c)$.

b. $E(N,m,d) = F(N,m,d) - \dfrac{(d-m)}{m}[N^2 - mN + c(m-c)]$.

c. *If $N \leq m$, then* $E(N,m,d) = F(N,m,d) = N(N-1)$.

Proof. Writing an arbitrary integer $j \geq 0$ as $j = i + m\ell$ for $0 \leq i \leq m-1$, we compute

$$\begin{aligned} F(N,m,d) &= 2\sum_{j=0}^{N-1} f(j,m,d) = 2\sum_{i=0}^{c-1}\sum_{\ell=0}^{k} f(i+m\ell,m,d) + 2\sum_{i=c}^{m-1}\sum_{\ell=0}^{k-1} f(i+m\ell,m,d) \\ &= 2\sum_{i=0}^{c-1}\sum_{\ell=0}^{k}(i+e(\ell,d)) + 2\sum_{i=c}^{m-1}\sum_{\ell=0}^{k-1}(i+e(\ell,d)) \\ &= \sum_{i=0}^{c-1}\left[2(k+1)i + E(k+1,d)\right] + \sum_{i=c}^{m-1}\left[2ki + E(k,d)\right] \\ &= cE(k+1,d) + (m-c)E(k,d) + (k+1)c(c-1) + km(m-1) - kc(c-1). \end{aligned}$$

Part (a) of the Lemma now follows by rewriting the last three terms as

$$c(c-1)+mk(m-1)=c(c-m)+(c+mk)(m-1)=(m-1)N-c(m-c).$$

Next, we compute

$$\begin{aligned} E(N,m,d) &= 2\sum_{j=0}^{N-1} e(j,m,d) = 2\sum_{j=0}^{N-1} f(j,m,d) - 2(d-m)\left[m\sum_{\ell=0}^{k-1}\ell + \sum_{j=0}^{c-1} k\right] \\ &= F(N,m,d) - k(d-m)[m(k-1)+2c] \end{aligned}$$

Writing $k=(N-c)/m$, the last term becomes

$$-\frac{(N-c)}{m}(d-m)(N+c-m) = -\frac{(d-m)}{m}[N^2-mN+c(m-c)],$$

proving part (b). Finally, part (c) is immediate from the fact that $e(j,m,d)=f(j,m,d)=j$ for $0\le j\le m-1$. ☐

Lemma 3.3. *Let N,m,d be integers satisfying $N\ge 1$, $d\ge 2$, and $1\le m\le d$. Write $N=c+mk$ with $0\le c\le m-1$ and $k\ge 0$. Then:*

a. $(m-c)\log_d\left(\frac{mk}{N}\right) + c\log_d\left(\frac{mk+m}{N}\right) \le 0.$

b. *If $N\ge d$, then* $(d-1)\log_d\left(\frac{mk+m}{N}\right) - (m-c) \le 0.$

Proof. The function $\log_d(x)$ is of course concave down. Letting $x_1 = mk/N$ and $x_2 = (mk+m)/N$, then, we have $x_1\le 1 < x_2$, and therefore $\log_d(1)\ge L(1)$, where

$$L(x) = \frac{1}{x_2-x_1}\left[(x_2-x)\log_d(x_1) + (x-x_1)\log_d(x_2)\right]$$

is the line through $(x_1,\log_d(x_1))$ and $(x_2,\log_d(x_2))$. That is,

$$0 \ge \frac{1}{m}\left[(m-c)\log_d\left(\frac{mk}{N}\right) + c\log_d\left(\frac{mk+m}{N}\right)\right],$$

proving part (a). For part (b), we have

$$(d-1)\log_d\left(\frac{mk+m}{N}\right) = \frac{(d-1)}{\log d}\cdot\log\left(1+\frac{m-c}{N}\right) \le \frac{(d-1)}{\log d}\cdot\frac{(m-c)}{N}.$$

However, $\log d = -\log[1-(d-1)/d] \ge (d-1)/d$, and since $N\ge d$,

$$(d-1)\log_d\left(\frac{mk+m}{N}\right) \le (d-1)\cdot\frac{d}{d-1}\cdot\frac{m-c}{N} = \frac{d}{N}(m-c) \le (m-c).$$ ☐

Lemma 3.4. *Let N,m,d be integers satisfying $N\ge 1$, $d\ge 2$, and $1\le m\le d-1$. Then:*

a. *$E(N,d)\le (d-1)N\log_d N$, with equality if N is a power of d.*

b. *$E(N,m,d) \le (d-1)N\left[\log_d N + 1 - \log_d m - \frac{(d-m)}{m(d-1)}N\right]$, with equality if N/m is a power of d.*

c. *$F(N,m,d)\le (d-1)N\log_d N$.*

d. *For $N\ge m$, $F(N,m,d) \le (d-1)N\left[\log_d N - \log_d m + \frac{m-1}{d-1}\right]$, with equality if N/m is a power of d.*

Proof. Fix $d \geq 2$. If $N = 1$, then both sides of part (a) are clearly zero. If $2 \leq N \leq d$, then $E(N, d) = N(N-1)$ by Lemma 3.2.c (with $m = d$) and equation (3). Because $(\log x)/(x-1)$ is a decreasing function of the real variable $x > 1$, we have $(\log d)/(d-1) \leq (\log N)/(N-1)$, with equality for $N = d$. Part (a) then follows for $1 \leq N \leq d$.

For $N \geq d + 1$, we proceed by induction on N, assuming part (a) holds for all positive integers up to $N - 1$. Write $N = c + dk$, where $0 \leq c \leq d - 1$, so that $1 \leq k \leq N - 2$. By Lemma 3.2.a (with $m = d$) and equation (3), we have

$$\begin{aligned} E(N, d) &= (d - c)E(k, d) + cE(k + 1, d) + (d - 1)N - c(d - c) \\ &\leq (d - c)(d - 1)k \log_d k + c(d - 1)(k + 1) \log_d(k + 1) + (d - 1)N - c(d - c) \\ &= (d - c)(d - 1)k \log_d(dk) + c(d - 1)(k + 1) \log_d(dk + d) - c(d - c) \end{aligned}$$

where the final equality is because $N = (d - c)k + c(k + 1)$, and the inequality (which is equality if N is a power of d) is by the inductive hypothesis, since $k, k + 1 \leq N - 1$. More generally, adding and subtracting $(d - 1)N \log_d N$, we have

$$\begin{aligned} E(N, d) \leq (d - 1)N \log_d N &+ (d - c)(d - 1)k \log_d \left(\frac{dk}{N}\right) \\ &+ c(d - 1)(k + 1) \log_d \left(\frac{dk + d}{N}\right) - c(d - c) \\ = (d - 1)N \log_d N &+ c \left[(d - 1) \log_d \left(\frac{dk + d}{N}\right) - (d - c)\right]. \\ &+ (d - 1)k \left[(d - c) \log_d \left(\frac{dk}{N}\right) + c \log_d \left(\frac{dk + d}{N}\right)\right] \end{aligned}$$

By Lemma 3.3 with $m = d$, the quantities in brackets are nonpositive, and part (a) follows.

If $m = 1$, then parts (c–d) are immediate from part (a) and equation (3). Moreover, by Lemma 3.2.a–b (with $m = 1$) and part (a),

$$E(N, 1, d) = E(N, d) - (d - 1)N(N - 1) \leq (d - 1)N[\log_d N + 1 - N],$$

with equality if N is a power of d. This is exactly part (b) for $m = 1$. Thus, we may assume for the remainder of the proof that $2 \leq m \leq d$.

We now turn to part (d). If $N = m$, then by Lemma 3.2.c, we have $F(m, m, d) \leq m(m - 1)$, which exactly equals the desired right hand side. For $N \geq m + 1$, write $N = c + mk$, where $k \geq 1$ and $0 \leq c \leq m - 1$. By Lemma 3.2.a,

$$F(N, m, d) = (m - c)E(k, d) + cE(k + 1, d) + (m - 1)N - c(m - c). \tag{4}$$

If $m + 1 \leq N \leq d - 1$, then $k \leq d - 1$, so that by Lemma 3.2.c, equation (4) becomes

$$\begin{aligned} F(N, m, d) &= (m - c)k(k - 1) + ck(k + 1) + (m - 1)N - c(m - c) \\ &= mk^2 - mk + 2ck + (m - 1)N - c(m - c) = (m + k - 1)N - (m - c)(k + c) \\ &= m^{-1} \left[(N + m^2 - c - m)N - (m - c)(N - c + cm)\right] \\ &= m^{-1} \left[(N + m^2 - 2m)N - c(m - c)(m - 1)\right] \leq N \left[(N/m) + m - 2\right], \end{aligned}$$

where we have substituted $k = (N - c)/m$ along the way. Thus, we must show

$$N \left[(N/m) + m - 2\right] \leq N \left[(d - 1) \log_d(N/m) + m - 1\right].$$

Equivalently, we must show

$$\frac{\log d}{d-1} \le \frac{\log(N/m)}{(N/m)-1},$$

which is true because $(\log x)/(x-1)$ is a decreasing function of $x > 1$, and $1 < N/m < d$.

If $N \ge d$ in part (d), then $k \ge 1$. Applying part (a) to equation (4), we obtain

$$F(N,m,d) \le (m-c)(d-1)k\log_d k + c(d-1)(k+1)\log_d(k+1) + (m-1)N - c(m-c), \tag{5}$$

with equality if $c = 0$ and k is a power of d, whence we immediately obtain the statement of the Lemma for $N = md^i$. More generally, (5) becomes

$$\begin{aligned} F(N,m,d) &\le (d-1)N\left(\log_d \frac{N}{m} + \frac{m-1}{d-1}\right) + (m-c)(d-1)k\log_d\left(\frac{mk}{N}\right) \\ &\qquad + c(d-1)(k+1)\log_d\left(\frac{mk+m}{N}\right) - c(m-c) \\ &= (d-1)N\left(\log_d \frac{N}{m} + \frac{m-1}{d-1}\right) + c\left[(d-1)\log_d\left(\frac{mk+m}{N}\right) - (m-c)\right] \\ &\qquad + (d-1)k\left[(m-c)\log_d\left(\frac{mk}{N}\right) + c\log_d\left(\frac{mk+m}{N}\right)\right]. \end{aligned}$$

Part (d) now follows from Lemma 3.3, as before.

For part (c), if $1 \le N \le m$, then by part (a) and Lemma 3.2.c,

$$F(N,m,d) = N(N-1) = E(N,d) \le (d-1)N\log_d N,$$

as desired. The remaining case, that $N \ge m$, will follow from part (d) provided

$$m-1 \le (d-1)\log_d m.$$

However, this is the same as showing that $\log d/(d-1) \le \log m/(m-1)$, which once again follows from the fact that $(\log x)/(x-1)$ is decreasing for $x > 1$.

Last, we turn to part (b). If $1 \le N \le m$, then $E(N,m,d) = N(N-1)$ by Lemma 3.2.c. Thus, we wish to show that

$$N-1 \le (d-1)\left(1 + \log_d \frac{N}{m}\right) - (d-m)\frac{N}{m},$$

which is to say

$$\frac{dN}{m} - 1 \le (d-1)\log_d\left(\frac{dN}{m}\right),$$

with equality when $N = m$. Yet again, this inequality follows immediately from the facts that $(\log x)/(x-1)$ is decreasing for $x > 1$ and that $1 \le dN/m \le d$.

It only remains to consider $N \ge m+1$. Writing $N = c + mk$, where $k \ge 1$ and $0 \le c \le m-1$, and invoking Lemma 3.2.b, we have

$$\begin{aligned} E(N,m,d) &= F(N,m,d) - \frac{(d-m)}{m}[N^2 - mN + c(m-c)] \\ &\le F(N,m,d) - \frac{(d-m)}{m}N^2 + (d-m)N, \end{aligned}$$

with equality for $c = 0$. By part (d), we obtain

$$E(N,m,d) \le (d-1)N\left[\log_d N - \log_d m + \frac{m-1}{d-1}\right] - \frac{(d-m)}{m}N^2 + (d-m)N$$
$$= (d-1)N\left[\log_d N - \log_d m + 1 - \frac{d-m}{m(d-1)}N\right],$$

with equality if N is of the form $N = md^i$. $\square$

Besides the preceding integer quantities and their bounds, we will need the following bound involving a certain family of real-valued functions.

Lemma 3.5. *Let $d \ge 2$ be an integer, and let A, B, t be positive real numbers such that*

$$(d-1)A \ge d^{B-1} \qquad \text{and} \qquad t \ge 1.$$

Define $\eta : (0,\infty) \to \mathbb{R}$ by

$$\eta(x) = t\log_d x - Ax + B.$$

Set the real number $M(A,B,t)$ to be

$$M(A,B,t) = \frac{t}{A}\left(\log_d t + \log_d(\max\{1, \log_d t\}) + 3\right).$$

Then $\eta(x) < 0$ for all $x \ge M(A,B,t)$.

Proof. By differentiating, we see that η is decreasing for $x \ge t/(A\log d)$, and hence for $x \ge M(A,B,t)$. Thus, it suffices to show that $\eta(M(A,B,t)) < 0$.

First, suppose that $t < d$. Then

$$\eta(M(A,B,t)) = t\log_d t + t\log_d\left[A^{-1}(\log_d t + 3)\right] - t\log_d t - 3t + B$$
$$= t\log_d\left[A^{-1}d^{B-3}(\log_d t + 3)\right] - B(t-1) \le t\log_d\left[(d-1)(\log_d t + 3)/d^2\right],$$

where the inequality is because $A^{-1}d^{B-1} \le (d-1)$ and $B > 0$, by hypothesis. Since $t < d$, the quantity inside square brackets is strictly less than $4(d-1)/d^2 \le 1$. Thus, $\eta(M(A,B,t)) < t\log_d(1) = 0$, and we are done.

Second, if $t \ge d$, then by a similar computation,

$$\eta(M(A,B,t)) = t\log_d\left[A^{-1}d^{B-3}u^{-1}(u + \log_d u + 3)\right] - B(t-1)$$
$$< t\log_d\left[(d-1)(u + \log_d u + 3)/(d^2u)\right]$$

where $u = \log_d t \ge 1$. Writing $H(u) = (d-1)(u + \log_d u + 3)/(d^2u)$, it suffices to show that $H(u) \le 1$ for $u \ge 1$. Differentiating, it is easy to see that H is decreasing for such u. Since $H(1) = 4(d-1)/d^2 \le 1$, we are done. $\square$

4. Transfinite Diameters and Bad Primes

Given a metric space X and an integer $N \ge 2$, the N^{th} diameter of X is defined to be

$$\mathbf{d}_N(X) = \sup_{x_1,\ldots,x_N \in X} \prod_{i \ne j} d_X(x_i, x_j)^{1/[N(N-1)]},$$

which measures the maximal average distance between any two of N points in X. (See [15], for example, for a computation of the N^{th} diameter of the interval $[0,1]$.) This quantity is usually used to define the *transfinite diameter* of X,

$$\mathbf{d}(X) = \lim_{N\to\infty} \mathbf{d}_N(X),$$

which converges because $\{\mathbf{d}_N(X)\}_{N\geq 2}$ is a decreasing sequence. If X is a nice enough (e.g., compact) subset of a valued field, then the transfinite diameter coincides with the Chebyshev constant and the logarithmic capacity of X; see Section 5.4 of [1], or Chapters 3 and 4 of [32]. Baker and Hsia used this equality in [3] to compute the transfinite diameter of filled Julia sets of polynomials, even when those sets are not compact. (Their result of $|a_d|_v^{-1/(d-1)}$, where d is the degree and a_d the lead coefficient of the polynomial, was already well known for $\mathbb{C}_v = \mathbb{C}$.) See [32] for more on transfinite diameters and capacities in $\mathbb{C}_v$.

However, in this paper we will be interested in the N^{th} diameters $\mathbf{d}_N(X)$ themselves, rather than the transfinite diameter. In particular, the following Lemma contains our main bound for $\mathbf{d}_N(\mathcal{K}_v)^{N(N-1)}$, where $\mathcal{K}_v$ is the filled Julia set of a polynomial $\phi \in \mathbb{C}_v[z]$. The proof uses an estimate involving van der Monde determinants similar to a bound that appears in the proof of Lemme 5.4.2 in [1].

Lemma 4.1. *Let $\mathbb{C}_v$ be a complete, algebraically closed field with absolute value $|\cdot|_v$. Let $\phi \in \mathbb{C}_v[z]$ be a polynomial of degree $d \geq 2$ with lead coefficient $a_d \in \mathbb{C}_v$. Denote by $\mathcal{K}_v$ the filled Julia set of ϕ in $\mathbb{C}_v$, and let r'_v be the radius of the smallest disk that contains $\mathcal{K}_v$. Set $r_v = |a_d|_v^{1/(d-1)} r'_v$.*

Then for any integer $N \geq 2$ and any set $\{x_1, \ldots, x_N\} \subseteq \mathcal{K}_v$ of N points in $\mathcal{K}_v$,

$$\prod_{i\neq j} |x_i - x_j|_v \leq |a_d|_v^{-N(N-1)/(d-1)} \max\{1, |N|_v^N\} r_v^{E(N,d)},$$

where $E(N,d)$ is twice the sum of all base-d coefficients of all integers from 0 to $N-1$, as in Definition 3.1.

Proof. Choose $\alpha \in \mathbb{C}_v$ such that $\alpha^{d-1} = a_d$, and let $\psi(z) = \alpha\phi(\alpha^{-1}z)$. Then ψ is a monic polynomial with filled Julia set $\mathcal{K}'_v = \alpha\mathcal{K}_v$, and the smallest disk containing $\mathcal{K}'_v$ has radius r_v. If the Lemma holds for ψ, then for $x_1, \ldots, x_N \in \mathcal{K}_v$, we have $\alpha x_i \in \mathcal{K}'_v$, and therefore

$$\prod_{i\neq j} |x_i - x_j|_v = |\alpha|_v^{-N(N-1)} \prod_{i\neq j} |\alpha x_i - \alpha x_j|_v \leq |\alpha|_v^{-N(N-1)} \max\{1, |N|_v^N\} r_v^{E(N,d)},$$

as desired. Thus, it suffices to prove the Lemma in the case that ϕ is monic.

We will now construct a sequence $\{f_j\}_{j=1}^{\infty}$ of monic polynomials over $\mathbb{C}_v$ such that each f_j has degree j and such that $|f_j(x)|_v$ is not especially large for any $x \in \mathcal{K}_v$.

First, let $\overline{D}(a, r_v)$ be the smallest disk containing $\mathcal{K}_v$, where $a \in \mathbb{C}_v$ and r_v is as in the statement of the Lemma. For any integer $j \geq 0$ written in base-d notation as

$$j = c_0 + c_1 d + c_2 d^2 + \cdots + c_M d^M,$$

with $c_i \in \{0, 1, \ldots, d-1\}$, define

$$f_j(z) = \prod_{i=0}^{M} [\phi^i(z) - a]^{c_i}.$$

Clearly, f_j is monic of degree j. Moreover, for $x \in \mathcal{K}_v$, we have $\phi^i(x) \in \mathcal{K}_v$, and therefore

$$|f_j(x)|_v \leq \prod_{i=0}^{M} r_v^{c_i} = r_v^{e(j,d)},$$

where $e(j,d)$ is as in Definition 3.1.

Given $x_1, \ldots, x_N \in \mathcal{K}_v$, denote by $V(x_1, \ldots, x_N)$ the corresponding van der Monde matrix (i.e., the $N \times N$ matrix with (i,j) entry x_i^{j-1}). Recall that

$$\prod_{i \neq j} |x_i - x_j|_v = |\det V(x_1, \ldots, x_N)|_v^2.$$

Because f_{N-1} is monic, we may replace the last column of the matrix by a column with entry $f_{N-1}(x_i)$ in the ith row, without changing the determinant. We may then replace the second to last column by a column with entry $f_{N-2}(x_i)$ in the ith row, and so on. Thus, if we denote by $A(x_1, \ldots, x_N)$ the matrix with (i,j) entry $f_{j-1}(x_i)$, then

$$\det V(x_1, \ldots, x_N) = \det A(x_1, \ldots, x_N).$$

If $\mathbb{C}_v = \mathbb{C}$ is archimedean, then by Hadamard's inequality applied to the columns of A,

$$|\det A(x_1, \ldots, x_N)|^2 \leq \prod_{j=0}^{N-1} \left(|f_j(x_1)|^2 + \cdots + |f_j(x_N)|^2\right) \leq \prod_{j=0}^{N-1} N r_v^{2e(j,d)} = N^N r_v^{E(N,d)}.$$

Similarly, if $\mathbb{C}_v$ is non-archimedean, then by the non-archimedean version of Hadamard's inequality (see, for example, [1], Preuve du Lemme 5.3.4), we have

$$|\det A(x_1, \ldots, x_N)|^2 \leq \prod_{j=0}^{N-1} \max_{i=1,\ldots,N} |f_j(x_i)|^2 \leq \prod_{j=0}^{N-1} r_v^{2e(j,d)} = r_v^{E(N,d)}. \qquad \square$$

Remark 4.2. We can recover the Baker and Hsia bound $\mathbf{d}(\mathcal{K}_v) \leq |a_d|_v^{-1/(d-1)}$ immediately from Lemmas 4.1 and 3.4.a. (The opposite inequality is more subtle, however.)

Remark 4.3. There are many cases for which the bound of Lemma 4.1 is sharp. In particular, for non-archimedean v, degree $d \geq 2$ with $|d-1|_v = 1$, and $c \in \mathbb{C}_v$ with $|c| > 1$, recall that the function $\phi(z) = z^d - c^{d-1}z$ of Example 2.3 has $\mathcal{K}_v$ homeomorphic to a Cantor set on d pieces. For arbitrary $N \geq 2$, one can distribute N points in $\mathcal{K}_v$ in the following way. Write $N = \sum_{i=0}^{M} c_i d^i$, and put c_M points in each of the d^M pieces at level M, maximally far apart in each piece; then put c_{M-1} in each of the d^{M-1} pieces at level $M-1$, each as far as possible from the existing points; and so on. Keeping track of the radii of the disks at each level, one can show that $\prod_{i \neq j} |x_i - x_j|_v = r_v^{E(N,d)}$ exactly.

In many other cases, however, the bound is not quite sharp, though it appears to be approximately the right order of magnitude. In the archimedean case, of course, the Hadamard inequality introduces some error. Still, the greater factor seems to be the choice of the monic polynomial f_j. When j is a power of d, computations suggest that our choice of f_j is very close to sharp, if not actually sharp. However, when j is not a power of d, our construction of f_j as a product of smaller factors is in general not optimal, even in the non-archimedean setting. For example, if $\phi(z) = z^3 - az^2$ is the map of Example 2.4 (non-archimedean, with $d = 3$, $|a|_v > 1$, and $|2|_v = 1$), then the function $f_6(z) = (\phi(z))^2$ of the proof has $|f_6(z)|_v$ growing as large as r^2 on $\mathcal{K}_v$; but the function $\tilde{f}_6(z) = (\phi(z)) \cdot (\phi(z) - a)$

has $|\tilde{f}_6(z)|_v \le r$. Ultimately, while the exponent $E(N,3)$ of Lemma 4.1 is essentially $2N\log_3 N$, the actual exponent for this ϕ should be something more like $(4/3)N\log_3 N$.

In the archimedean case, the Chebyshev polynomials $\{\psi_j\}_{j\ge 1}$ provide an even stronger example of this phenomenon. More precisely, if $\mathbb{C}_v = \mathbb{C}$ and $\phi(z) = \psi_2(z) = z^2 - 2$, then $\mathcal{K}_v$ is simply the interval $[-2,2]$ in the real line. For $j \ge 1$, the j^{th} Chebyshev polynomial ψ_j has $|\psi_j| \le 2$ on $\mathcal{K}_v$, as compared with the proof's bound of $2^{c_0+c_1+\cdots}$ for $|f_j|$.

In general, however, knowing nothing about the polynomial other than its degree and the radius r_v, we cannot substantially improve on Lemma 4.1.

5. A Partition of the Filled Julia Set: Non-archimedean Case

The key to the Main Theorem, as described in the introduction, is to divide the filled Julia set at a particular bad prime into two smaller pieces X_1 and X_2. As a result, the product $\prod_{i\ne j}|x_i - x_i|_v$, when restricted to $\{x_i\} \subseteq X_k$ (for fixed $k = 1,2$), will be substantially smaller than the bound of Lemma 4.1. We begin with non-archimedean primes.

Lemma 5.1. *Let $\mathbb{C}_v$ be a complete, algebraically closed field with non-archimedean absolute value $|\cdot|_v$. Let $\phi \in \mathbb{C}_v[z]$ be a polynomial of degree $d \ge 2$ with lead coefficient $a_d \in \mathbb{C}_v$. Denote by $\mathcal{K}_v$ the filled Julia set of ϕ in $\mathbb{C}_v$, and let r'_v be the radius of the smallest disk U_0 that contains $\mathcal{K}_v$. Set $r_v = |a_d|_v^{1/(d-1)} r'_v$, and suppose that $r_v > 1$.*

Then there are disjoint sets $X_1, X_2 \subseteq \mathcal{K}_v$ and positive integers m_1, m_2 with the properties that $X_1 \cup X_2 = \mathcal{K}_v$, that $m_1 + m_2 = d$, that for $k = 1,2$, $\phi : X_k \twoheadrightarrow \mathcal{K}_v$ is m_k-to-1, and that for $k = 1,2$, for any integer $N \ge 2$, and for any set $\{x_1,\ldots,x_N\} \subseteq X_k$ of N points in X_k,

$$\prod_{i\ne j}|x_i - x_j|_v \le |a_d|_v^{-N(N-1)/(d-1)} r_v^{E(N,m_k,d)},$$

where $E(N,m_k,d)$ is as in Definition 3.1.

Proof. As in the proof of Lemma 4.1, we may assume that ϕ is monic.

By Lemma 2.5, U_0 is a closed disk of radius $r_v \in |\mathbb{C}_v^\times|_v$. We may write $U_0 = \overline{D}(a, r_v)$ for some point $a \in \mathcal{K}_v$, since $\mathcal{K}_v$ is nonempty, and since any point of a non-archimedean disk is a center. Pick $b \in \phi^{-1}(a)$. Note that $b \in \mathcal{K}_v \subseteq U_0$.

Write $U_1 = \phi^{-1}(U_0)$. By Lemma 2.7, $U_1 = D_1 \cup \cdots \cup D_\ell$ for some disjoint closed disks $\{D_i\}$, with $2 \le \ell \le d$. Moreover, $\phi : D_i \twoheadrightarrow U_0$ maps d_i-to-one for some positive integers $\{d_i\}$ with $d_1 + \cdots + d_\ell = d$. Define

$$W_1 = \{x \in U_1 : |x - b|_v < r_v\}, \qquad \text{and} \qquad W_2 = U_1 \setminus W_1,$$

so that $W_1 \cap W_2 = \emptyset$ and $W_1 \cup W_2 = U_1$. If $W_2 = \emptyset$, then $\mathcal{K}_v \subseteq D(b, r_v) \subsetneq U_0$, contradicting the minimality of U_0. (The second inclusion is strict because $r_v \in |\mathbb{C}_v^\times|_v$.) Thus, since $b \in W_1$, both W_1 and W_2 are nonempty.

Furthermore, W_1 and W_2 are both finite unions of disks D_i above. Hence, there are integers $m_1, m_2 \ge 1$ so that each W_k maps m_k-to-one onto U_0, with $m_1 + m_2 = d$. Let $X_k = W_k \cap \mathcal{K}_v$ for $k = 1,2$. Since $\phi^{-1}(\mathcal{K}_v) = \mathcal{K}_v$, ϕ must map X_k m_k-to-one onto $\mathcal{K}_v$.

For any integer $i \ge 1$, observe that the polynomial $\phi^i(z) - a$ is monic of degree d^i. Moreover, since the equation $\phi^{i-1}(z) = a$ has exactly d^{i-1} roots (counting multiplicity), all of which lie in U_0, it follows that $\phi^i(z) = a$ has $m_1 d^{i-1}$ roots in W_1 and $m_2 d^{i-1}$ roots in W_2, counting multiplicity. Thus, we may write

$$\phi^i(z) - a = g_i(z) h_i(z)$$

where g_i is monic of degree $m_1 d^{i-1}$ with all its roots in W_1, and h_i is monic of degree $m_2 d^{i-1}$ with all its roots in W_2. In addition, define $g_0(z) = h_0(z) = z - a$.

We will now use the polynomials g_i to compute the bounds given in the Lemma for X_1; the proof for X_2 is similar, using h_i. To simplify notation, write $X = X_1$ and $m = m_1$.

For any integer $j \geq 0$, write $j = c_0 + mk$, and write k in base-d notation, so that

$$j = c_0 + m(c_1 + c_2 d + c_3 d^2 + \cdots + c_M d^{M-1}),$$

with $c_0 \in \{0, 1, \ldots, m-1\}$, and with $c_i \in \{0, 1, \ldots, d-1\}$ for $i \geq 1$. Define

$$f_j(z) = \prod_{i=0}^{M} [g_i(z)]^{c_i}.$$

Clearly, f_j is monic of degree j. Meanwhile, for $x \in X$ and $i \geq 1$, observe that $\phi^i(x) \in \mathcal{K}_v$, and therefore $|\phi^i(x) - a| \leq r_v$. On the other hand, all roots of h_i lie in W_2, which is distance r_v from x; therefore, $|h_i(x)| = r_v^{(d-m)d^{i-1}}$. It follows that

$$|g_i(x)|_v \leq r_v^{1-(d-m)d^{i-1}}$$

for all $i \geq 1$. In addition, since $X \subseteq U_0$, we have $|g_0(x)| \leq r_v$. Thus,

$$|f_j(x)|_v \leq r_v^{c_0} \prod_{i=1}^{M} r_v^{c_i(1-(d-m)d^{i-1})} = r_v^e,$$

where $e = e(j, m, d)$ in the notation of Definition 3.1.

By the same van der Monde determinant argument as in the proof of Lemma 4.1, it follows that if $N \geq 2$ and $x_1, \ldots, x_N \in X$, then

$$\prod_{i \neq j} |x_i - x_j|_v \leq r_v^{E(N,m,d)}. \qquad \square$$

Remark 5.2. In some cases, $\mathcal{K}_v$ can be split into more than two pieces, each much smaller than the X_1, X_2 of Lemma 5.1. For example, the filled Julia set of the map $\phi(z) = z^d - c^{d-1} z$ of Example 2.3 breaks naturally into d pieces. Adapting the method of the Lemma for each piece, we could ultimately replace the coefficient $d^2 - 2d + 2$ in Theorem 7.1 by d.

However, as previously noted, most polynomials are not so simple. Indeed, the filled Julia set of $\phi(z) = z^d - az^{d-1}$ from Example 2.4 splits into only two pieces. (Of course, if we take a higher preimage U_n in that example, we get more than two pieces; but because of the large radii, there appears to be no improvement gained by using $n > 1$.) Even an application of the arguments of Remark 4.3 would result in only a slight decrease in the coefficient of $N \log_d N$ in the exponent (cf. Lemma 3.4.b). Unfortunately, a real improvement would require an increase in the size of the (negative) coefficient of N^2, not the $N \log_d N$ term.

6. A Partition of the Filled Julia Set: Archimedean Case

The final tool needed for Theorem 7.1 is an archimedean analogue of Lemma 5.1. Roughly the same argument works, but only if the diameter of the filled Julia set $\mathcal{K}$ is large enough. This phenomenon is familiar to complex dynamicists. For example, given $\phi(z) = z^2 + c \in \mathbb{C}[z]$, if the diameter of $\mathcal{K}$ is small, then c lies in the Mandelbrot set, in which case $\mathcal{K}$ is connected. However, once the diameter is large enough, c leaves the Mandelbrot set and $\mathcal{K}$ becomes disconnected. In fact, as the diameter grows, the various pieces of $\mathcal{K}$ shrink.

We begin with the following preliminary result.

Lemma 6.1. *Let $\phi \in \mathbb{C}[z]$ be a polynomial of degree $d \geq 2$ with lead coefficient $a_d \in \mathbb{C}$. Denote by $\mathcal{K}$ the filled Julia set of ϕ in $\mathbb{C}$, and let $U_0 = \overline{D}(a, r')$ be the smallest disk that contains $\mathcal{K}$. Set $r = |a_d|^{1/(d-1)} r'$, and suppose that*

$$r > \begin{cases} 3, & \text{if } d = 2, \text{ or} \\ 2+\sqrt{3}, & \text{if } d \geq 3. \end{cases}$$

Then $\mathcal{K}$ is contained in a union of d open disks of radius $|a_d|^{-1/(d-1)}$.

Proof. As in the proof of Lemma 4.1, we may assume that ϕ is monic. Denote by $b_1, \ldots, b_d$ the (possibly repeated) roots of $\phi(z) = a$, and let $\overline{D}(c, s)$ be the smallest disk containing $b_1, \ldots, b_d$. (Here, we break our convention and allow $s = 0$ if $b_1 = \cdots = b_d$.) Our first goal is to show that $s \geq r - 1$.

Because $\mathcal{K}$ is not contained in $D(c, r)$, there must be some $y_0 \in \mathcal{K}$ such that $|y_0 - c| \geq r$. Let $Y = \overline{D}(c, s) \cap D(y_0, |y_0 - c|)$. We claim that Y is contained in a disk of radius strictly less than s (or that Y is empty, if $s = 0$). Indeed, if $|y_0 - c| < s$, then $Y \subseteq D(y_0, |y_0 - c|)$ trivially. Otherwise, $|y_0 - c| \geq s$, and since the center c of the first disk lies on the boundary of the second, the intersection Y is contained in a strictly smaller disk. (For example, center the new disk at the midpoint of the two intersection points of the two boundary circles.)

By the minimality of s, then, not all of $b_1, \ldots, b_d$ can be in Y. Thus, there is some $1 \leq i \leq d$ such that $|y_0 - b_i| \geq |y_0 - c| \geq r$. Without loss, assume that $|y_0 - b_1| \geq r$.

For all $i \geq 2$, we have $|y_0 - b_i| \geq r - s$, because $b_i \in \overline{D}(c, s)$. Since $y_0 \in \mathcal{K}$, we have $\phi(y_0) \in \mathcal{K}$, and therefore $|\phi(y_0) - a| \leq r$. If $r - s \geq 0$, then, we have

$$r \geq |\phi(y_0) - a| = \prod_{i=1}^{d} |y_0 - b_i| = |y_0 - b_1| \cdot \prod_{i=2}^{d} |y_0 - b_i| \geq r \cdot (r - s)^{d-1},$$

from which we obtain $r - s \leq 1$. Regardless of the sign of $r - s$, then, we have $s \geq r - 1$.

Re-index $\{b_1, \ldots, b_d\}$ (possibly changing the previous role of b_1) so that b_1 and b_d are distance $\max\{|b_i - b_j|\}$ apart, and so that for all $i = 1, \ldots, d-1$, we have $|b_{i+1} - b_1| \geq |b_i - b_1|$. Note that $|b_d - b_1| \geq \sqrt{3}s \geq \sqrt{3}(r - 1)$; see, for example, [34], Exercise 6-1.

If $d = 2$, we can improve this lower bound. In that case, the smallest disk containing b_1 and b_2 is the closed disk centered at $(b_1 + b_2)/2$ of radius $|b_1 - b_2|/2$. That is, $s = |b_1 - b_2|/2$. It follows that $|b_1 - b_2| = 2s \geq 2(r - 1)$.

For all degrees $d \geq 2$, we have $r > 3$, so that $s > 2$, and therefore the two disks $\overline{D}(b_1, 1)$ and $\overline{D}(b_d, 1)$ are disjoint. Moreover, as y ranges through $\mathbb{C} \setminus [D(b_1, 1) \cup D(b_d, 1)]$, the minimum value of $|y - b_1| \cdot |y - b_d|$ is $|b_1 - b_d| - 1$, attained at only two points, namely the point on the boundary of each disk closest to the other disk.

Let $U_1 = \phi^{-1}(U_0)$. Since $\mathcal{K} = \phi^{-1}(\mathcal{K}) \subseteq U_1$, it suffices to show that

$$U_1 \subseteq \bigcup_{i=1}^{d} D(b_i, 1). \tag{6}$$

If not, then there is some $y \in U_1 \setminus \bigcup D(b_i, 1)$. If $d \geq 3$, then by the above computations, we have $|y - b_1| \cdot |y - b_d| \geq \sqrt{3}(r-1) - 1$. Since $\phi(y) \in U_0$, we obtain

$$r \geq |\phi(y) - a| = \prod_{i=1}^{d} |y - b_i| \geq (\sqrt{3}(r-1) - 1) \prod_{i=2}^{d-1} |y - b_i| \geq (\sqrt{3}(r-1) - 1),$$

contradicting the hypothesis that $r > 2 + \sqrt{3}$. Similarly, if $d = 2$, then

$$r \geq |\phi(y) - a| = |y - b_1| \cdot |y - b_2| > 2(r-1) - 1,$$

contradicting the hypothesis that $r > 3$, and proving the Lemma. □

Remark 6.2. Because $\mathcal{K}_v$ is compact for archimedean v, the conclusion of Lemma 6.1 implies that $\mathcal{K}$ is in fact contained in d closed disks of radius strictly less than $|a_d|_v^{-1/(d-1)}$. This fact will be useful in Cases 2 and 3 of the proof of Theorem 7.1.

We are now prepared to present our archimedean version of Lemma 5.1.

Lemma 6.3. *Let $\phi \in \mathbb{C}[z]$ be a polynomial of degree $d \geq 2$ with lead coefficient $a_d \in \mathbb{C}$. Denote by $\mathcal{K}$ the filled Julia set of ϕ in $\mathbb{C}$, and let r' be the radius of the smallest disk U_0 that contains $\mathcal{K}$. Set $r = |a_d|^{1/(d-1)} r'$ and*

$$C_d = d^{-(d-2)/(d-1)} \leq \min\left\{1, \frac{1.2}{d-1}\right\}.$$

Suppose that

$$r \geq \begin{cases} 4 & \text{if } d = 2 \\ \dfrac{\sqrt{3} + 2(d-1)}{\sqrt{3} - (d-1)C_d}, & \text{if } d \geq 3. \end{cases}$$

Then there are disjoint sets $X_1, X_2 \subseteq \mathcal{K}$ and positive integers m_1, m_2 with the properties that $X_1 \cup X_2 = \mathcal{K}$, that $m_1 + m_2 = d$, that for $k = 1, 2$, $\phi : X_k \twoheadrightarrow \mathcal{K}_v$ is m_k-to-1, and that for $k = 1, 2$, for any integer $N \geq 2$, and for any set $\{x_1, \ldots, x_N\} \subseteq X_k$ of N points in X_k,

$$\prod_{i \neq j} |x_i - x_j| \leq N^N |a_d|^{-N(N-1)/(d-1)} C_d^{-F(N,m_k,d)} (C_d r)^{E(N,m_k,d)},$$

where $E(N, m_k, d)$ and $F(N, m_k, d)$ are as in Definition 3.1.

Proof. As in the proof of Lemma 4.1, we may assume that ϕ is monic. It is easy to check that $C_d \leq \min\{1, 1.2/(d-1)\}$ (the closest approach for $d \geq 3$ occurs at $d = 5$), and that the lower bound $(\sqrt{3} + 2(d-1))/(\sqrt{3} - (d-1)C_d)$ (respectively, 4) for r is greater than $2 + \sqrt{3}$ (respectively, 3), so that we may invoke Lemma 6.1.

Write $U_0 = \overline{D}(a, r)$, and define and order $b_1, \ldots, b_d$ as in the proof of Lemma 6.1, so that $|b_1 - b_d| \geq \sqrt{3}(r-1)$ (or $|b_1 - b_d| \geq 2(r-1)$, if $d = 2$).

If $d \geq 3$, observe that for some $m = 1, \ldots, d-1$, we have

$$|b_{m+1} - b_1| \geq |b_m - b_1| + 2 + C_d r.$$

For if not, then

$$\sqrt{3}(r-1) \leq |b_d - b_1| < (d-1)\left[2 + C_d r\right],$$

so that $[\sqrt{3} - (d-1)C_d] r < \sqrt{3} + 2(d-1)$, contradicting the hypotheses.

If $d = 2$, we have $|b_2 - b_1| \geq 2r - 2 \geq 2 + r$, since $r \geq 4$. Let $m = 1$ in this case.

Let $U_1 = \phi^{-1}(U_0)$, and set $W_1 = \overline{D}(b_1, |b_m - b_1| + 1) \cap U_1$ and $W_2 = U_1 \setminus W_1$. Observe that $\operatorname{dist}(W_1, W_2) \geq C_d r$. Indeed, if $y_1 \in W_1$ and $y_2 \in W_2$, then $y_1 \in \overline{D}(b_i, 1)$ and $y_2 \in \overline{D}(b_j, 1)$ for some $1 \leq i \leq m$ and some $m+1 \leq j \leq d$; therefore

$$|y_2 - y_1| \geq |b_j - b_1| - |b_i - b_1| - 2 \geq |b_{m+1} - b_1| - |b_m - b_1| - 2 \geq C_d r.$$

Since W_1 contains m preimages of a and W_2 contains the other $d-m$, it follows that ϕ maps W_1 m-to-1 onto the connected set U_0, and it maps W_2 $(d-m)$-to-1 onto U_0.

Let $X_1 = W_1 \cap \mathcal{K}$, $X_2 = W_2 \cap \mathcal{K}$, $m_1 = m$, and $m_2 = d - m$. By the previous paragraph, X_1 and X_2 satisfy all of the mapping properties claimed in the Lemma. For any integer $i \geq 0$, define $g_i(z)$ and $h_i(z)$ as in the proof of Lemma 5.1. That is, for $i \geq 1$, write

$$\phi^i(z) - a = g_i(z) h_i(z),$$

where g_i is a monic polynomial of degree $m_1 d^{i-1}$ with all of its roots in W_1, and h_i is a monic polynomial of degree $m_2 d^{i-1}$ with all of its roots in W_2. For $i = 0$, define $g_0(z) = h_0(z) = z - a$. We will now compute the bounds given in the Lemma for X_1; the proof for X_2 is similar. Write $X = X_1$ and $m = m_1$.

As in the proof of Lemma 5.1, we may write any integer $j \geq 0$ as

$$j = c_0 + m(c_1 + c_2 d + c_3 d^2 + \cdots + c_M d^{M-1}),$$

with $c_0 \in \{0, 1, \ldots, m-1\}$, and with $c_i \in \{0, 1, \ldots, d-1\}$ for $i \geq 1$. Similarly, define

$$f_j(z) = \prod_{i=0}^{M} [g_i(z)]^{c_i},$$

which is clearly monic of degree j. As before, for any $x \in X$, we have $|\phi^i(x) - a| \leq r$. Similarly, the roots of h_i, which all lie in W_2 (for $i \geq 1$), are distance at least $C_d r$ from x. Thus, $|h_i(x)| \geq (C_d r)^{(d-m)d^{i-1}}$, and hence

$$|g_i(x)| \leq r (C_d r)^{-(d-m)d^{i-1}} = C_d^{-1} (C_d r)^{1-(d-m)d^{i-1}}$$

for all $i \geq 1$. Moreover, since $x \in U_0$, we have $|g_0(x)| \leq r$. We obtain

$$|f_j(x)| \leq r^{c_0} \prod_{i=1}^{M} C_d^{-c_i} (C_d r)^{c_i(1-(d-m)d^{i-1})} = C_d^{-(c_0 + c_1 + \cdots + c_M)} (C_d r)^e,$$

where $e = e(j, m, d)$ in the notation of Definition 3.1. The Lemma then follows by the van der Monde determinant argument of the proof of Lemma 4.1. □

Remark 6.4. Later, in the proof of Theorem 7.1, we will consider the quantity $C_d r$, rather than the radius r, at the archimedean primes. It is easy to prove that the lower bound for r given in Lemma 6.3 is guaranteed to hold provided $C_d r \geq 4 + \sqrt{3}$. (In fact, $4 + \sqrt{3}$ is the value of $C_d(\sqrt{3} + 2(d-1))/(\sqrt{3} - (d-1)C_d)$ at $d = 3$.) For $d = 2$, we also note the more obvious facts that $C_2 = 1$ and that the corresponding sufficient lower bound for $C_2 r$ is 4.

Remark 6.5. The bounds in Lemma 6.1 and Lemma 6.3 are not sharp. Besides the fact that most of the comments from Remarks 4.3 and 5.2 apply here, our geometric arguments could also be improved. For example, in the proof of Lemma 6.1, if we considered $\overline{D}(c, s) \cap D(y_0, t)$ instead of $\overline{D}(c, s) \cap D(y_0, |y_0 - c|)$, where $t = \sqrt{|y_0 - c|^2 + s^2}$, we could show that some b_i satisfies $|y_0 - b_i| \geq t$. Related arguments could show that two or more points b_i, b_j must make the product $|y_0 - b_i| \cdot |y_0 - b_j|$ larger than we proved. Similarly, it should be possible to increase the $\sqrt{3}$ factor to something closer to 2 by considering the geometric arrangement of the $\{b_i\}$ more delicately.

7. The Global Bound

At last, we are prepared to state and prove a precise version of the Main Theorem.

Theorem 7.1. *Let K be a global field, and let $\phi \in K[z]$ be a polynomial of degree $d \geq 2$. Let $s_\infty \geq 0$ be the number of archimedean primes of K, and let $s \geq s_\infty$ be the number of bad (i.e., not potentially good) primes of ϕ in M_K, including all archimedean primes.*

If K is a function field, i.e., if $s_\infty = 0$, let q be the size of the smallest residue field of a prime $v \in M_K$. If K is a number field, let $D = [K:\mathbb{Q}]$, and let

$$\sigma = \begin{cases} 7 & \text{if } d = 2, \\ \dfrac{2 \cdot 33^{(d-1)(d-2)}}{(d-1)(d-2)} & \text{if } d \geq 3. \end{cases}$$

Set

$$t = \begin{cases} s & \text{if } s_\infty = 0, \\ s - s_\infty & \text{if } s_\infty > 0 \text{ and } s \leq \sigma D, \\ s + \dfrac{D \log d}{4 \log 2} & \text{if } s_\infty > 0 \text{ and } s > \sigma D, \end{cases}$$

and

$$\beta = \begin{cases} 9 & \text{if } s_\infty > 0,\ s \leq \sigma D, \text{ and } d = 2, \\ \max\{11, 2d\} & \text{if } s_\infty > 0,\ s \leq \sigma D, \text{ and } d \geq 3, \\ 1 & \text{otherwise.} \end{cases}$$

Then ϕ has no more than $M + 1$ K-rational preperiodic points in $\mathbb{P}^1(K)$, where

$$M = \begin{cases} q & \text{if } s = s_\infty = 0, \\ \beta^D & \text{if } s = s_\infty > 0, \\ \beta^D (d^2 - 2d + 2)(t \log_d t + 3t) & \text{if } 0 < t < d, \\ \beta^D (d^2 - 2d + 2)(t \log_d t + t \log_d \log_d t + 3t) & \text{otherwise.} \end{cases}$$

Proof. For each prime $v \in M_K$, let $n_v \geq 1$ be the exponent so that the product formula (1) holds for all $x \in K^\times$. Let S be the (finite) set of primes of K of bad reduction of ϕ, including all the archimedean primes; that is, $\#S = s$. Let $a_d \in K$ be the lead coefficient of ϕ. For each prime $v \in M_K$, let $\mathcal{K}_v \subseteq \mathbb{C}_v$ denote the filled Julia set of ϕ in $\mathbb{C}_v$, let r'_v be the radius of the smallest disk in $\mathbb{C}_v$ containing $\mathcal{K}_v$, and let $r_v = |a_d|_v^{1/(d-1)} r'_v$.

For each non-archimedean prime v, let $R_v = r_v^{n_v}$. For each archimedean prime v, let $R_v = (C_d r_v)^{n_v}$, where $C_d = d^{-(d-2)/(d-1)} \leq 1$, as in the statement of Lemma 6.3. We consider four cases, some of which overlap with others.

Case 0. The simplest case is that K is a function field and $S = \emptyset$; that is, there are no archimedean primes, and all primes have potentially good reduction. Let $w \in M_K$ be a prime whose residue field has only q elements, and suppose that there are $q + 1$ distinct K-rational preperiodic points $\{x_1, \ldots, x_{q+1}\}$ besides the point at ∞.

By Lemma 2.5.c, we have $|x_i - x_j|_v \leq |a_d|_v^{-1/(d-1)}$ for every $v \in M_K$ and every $i, j \in \{1, \ldots, n\}$. Moreover, by the pigeonhole principle, there must be some distinct $i, j \in \{1, \ldots, n\}$ such that $|x_i - x_j|_w < |a_d|_w^{-1/(d-1)}$. Hence,

$$1 = \prod_{v \in M_K} |x_i - x_j|_v^{n_v} < \prod_{v \in M_K} \left[|a_d|_v^{-1/(d-1)} \right]^{n_v} = 1,$$

which is a contradiction. Thus, there are at most q finite K-rational preperiodic points.

Case 1. Choose $w \in M_K$ such that $R_w \geq R_v$ for all $v \in M_K$. (Such w exists because $R_v = 1$ for all but finitely many $v \in M_K$.) In this main case, we suppose that:

- $R_w > 1$.
- If K is a number field, then $R_w \geq 16$ and $s - s_\infty \geq 1$.
- If w is archimedean, then the lower bounds of Lemma 6.3 hold for r_w.

In particular, we may choose integers m_1, m_2 and sets $X_1, X_2 \subseteq \mathcal{K}_w$ for ϕ according to Lemma 5.1 (if w is non-archimedean) or Lemma 6.3 (if w is archimedean).

For each index $k = 1, 2$, set

$$A_k = \frac{d - m_k}{m_k(d-1)}, \qquad B_k = 1 - \log_d m_k, \qquad \text{and} \qquad N_k = M(A_k, B_k, t),$$

where $M(\cdot, \cdot, \cdot)$ is as in Lemma 3.5, and where t is as in the statement of the Theorem. We claim that there are fewer than N_k K-rational preperiodic points in X_k.

To prove the claim, fix $k = 1, 2$, and let $m = m_k$, $A = A_k$, $B = B_k$, and $N = N_k$. Suppose there are N distinct K-rational preperiodic points $x_1, \ldots, x_N$ in X_k. Then by the product formula applied to both $\prod_{i \neq j}(x_i - x_j)$ and a_d,

$$\begin{aligned} 1 &= \prod_{v \in M_K} \left| \prod_{i \neq j} (x_i - x_j) \right|_v^{n_v} = \prod_{v \in M_K} \left[|a_d|_v^{N(N-1)/(d-1)} \prod_{i \neq j} |x_i - x_j|_v \right]^{n_v} \\ &\leq \prod_{v \in S} \left[|a_d|_v^{N(N-1)/(d-1)} \prod_{i \neq j} |x_i - x_j|_v \right]^{n_v}, \end{aligned} \tag{7}$$

where the inequality is because $|x - y|_v \leq |a_d|_v^{-1/(d-1)}$ for all $v \in M_K \setminus S$ and $x, y \in \mathcal{K}_v$, by Lemma 2.5.c, and because $x_1, \ldots, x_N \in \mathcal{K}_v$ for every $v \in M_K$.

If w is non-archimedean, then by Lemma 4.1 and Lemma 5.1, (7) becomes

$$1 \leq N^{DN} r_w^{n_w E(N,m,d)} \prod_{v \in S \setminus \{w\}} r_v^{n_v E(N,d)} = N^{DN} C_d^{-DE(N,d)} R_w^{E(N,m,d)} \prod_{v \in S \setminus \{w\}} R_v^{E(N,d)},$$

where we set $D = 0$ if K is a function field. (The appearance of D in the exponent comes from equation (2).) Because $R_w \geq R_v$ and $E(N,d) \geq 0$, we can replace each R_v by R_w; and because $R_w, C_d^{-1} \geq 1$, we can apply Lemma 3.4.a–b to obtain

$$1 \leq N^{DN} C_d^{-(d-1)DN \log_d N} R_w^{(d-1)N[s \log_d N - AN + B]}. \tag{8}$$

Similarly, if w is archimedean, then by Lemma 4.1 and Lemma 6.3, (7) becomes

$$\begin{aligned} 1 &\leq N^{DN} C_d^{-n_w F(N,M,d)} (C_d r_w)^{n_w E(N,m,d)} \prod_{v \in S \setminus \{w\}} r_v^{n_v E(N,d)} \\ &= N^{DN} C_d^{-(D-n_w)E(N,d) - n_w F(N,m,d)} R_w^{E(N,m,d)} \prod_{v \in S \setminus \{w\}} R_v^{E(N,d)}. \end{aligned}$$

Replacing each R_v by R_w as before, and applying Lemma 3.4a–c, we obtain exactly inequality (8) once more.

Meanwhile, we compute

$$N^{DN} C_d^{-(d-1)DN \log_d N} = d^{DN \log_d N} d^{(d-2)(DN \log_d N)} = d^{(d-1)DN \log_d N}. \tag{9}$$

If K is a number field, our assumption that $R_w \geq 16$ means that $d \leq R_w^{1/\log_d 16}$. Combining (8) and (9), then, we obtain

$$1 \leq R_w^{(d-1)N[t \log_d N - AN + B]}, \tag{10}$$

where $t = s + D \log d/(4 \log 2)$, as in the statement of the Theorem. The same inequality follows for function fields with $t = s$, since $D = 0$ in that case. By our definitions of A, B, and t, the hypotheses of Lemma 3.5 hold. Thus, by that Lemma and our choice of N, we have $t \log_d N - AN + B < 0$, so that $1 < 1$, which is a contradiction, proving the claim that there are fewer than N_k K-rational preperiodic points in X_k. (However, since N_k need not be an integer, we cannot claim that there are at most $N_k - 1$ such points.)

The total number of finite K-rational preperiodic points is the number in X_1 plus the number in X_2. That is, there are fewer than $N_1 + N_2$ such points. That upper bound is

$$N_1 + N_2 = M(A_1, B_1, t) + M(A_2, B_2, t). \tag{11}$$

From the definition of $M(A, B, t)$ in Lemma 3.5, it is easy to check that, as m_1 varies from 1 to $d-1$, the largest value of $N_1 + N_2$ in equation (11) is attained at $m_1 = 1$ and $m_2 = d-1$ (or vice versa). In that case, the bound is

$$N_1 + N_2 = (d^2 - 2d + 2) \left(t \log_d t + t \log_d(\max\{1, \log_d t\}) + 3t \right).$$

Adding 1 for the point at ∞, we obtain the bound stated in the Theorem, with $\beta = 1$.

Case 2. Next, suppose that K is a number field and $d = 2$. Write S_∞ for the set of archimedean primes of M_K, and let $s_\infty = \#S_\infty$. We will remove the archimedean primes from the picture by covering the filled Julia set at each such prime $v \in S_\infty$ by at most 9^{n_v} disks of diameter less than $|a_d|_v^{-1}$. To simplify notation, let $\mathcal{K}'_v = a_d \mathcal{K}_v$; we wish to cover $\mathcal{K}'_v$ by disks of diameter less than 1.

For any real prime $v \in S_\infty$, the set $\mathcal{K}'_v$ is contained either in a single interval of length 6 or in two intervals of length less than 2, by Lemma 6.1 and Remark 6.2. (In fact, the bound of 6 could be reduced to 4, but we will not need that stronger bound here.) In particular, $\mathcal{K}'_v$ is contained in a union of seven or fewer intervals of length strictly less than 1.

For a complex prime $v \in S_\infty$, the same Lemma implies that $\mathcal{K}'_v$ is contained either in a single disk of radius 3 or in two disks of radius less than 1. Each disk of radius 1 can easily be covered by nine disks of diameter slightly less than 1. Similarly, the disk of radius 3 can be covered by a square of side length 6. That square can then be divided into 81 squares of side length 2/3, each of which fits inside a disk of diameter less than 1.

Scaling back by $|a_d|_v^{-1}$, then, we have at each archimedean prime $v \in S_\infty$ at most 9^{n_v} disks of diameter less than $|a_d|_v^{-1}$ which together cover $\mathcal{K}_v$, as promised. In total, then, we have at most 9^D choices of one disk for each archimedean prime.

For any such choice $\mathbb{D} = \{D_v : v \in S_\infty\}$ of one disk of diameter less than $|a_d|_v^{-1}$ for each archimedean prime v, let $\mathcal{P}_\mathbb{D}$ denote the set of K-rational preperiodic points x for which $x \in D_v$ for every $v \in S_\infty$. We will bound the size of $\mathcal{P}_\mathbb{D}$.

If $S = S_\infty$, then each set $\mathcal{P}_\mathbb{D}$ can contain at most one point. Indeed, if there were distinct points $x, y \in \mathcal{P}_\mathbb{D}$, then

$$1 = \prod_{v \in M_K} |x - y|_v^{n_v} < \prod_{v \in M_K} \left[|a_d|_v^{-1} \right]^{n_v} = 1,$$

by Lemma 2.5.c, with the strict inequality coming from the fact that the diameter at each archimedean prime is strictly less than $|a_d|_v^{-1}$. Since there are 9^D choices of $\mathbb{D}$, there are at most 9^D finite K-rational preperiodic points.

On the other hand, if $S \supsetneq S_\infty$, then choose $w \in M_K \setminus S_\infty$ such that $R_w \geq R_v$ for all $v \in M_K \setminus S_\infty$. By Lemma 2.5.c, $r_w > 1$, so that we may apply Lemma 5.1 at w.

Now fix $\mathbb{D}$ and follow the argument of Case 1, but restricted to $\{x_i\} \subseteq \mathcal{P}_\mathbb{D}$. At each archimedean prime $v \in S_\infty$ we have $|x_i - x_j|_v \leq |a_d|_v^{-1}$. Therefore, by Lemmas 4.1, 5.1, and 3.4.a–b, inequality (7) becomes

$$1 \leq R_w^{(d-1)N[t \log_d N - AN + B]},$$

where $t = s - s_\infty$. Following the rest of the argument of Case 1 (from inequality (10) on), and multiplying by 9^D (the number of choices $\mathbb{D}$), we obtain the desired bounds.

Case 3. If K is a number field and $d \geq 3$, we proceed roughly as in Case 2. Again, write S_∞ for the set of archimedean primes of M_K, let $s_\infty = \# S_\infty$, and let $\mathcal{K}'_v = \alpha \mathcal{K}_v$, where $\alpha^{d-1} = a_d$. This time, we will cover $\mathcal{K}'_v$ by at most β^{n_v} disks of diameter less than 1, where $\beta = \max\{11, 2d\}$.

For a real prime $v \in S_\infty$, Lemma 6.1 and Remark 6.2 imply that $\mathcal{K}'_v$ is contained either in a single interval of length $4 + 2\sqrt{3}$ or in d intervals of length less than 2. In particular, $\mathcal{K}'_v$ is contained in a union of $\max\{8, 2d\} \leq \beta$ or fewer intervals of length less than 1.

For a complex prime $v \in S_\infty$, the same Lemma implies that $\mathcal{K}_v$ is contained either in a single disk of radius $2+\sqrt{3}$ or in d disks of radius less than 1. As before, each disk of radius 1 can be covered by nine disks of diameter less than 1. Similarly, the disk of radius $2 + \sqrt{3}$ can be covered by a square of side length $4 + 2\sqrt{3}$. That square can be divided into 121 squares of side length $(4 + 2\sqrt{3})/11$, each of which fits inside a disk of diameter less than 1. (In fact, using a hexagonal tiling, one could cover the big disk by 84 disks of diameter less than 1, but the messy proof gives only a minor improvement over 121.) Thus, $\mathcal{K}'_v$ can be covered by a union of $\max\{121, 9d\} \leq \beta^2$ disks of diameter less than 1.

The rest of Case 3 then follows Case 2, with β^D in place of 9^D. This completes our analysis of the four cases.

Final step. If K is a function field, we are done; indeed, by Lemma 2.5.c, Cases 0 and 1 cover all the possibilities.

If K is a number field, we will now show that for $s > \sigma D$, we are automatically in Case 1. Because $n_v \leq 2$ for an archimedean prime v, and by Remark 6.4, we need only show there is some $w \in M_K$ such that $R_w \geq 4^2$ if $d = 2$, or such that $R_w \geq (4 + \sqrt{3})^2$ if $d \geq 3$.

From basic algebraic number theory, there are at most D primes of K above any given prime of $\mathbb{Q}$. Given an integer $m \geq 1$, let p_m denote the m^{th} prime in $\mathbb{Q}$. (That is, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on.) Thus, if $s - s_\infty > D(m-1)$, there must be some $w \in S \setminus S_\infty$ lying above a prime $p \geq p_m$ of $\mathbb{Q}$. Since $D \geq s_\infty$, we get such a w provided $s > mD$.

Meanwhile, by Lemma 2.8, given $w \in S \setminus S_\infty$ lying over a prime p of $\mathbb{Q}$, we have

$$R_w \geq |\pi_w|_w^{-n_w} \geq p \qquad \text{if } d = 2,$$

or

$$R_w^{(d-1)(d-2)} \geq |\pi_w|_w^{-n_w} \geq p \qquad \text{if } d \geq 3,$$

where π_w is a uniformizer at w. (The Lemma applies because if $\mathcal{K}_w \cap K = \emptyset$, then there are no finite K-rational preperiodic points at all, and the conclusion of the Theorem is trivial.) For $d = 2$, then, the condition $R_w \geq 16$ is guaranteed provided $s \geq 7D + 1$, since 17 is the seventh prime of $\mathbb{Q}$. Thus, $s > 7D = \sigma D$ suffices for $d = 2$.

For $d \geq 3$, the elementary estimate in Theorem 4.7 of [2] says that $p_m > (1/6) m \log m$ for any integer $m \geq 1$. It is easy to check that $m = \lfloor \sigma \rfloor$ satisfies $m \log m \geq 6(4+\sqrt{3})^{2(d-1)(d-2)}$, where $\sigma = 2 \cdot 33^{(d-1)(d-2)}/[(d-1)(d-2)]$ as in the statement of the Theorem. (The 33 appears because it is the smallest integer larger than $(4+\sqrt{3})^2$.) Thus, $s > \sigma D$ implies $R_w \geq (4+\sqrt{3})^2$, once again forcing Case 1. □

Remark 7.2. If, for a given polynomial ϕ, we know that we are in Case 1 (say, by inspection of the filled Julia set at one prime), then we can set $\beta = 1$ in the statement of the Theorem, even if $s \leq \sigma D$. In particular, for a fixed function ϕ, the conditions of Case 1 are preserved if one passes to a finite extension of K. Thus, one would not have to worry about the growth of s relative to σD as one traveled up a tower of number fields, even though one cannot expect s to increase as fast as σD in general.

Remark 7.3. Our covering methods in Cases 2 and 3 are rather crude, and it should be possible to cover $\mathcal{K}_v \subseteq \mathbb{C}$ more efficiently. For example, instead of disks of diameter 1, one could use larger sets Y for which $\prod_{1 \leq i,j \leq L} |y_i - y_j|_v \leq 1$ for some fixed small integer L. Such a covering should improve the coefficient β^D in the final bound.

Even without any extra work, the coefficient can be reduced in special cases. For example, if K is a totally real number field, then the cutoff σ would be much smaller, since we would only need $R_w \geq 4$ (if $d = 2$), or $R_w \geq 4 + \sqrt{3}$ (if $d \geq 3$), rather than 4^2 or $(4+\sqrt{3})^2$. Moreover, if K is totally real and $d = 2$, then each archimedean $\mathcal{K}_v \cap \mathbb{R}$ is contained in a union of four intervals of length 1. (See, for example, Lemma 6.4 and Proposition 6.6 of [9].) Thus, the coefficient $\beta^D = 9^D$ could be replaced by 4^D, with one exception.

The one exception is if all non-archimedean primes have good reduction and the archimedean filled Julia set is an interval of length 4. This occurs for the Chebyshev polynomial $\phi(z) = z^2 - 2$, which has filled Julia set $[-2, 2]$. In this special case, after removing the points ∞ and 2, the rest of the preperiodic points can be covered by four half-open intervals of length 1 at each archimedean prime. Since there are no non-archimedean bad primes, we obtain a bound of $2 + 4^D$ for the total number of preperiodic points in $\mathbb{P}^1(K)$.

Remark 7.4. Another approach to finding a cutoff σ which forces Case 1 would be to consider the set $T \subseteq S$ consisting of non-archimedean bad primes v at which there are actually K-rational preperiodic points x, y for which $|x - y|_v > |a_d|_v^{-1/(d-1)}$. For such primes, the exponent of $-1/[(d-1)(d-2)]$ in Lemma 2.8 could be improved to $-1/(d-1)$.

Unfortunately, there may not be very many such primes. As a result, although the exponent of $(d-1)(d-2)$ in the definition of σ could be improved to $(d-1)$, it would come at the expense of introducing a extra factor like β^D into the formula for σ.

Remark 7.5. For large degrees d, one can obtain slightly smaller bounds by using more than one big bad prime w. There is, of course, a trade-off. While using $\ell \geq 2$ big primes w ultimately increases the coefficient A of $-N$ in the exponent of (8), it also increases the number of pieces $\{X_k\}$ from 2 to 2^ℓ. It appears that the optimal number of such primes to use is $\ell \approx 2\log_2(d-1)$. The improved bound for the number of rational preperiodic points would be roughly the old bound divided by $2\log_2(d-1)$, for large d. However, the proof would be vastly more complicated, especially in dealing with the archimedean primes, and it would give only a slight improvement in the bound.

We close by presenting a slight strengthening of Theorem 7.1 in the simplest case.

Example 7.6. Let $K = \mathbb{Q}$ (so that $D = s_\infty = 1$, and $n_v = 1$ for all $v \in M_\mathbb{Q}$) and $d = 2$. That is, we wish to bound the number of rational preperiodic points of a quadratic polynomial $\phi \in \mathbb{Q}[z]$. It is of course well known that any such polynomial is conjugate over $\mathbb{Q}$ to one of the form $\phi_c(z) = z^2 + c$, with $c \in \mathbb{Q}$.

Let us suppose that ϕ_c has at least one preperiodic point in $\mathbb{Q}$. This supposition implies that $c = j/m^2$ for some relatively prime integers $j, m \in \mathbb{Z}$, and that $-\infty < c \leq 1/4$; see, for example, Proposition 6.7 of [9]. (One can also easily establish that j must satisfy one of approximately 2^s congruences modulo m, but we do not need that here.) For non-archimedean primes v of $\mathbb{Z}$, we have $R_v = |m|_v^{-1}$ if v is odd, and $R_2 = \max\{|m/2|_2^{-1}, 1\}$. (Note that if $4 \nmid m$, then ϕ_c has good reduction at $v = 2$, after a change of coordinates.) In addition, for $c < 0$, we have $R_\infty = (1 + \sqrt{1-4c})/2$.

By Remark 7.3, the β^D coefficient becomes 1 if there is some prime v with $R_v \geq 4$. Still assuming that there is at least one preperiodic point in $\mathbb{Q}$, Lemma 2.8 says that such a prime must exist unless the only bad primes are ∞, 2, and 3, and $R_2, R_3, R_\infty < 4$. By our characterization of R_v above, this means that the denominator m is a divisor of 12, and that $-12 < c \leq 1/4$. There are only finitely many rational numbers of the form $c = j/144$ between -12 and $1/4$, and a simple computer search shows none of the corresponding polynomials ϕ_c has more than eight preperiodic points in $\mathbb{Q}$. (For five such values of c, namely $-21/16$, $-29/16$, $-91/36$, $-133/144$, and $-1333/144$, there are exactly eight preperiodic points in $\mathbb{Q}$. Incidentally, there are infinitely many values $c \in \mathbb{Q}$ for which ϕ_c has at least eight preperiodic point in $\mathbb{Q}$, by Theorem 2 of [28].)

For all other c, we are essentially in Case 1, except that we can only assume $R_w \geq 4$, rather than $R_w \geq 16$. The only effect this has on the proof of Case 1 is to change the value of t in inequality (10) to $t = s + D\log d/(2\log 2) = s + 1/2$. If $s = 1$, then only the archimedean prime is bad, and in light of Remark 7.3, there are at most five preperiodic points in $\mathbb{Q}$; in fact, there are at most four for $s = 1$ and $c \neq -2$. The only remaining possibility is that $s \geq 2$, in which case the number of preperiodic points in $\mathbb{Q}$ is at most

$$(2s+1)\left[\log_2(2s+1) + \log_2(\log_2(2s+1) - 1) + 2\right].$$

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