one gets $(I + T)u = f$. To complete the proof, let us establish that $\sup_j \|u_j\| < \infty$. Assuming that this is false, one can choose a subsequence, denoted by u_j again, such that $\|u_j\| > j$. Let $z_j := u_j/\|u_j\|$. Then $\|z_j\| = 1$, z_j is orthogonal to $N(I + T)$, and $z_j + Tz_j = f_j/\|u_j\| \to 0$. As before, it follows that $z_j \to z$ in H, and passing to the limit in the equation for z_j one gets $z + Tz = 0$. Since z is orthogonal to $N(I + T)$, it follows that $z = 0$. This is a contradiction since $\|z\| = \lim_{j\to\infty} \|z_j\| = 1$. This contradiction proves the desired estimate and the proof is completed.

This proof is valid for any compact linear operator T. If T is a finite-rank operator, then the closedness of $R(I + T)$ follows from a simple observation: finite-dimensional linear spaces are closed.

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An Elementary Product Identity in Polynomial Dynamics

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1. A SURPRISING IDENTITY. In this article we present a surprising but completely elementary product formula concerning periodic cycles in the iteration of polynomials. A list of complex numbers $\{x_1, \ldots, x_n\}$ is said to be a *cycle of length n* for a polynomial f if all the x_i's are distinct and if

$$x_2 = f(x_1), \quad x_3 = f(x_2), \quad \ldots \quad x_n = f(x_{n-1}), \quad \text{and } x_1 = f(x_n).$$

Thus, repeatedly evaluating $f(z)$ at the points $x_1, \ldots, x_n$ causes them to cycle.

The simplest case of our identity concerns cycles in the much-studied family of quadratic polynomials of the form $f(z) = z^2 + c$.

Theorem 1. *Let $c \in \mathbb{C}$ be given, and let $f(z) = z^2 + c$. Let $n \geq 2$ be an integer, and let $\{x_1, \ldots, x_n\}$ be a cycle of length n. Then*

$$\prod_{i=1}^{n} (f(x_i) + x_i) = 1.$$

This result was originally suggested by experimental computations using PARI/GP. Given the numbers that tend to arise along the way, it was quite unexpected, as the following example helps to illustrate.

Example 1. Consider $f(z) = z^2 - 5$, and let $n = 3$. There are two cycles of length 3, each consisting of roots of the polynomial

$$\frac{f \circ f \circ f(z) - z}{f(z) - z} = z^6 + z^5 - 14z^4 - 9z^3 + 61z^2 + 16z - 79.$$

In particular, $\{x_1, x_2, x_3\}$ is such a cycle, where

$$x_1 = -2.74766489\ldots, \quad x_2 = 2.54966236\ldots, \quad \text{and} \quad x_3 = 1.50077817\ldots.$$

A computation gives

$$\begin{aligned} s_1 &= x_1 + x_2 = -0.19800253\ldots \\ s_2 &= x_2 + x_3 = 4.05044053\ldots \\ s_3 &= x_3 + x_1 = -1.24688672\ldots, \end{aligned}$$

and $s_1 s_2 s_3 = 1$, as predicted by Theorem 1.

In spite of the messy numbers that can arise in practice, the proof of the theorem is short and simple.

Proof of Theorem 1. For any distinct $u, v \in \mathbb{C}$, we have

$$u + v = \frac{u^2 - v^2}{u - v} = \frac{(u^2 + c) - (v^2 + c)}{u - v} = \frac{f(u) - f(v)}{u - v}. \tag{1}$$

By hypothesis, $x_{i+1} \neq x_i$ for every i. In particular, if we denote $f(x_n)$ by x_{n+1}, then

$$\prod_{i=1}^{n}(x_{i+1} + x_i) = \prod_{i=1}^{n} \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{\prod_{i=1}^{n}(f(x_{i+1}) - x_{i+1})}{\prod_{i=1}^{n}(f(x_i) - x_i)} = 1,$$

where the first equality is by (1) and the last is by observing that the product in the numerator has exactly the same terms as in the denominator, but in rotated order. ∎

Our goal is to formulate and prove a generalization (Theorem 2) for all monic polynomials over any algebraically closed field. We also present some implications for algebraic units in Theorem 3.

The following terminology and notation is standard. Let K be an algebraically closed field, and let $f(z) \in K[z]$ be a polynomial with coefficients in K. For a positive integer n, we denote the n-th iterate of f by f^n; that is, $f^1 = f$, $f^2 = f \circ f$, and in general,

$$f^n = f \circ f \circ \cdots \circ f.$$

The set $\{f^n\}$ is the (*discrete*) *dynamical system* of f. A point $x \in K$ is *fixed* if $f(x) = x$. More generally, $x \in K$ is *periodic* (of *period* n) if $f^n(x) = x$ for some $n \geq 1$. If x is periodic, then the *minimal period* of x is the smallest positive integer n with $f^n(x) = x$. If x is periodic of (minimal) period n, the set

$$\{x, f(x), f^2(x), \ldots, f^{n-1}(x)\}$$

is called a *cycle* of (minimal) period n, or of length n. Finally, a point x is *preperiodic* if $f^m(x)$ is periodic for some m.

For introductory reading on dynamical systems, see [**1**] or [**3**].

2. A GENERALIZATION. Theorem 1 is surprising (at least to the author) for two reasons. First of all, it involves the sum of a number x and its image $f(x)$. For an arbitrary point x, such a sum generally has no particular significance. Indeed, a change of coordinate $z \mapsto h(z)$ (for an invertible function h) corresponds to conjugating the original map f to $h \circ f \circ h^{-1}$. Then the sum in the new coordinate would be $h(f(x)) + h(x)$, which is usually not the same as $f(x) + x$ or even $h(f(x) + x)$. Normally, one would expect dynamical results to be independent of coordinate; but Theorem 1, in its current form, is not.

However, although sums do not behave well under coordinate changes, differences are another story. If $h(z)$ is an analytic change of coordinates (and therefore a bijective map from $\mathbb{C}$ to itself), then it must be of the form $h(z) = az + b$. If we restrict further and allow only translations $h(z) = z + b$, then it *is* true that $h(u) - h(v) = u - v$. Such restrictions on h are absolutely necessary if we require that f be a monic polynomial, even after the coordinate change. From this viewpoint, we should consider the product term of Theorem 1 to be $f(x_i) - (-x_i)$. The only snag is that, like summation, multiplication by -1 is not respected by h. To gain a proper understanding of Theorem 1, then, we need a dynamical, rather than algebraic, understanding of the quantity $-x_i$. The solution is to recognize that $-x_i$, while not a point in the periodic cycle, is preperiodic. More precisely, it is the other point (besides x_i) in the pre-image $f^{-1}(f(x_i))$.

The second reason Theorem 1 is surprising is that it need not hold for $n = 1$, which at first seems a strange exception. However, if we view the two quantities x_i and $f(x_i)$ merely as two periodic points (and momentarily forget that one is the image of the other), the $n \neq 1$ condition tells us that the two points are distinct. It turns out that they do not need to be in the same periodic cycle, nor even have the same minimal period. However, given any two periodic points x and w, we can always pick an integer n that is a period for both; any positive integer divisible by both minimal periods will do. With these ideas in mind, we can now generalize Theorem 1.

Theorem 2. *Let K be an algebraically closed field, let $f(z) \in K[z]$ be a monic polynomial of degree d, and let $x_1, w_1 \in K$ be distinct periodic points of f. Let*

$$w_i = f^{i-1}(w_1) \quad \text{and} \quad x_i = f^{i-1}(x_1) \quad \text{for } i = 2, 3, 4, \ldots.$$

Let n be an integer that is a period for both x_1 and w_1. For each $i = 1, \ldots, n$, let $y_{i,1}, y_{i,2}, \ldots, y_{i,d}$ be the d pre-images (repeated according to multiplicity) of x_{i+1} under f; we specify that $y_{i,d} = x_i$. Then

$$\prod_{i=1}^{n} \left[\prod_{j=1}^{d-1} (w_i - y_{i,j}) \right] = 1.$$

The two periodic points x_1 and w_1 need not be in the same cycle. They can even both be fixed points, as long as they are distinct.

The proof of Theorem 2 is short and elementary. The deepest fact involved is the Fundamental Theorem of Algebra, which allows us to describe precisely a monic polynomial when we know only the set of roots (with multiplicities).

Proof of Theorem 2. The product is vacuously 1 if $d < 2$, because then there are no terms on the left side. Therefore, we assume that $d \geq 2$.

For any fixed $i = 1, \ldots, n$, let $p_i(z) = f(z) - x_{i+1} \in K[z]$. Then $y_{i,1}, \ldots, y_{i,d}$ is a complete list of all the roots (counting multiplicity) of $p_i(z) = 0$. Furthermore, p_i is a monic polynomial. Thus, by the Fundamental Theorem of Algebra,

$$f(z) - x_{i+1} = p_i(z) = \prod_{j=1}^{d} \left(z - y_{i,j}\right).$$

Evaluating at $z = w_i$, we get

$$f(w_i) - x_{i+1} = \prod_{j=1}^{d} (w_i - y_{i,j}). \tag{2}$$

Recall that $f(w_i) = w_{i+1}$ and $y_{i,d} = x_i$ by definition. Additionally, $w_i \neq x_i$, because $f^{n+1-i}(w_i) = w_1 \neq x_1 = f^{n+1-i}(x_i)$. Dividing both sides of (2) by $(w_i - x_i)$ gives

$$\frac{w_{i+1} - x_{i+1}}{w_i - x_i} = \prod_{j=1}^{d-1} (w_i - y_{i,j}). \tag{3}$$

Now take the product of (3) over all $i = 1, \ldots, n$. Since $x_{n+1} = x_1$ and $w_{n+1} = w_1$ by definition, we get

$$\prod_{i=1}^{n} \left[\prod_{j=1}^{d-1} (w_i - y_{i,j}) \right] = \prod_{i=1}^{n} \frac{w_{i+1} - x_{i+1}}{w_i - x_i} = 1$$

because the product telescopes. ∎

3. DYNAMICAL UNITS. We now turn to some more advanced implications in algebraic number theory. Let R be an integral domain with fraction field L, let K be an algebraic closure of L, and let $f(z) \in R[z]$ be a polynomial with coefficients in R. If f is monic, any root of f in K is called an *algebraic integer*. The set of algebraic integers forms an integral domain $\mathcal{O}$ that contains R; $\mathcal{O}$ is called the *integral closure* of R in K. Elements of the unit group $\mathcal{O}^*$ are called, naturally, *algebraic units*. See [**2**, VII] for an introduction to algebraic integers.

Our product identity can be used to produce algebraic units from any two distinct periodic points of a monic polynomial. If $f(z)$ is a monic polynomial with integer coefficients, then all the periodic and preperiodic points are roots of monic polynomials with integer coefficients, and so they are algebraic integers over $\mathbb{Z}$. (For example, $f^2(z) - z$ is a monic polynomial with integer coefficients, and its roots are precisely the points of period 2.) The sum, difference, or product of two such (pre)periodic points is therefore also integral. Applying this information to the product in Theorem 2, we have the following general result.

Theorem 3. *Let R be an integral domain with field of fractions L, let K be an algebraic closure of L, and let $\mathcal{O}$ be the integral closure of R in K. Let $g(z) \in R[z]$ be a monic polynomial, and let $x, w \in K$ be distinct periodic points of g. Let m be a positive integer, and let $y \in K$ satisfy $g^m(y) = g^m(x)$. If $y = x$ and x is a critical point of g^m, or if $y \neq x$, then $w - y \in \mathcal{O}^*$.*

Example 2. Let w and x be any two distinct roots of

$$z^6 + z^5 - 14z^4 - 9z^3 + 61z^2 + 16z - 79.$$

Then $w + x$ is an algebraic unit. This is because, as we saw in Example 1, w and x are distinct periodic points of $f(z) = z^2 - 5$. If we let $y = -x$, then $y \neq x$ but $f(y) = f(x)$, so Theorem 3 applies with $m = 1$.

Proof of Theorem 3. Consider the monic polynomial $f = g^m$, and let $d = \deg f$. Our hypotheses tell us that $x_1 = x$ and $w_1 = w$ are distinct periodic points of f, and that $f(y) = f(x)$. In the notation of Theorem 2, let $\{y_{1,j}\}_{j=1}^{d}$ be the elements of $f^{-1}(f(x_1))$, counting multiplicity, with $y_{1,d} = x_1$. So $y = y_{1,j}$ for some $j = 1, \ldots, d$. If $y \neq x$, then $j \leq d - 1$. If $y = x$, then our hypotheses require that x is a critical point of f; therefore, x is a multiple root of $f(z) - f(x)$, and so we can assume $j \leq d - 1$.

In particular, $w - y$ is one of the terms that appears in the product identity of Theorem 2. We claim that every term in that identity is in $\mathcal{O}$; since the product is 1, it follows that every term in the product, including $w - y$, is a unit in $\mathcal{O}$.

It suffices to show that every w_i and $y_{i,j}$ is in $\mathcal{O}$. To do this, first note that f must have degree $d \geq 2$, because earlier we were able to choose a positive integer smaller than d. Now each w_i is a root of the polynomial $f^n(z) - z = 0$, which is a monic polynomial with coefficients in $\mathcal{O}$; and each $y_{i,j}$ is a root of the polynomial $f(z) - x_i = 0$, which is also monic with coefficients in $\mathcal{O}$. ■

In [**4**], Morton and Silverman proved a related result: If R is a Dedekind domain, if w and x are periodic points of g in *distinct cycles*, and if neither minimal period divides the other, then $w - x$ is a unit. Their proof proceeds by showing that $w - x$ has valuation 0 at every prime. On the other hand, our Theorem 3 allows w and x to be in either the same or different cycles, though the resulting unit involves not x but a pre-image y of x. It is also somewhat more general in that it applies over any integral domain, and the inverse of $w - y$ can be described in terms of similar dynamical quantities (namely, the remaining terms in the product identity).

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