

Examples of Wavelets for Local Fields

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ABSTRACT. It is well known that the Haar and Shannon wavelets in $L^2(\mathbb{R})$ are at opposite extremes, in the sense that the Haar wavelet is localized in time but not in frequency, whereas the Shannon wavelet is localized in frequency but not in time. We present a rich setting where the Haar and Shannon wavelets coincide and are localized both in time and in frequency.

More generally, if \mathbb{R} is replaced by a group G with certain properties, J. Benedetto and the author have proposed a theory of wavelets on G , including the construction of wavelet sets [2]. Examples of such groups G include the p -adic rational group $G = \mathbb{Q}_p$, which is simply the completion of \mathbb{Q} with respect to a certain natural metric topology, and the Cantor dyadic group $\mathbb{F}_2((t))$ of formal Laurent series with coefficients 0 or 1.

In this expository paper, we consider some specific examples of the wavelet theory on such groups G . In particular, we show that Shannon wavelets on G are the same as Haar wavelets on G . We also give several examples of specific groups (such as \mathbb{Q}_p and $\mathbb{F}_p((t))$, for any prime number p) and of various wavelets on those groups. All of our wavelets are localized in frequency; the Haar/Shannon wavelets are localized both in time and in frequency.

One of the principal goals of wavelet theory has been the construction of useful orthonormal bases for $L^2(\mathbb{R}^d)$. The group \mathbb{R}^d has been an appropriate setting both because of its use in applications and because of its special property of containing lattices such as \mathbb{Z}^d which induce discrete groups of translation operators on $L^2(\mathbb{R}^d)$. Of such wavelets, the easiest to describe are the Haar wavelet and the Shannon wavelet in $L^2(\mathbb{R})$. The Haar is a compactly supported step function in time but has noncompact support and slow decay in frequency. On the other hand, the Shannon is considered to be at the opposite extreme, being a compactly supported step function in frequency but with noncompact support and slow decay in time.

This distinction between Haar and Shannon wavelets appears to be caused by certain properties of the topological group \mathbb{R}^d . J. Benedetto and the author [2] have presented a theory of wavelets on $L^2(G)$, for groups G with different properties to be described in Section 2 below. In this paper, we shall present some of that theory and some examples. Our main result, Theorem 4.3, is the surprising fact that Haar and Shannon wavelets not only exist in $L^2(G)$, but the two are the same. Moreover,

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these Haar/Shannon wavelets are localized both in time and in frequency; in fact, both the wavelets and their transforms are compactly supported step functions.

To reach that goal, we must first generalize wavelet theory for \mathbb{R}^d to a more abstract setting. A number of authors have extended wavelet theories to locally compact groups G other than \mathbb{R}^d ; see, for example, [7, 10, 12, 15, 19, 21]. However, those generalizations required the group G to contain a discrete subgroup, just as \mathbb{R}^d contains the discrete lattice \mathbb{Z}^d . The more abstract generalized multiresolution analyses in [1, 22] are even broader, though the discrete group of translation operators is, naturally, a requirement and indeed one of the defining characteristics of a wavelet theory. Indeed, the standard constructions of multiresolution analysis and of wavelet sets in $L^2(\mathbb{R}^d)$ rely crucially on a discrete group of translation operators (as can be seen in expositions and investigations such as [3, 4, 9, 8, 20, 26]).

Beyond the identification of Haar and Shannon wavelets, we are also motivated by the study of analysis on local fields, which are of great importance in number theory; see [11, 16, 24, 25] for background. The study of Fourier analysis on local fields and on adèle groups arises in class field theory and in the study of certain zeta functions; see [5, 6, 23, 27], for example. However, no prior knowledge of such topics are needed to read this paper.

One simple example of a local field we wish to consider is the field $\mathbb{F}_p((t))$ of formal Laurent series over the field \mathbb{F}_p of p elements, where p is a prime; see Examples 2.5, 3.4, and 5.4. As an additive group, $\mathbb{F}_p((t))$ contains a compact open subgroup $\mathbb{F}_p[[t]]$ (consisting of formal power series) as well as a discrete subgroup Γ , with $\mathbb{F}_p((t)) = \mathbb{F}_p[[t]] + \Gamma$. Using the elements of Γ as translations, one can study wavelets in $L^2(\mathbb{F}_p((t)))$. For example, Lang [18] examined the MRA theory in the case $p = 2$; we shall consider this case briefly in Section 1 to motivate the generalizations to follow.

However, while some of the groups G we wish to consider contain a discrete lattice, others do not. For example, the p -adic field \mathbb{Q}_p , which we shall study in Examples 2.4, 3.3, 5.2, and 5.5, is a completion of \mathbb{Q} with respect to a certain metric and is a natural analogue of the real line \mathbb{R} . Unfortunately, as an additive group, \mathbb{Q}_p contains no nontrivial discrete subgroup, thus presenting a serious obstacle to the development of a theory of wavelets in $L^2(\mathbb{Q}_p)$. Kozyrev [17] produced one specific set of wavelet generators, analogous to Haar wavelets on \mathbb{R}^d , using a discrete set of translation operators which do not form a group. However, as we shall see, those operators allow the Haar wavelets, but they preclude the possibility of a general theory of wavelet sets or multiresolution analysis in $L^2(\mathbb{Q}_p)$.

Fortunately, although it lacks a lattice, \mathbb{Q}_p does contain the compact open subgroup \mathbb{Z}_p . Whereas \mathbb{R}^d contains a discrete lattice \mathbb{Z}^d with compact quotient $\mathbb{R}^d/\mathbb{Z}^d$, then, the group \mathbb{Q}_p contains a *compact* subgroup \mathbb{Z}_p with *discrete* quotient $\mathbb{Q}_p/\mathbb{Z}_p$. Thus, the situation for \mathbb{Q}_p is the reverse of that for \mathbb{R}^d , but it still permits the construction of a discrete *group* of operators on $L^2(\mathbb{Q}_p)$. The resulting operators $\tau_{[s]}$ are not simple translation operators, but one may compute with them using the results of Section 3 below.

Of course, one also needs a dilation, which we shall call an expansive automorphism, to be defined in Section 3. J. Benedetto and the author [2] therefore considered wavelets on any locally compact abelian group G with a compact open subgroup H and an expansive automorphism A . The resulting wavelets, developed independently of Kozyrev's work, form a large class of orthonormal wavelet bases

over $\mathbb{Q}_p, \mathbb{F}_p((t))$, and any other such group G . The theory also unifies both Lang's and Kozyrev's wavelets within a much broader setting.

Besides identifying Haar and Shannon wavelets in this context, the main purpose of this paper is to give examples of some of the groups and constructions considered in [2] in an expository setting. In Section 1, we present a concrete example of a Haar/Shannon wavelet, in the Cantor dyadic group $\mathbb{F}_2((t))$. In Section 2, we list some basic properties of a group G with compact open subgroup H and expansive automorphism A ; we also present several examples of such groups. In Section 3 we describe the discrete group of translation-like operators on $L^2(G)$ defined in [2], and we consider specific examples by revisiting several of the groups from Section 2. In Section 4 we define wavelets using those operators as in [2]. We also present an explicit analogue of Shannon wavelets on G in Theorem 4.2. Then, in Theorem 4.3, we prove the promised result that our Shannon wavelets *are* Haar wavelets. Finally, in Section 5, we present several examples of wavelet sets produced by the results of [2] over certain groups G .

1. A motivating example

We begin by observing the Haar/Shannon phenomenon in a specific and relatively simple context. Let $\mathbb{F}_2 = \{0, 1\}$ be the field of two elements, with operations $0 + 0 = 1 + 1 = 0, 1 + 0 = 0 + 1 = 1, 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, and $1 \cdot 1 = 1$. The field $G = \mathbb{F}_2((t))$ of formal Laurent series consists of all infinite formal sums

$$c_{n_0}t^{n_0} + c_{n_0+1}t^{n_0+1} + c_{n_0+2}t^{n_0+2} + \dots,$$

where $n_0 \in \mathbb{Z}$, each $c_n \in \mathbb{F}_2$, and no consideration is given to convergence or divergence. One can add or multiply elements of G as formal power series: addition follows the rule $b_n t^n + c_n t^n = (b_n + c_n)t^n$, and multiplication follows the rule $(c_n t^n) \cdot (c_m t^m) = (c_n \cdot c_m)t^{n+m}$. One can also put a metric topology on G by declaring the absolute value of $g \in G$ to be $|g| = 2^{-n_0}$, where n_0 is the smallest integer such that $c_{n_0} \neq 0$. (We will see the more general fields $\mathbb{F}_p((t))$ later in Examples 2.5, 3.4, and 5.4.)

Lang [18] considers the ‘‘Cantor dyadic group’’, which is the group $G = \mathbb{F}_2((t))$ under addition. He views G as the subgroup of $\prod_{n \in \mathbb{Z}} \mathbb{F}_2$ consisting of elements $(\dots, c_{-2}, c_{-1}, c_0, c_1, c_2, \dots)$ for which there exists an $n_0 \in \mathbb{Z}$ such that $c_n = 0$ for all $n < n_0$. The metric topology we have described on G coincides with the product topology G inherits from $\prod_{n \in \mathbb{Z}} \mathbb{F}_2$.

One can show that G is self-dual. Whereas the duality pairing on \mathbb{R} takes $x, y \in \mathbb{R}$ to $e^{2\pi ixy}$, the pairing on G takes $g_1, g_2 \in G$ to $e^{\pi ic}$, where $c = c_{-1}$ is the t^{-1} -coefficient of $g_1 \cdot g_2$. Thus, the pairing of g_1 and g_2 is 1 if $c = 0$, and -1 if $c = 1$.

G contains a discrete subgroup Γ consisting of elements for which $c_0 = c_1 = c_2 = \dots = 0$. (That is, Γ consists of all the formal sums which have only finitely many nonzero coefficients, and all nonnegative power terms $c_n t^n$ are zero.) The compact subgroup $H = \mathbb{F}_2[[t]]$ of G , consisting of all power series $c_0 + c_1 t + c_2 t^2 + \dots$, is a fundamental domain for Γ , in the sense that $H \cap \Gamma = \{0\}$ and $H + \Gamma = G$. It may be helpful to think of Γ as analogous to the discrete subgroup \mathbb{Z} of \mathbb{R} , while H is analogous to the fundamental domain $[-1/2, 1/2)$; however, in our case, H is itself a subgroup of G .

We may normalize a Haar measure μ on G with $\mu(H) = 1$. Note that $tH = \{c_1 t + c_2 t^2 + \dots\}$ is a subgroup of H of measure $1/2$, because H is the disjoint

union $(1 + tH) \cup tH$, and $\mu(tH) = \mu(1 + tH)$ by translation invariance. Note also that H is homeomorphic to the Cantor set; it consists of two disjoint pieces, tH and $1 + tH$, each of which is homeomorphic (via multiplication-by- t^{-1}) to H .

Similarly, $t^{-1}H = \{c_{-1}t^{-1} + c_0 + c_1t + \dots\}$ is a subgroup of G which contains H and which has measure 2. It is easy to see that

$$\dots \supset t^{-2}H \supset t^{-1}H \supset H \supset tH \supset t^2H \supset \dots,$$

and

$$\bigcap_{n \geq 0} t^n H = \emptyset, \quad \text{and} \quad \bigcup_{n \geq 0} t^{-n} H = G.$$

Thus, multiplication-by- t^{-1} is a dilation which expands areas by a factor of 2, just as multiplication-by-2 is a dilation of \mathbb{R} which expands areas by a factor of 2.

Lang [18, Example 5.2] observed that the function

$$f(g) = \begin{cases} 1, & \text{if } g \in tH; \\ -1, & \text{if } g \in 1 + tH; \\ 0, & \text{otherwise} \end{cases}$$

is a wavelet, in the sense that the set of dilated translates $\{f(t^n g - s) : n \in \mathbb{Z}, s \in \Gamma\}$ is an orthonormal basis for $L^2(G)$. Indeed, this function f is an obvious analogue of the usual Haar wavelet on \mathbb{R} . The surprising fact that motivates much of the rest of this paper is that using the duality pairing on G , one can easily compute that the Fourier transform of f is

$$\widehat{f}(\omega) = \begin{cases} 1, & \text{if } \omega \in t^{-1} + H; \\ 0, & \text{otherwise,} \end{cases}$$

which shows that \widehat{f} is a step function with compact support in the frequency domain, and that it is therefore natural to call f a Shannon wavelet. (The skeptical reader may observe that $t^{-1} + H$ may be written as the disjoint union of $t^{-1} + (tH)$ and $-t^{-1} + (1 + tH)$, just as the Shannon wavelet in $L^2(\mathbb{R})$ has Fourier transform equal to the characteristic function of $[-1, -1/2) \cup [1/2, 1)$.)

It turns out that the key property of G that allows the Haar/Shannon identification is the fact that the fundamental domain H is itself a subgroup of G . Moreover, the lattice Γ turns out to be unnecessary, provided we define operators to act as translations in some appropriate manner. Thus, we will be able to work with groups such as \mathbb{Q}_p , which have no such lattice. With these ideas in mind, we are now prepared to generalize.

2. The group, subgroup, and automorphism

We fix the following notation.

G	a locally compact abelian group
\widehat{G}	the dual group of G
H	a compact open subgroup of G
H^\perp	the annihilator of H in \widehat{G}
μ	Haar measure on G , normalized so that $\mu(H) = 1$
ν	Haar measure on \widehat{G} , normalized so that $\nu(H^\perp) = 1$

Given $x \in G$ and $\gamma \in \widehat{G}$, the pairing $(x, \gamma) \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ shall denote the action of γ on x . H^\perp , which by definition consists of all $\gamma \in \widehat{G}$ such that $(x, \gamma) = 1$ for each $x \in H$, is a compact open subgroup of \widehat{G} . The quotient groups G/H and \widehat{G}/H^\perp are discrete, with counting measure induced by μ and ν , respectively. Furthermore, the discrete group \widehat{H} is naturally isomorphic to \widehat{G}/H^\perp , and the compact group $\widehat{G/H}$ is naturally isomorphic to H^\perp . The normalizations of μ and ν are compatible in the sense that the Fourier inversion formula holds; see [14, §31.1(c)].

The study of wavelets in $L^2(\mathbb{R}^d)$ requires a dilation operator and a discrete group of translation operators. For the translations, one usually considers a discrete lattice $\Lambda \subset \mathbb{R}^d$ with a fundamental domain F which is also a neighborhood of the origin. However, our group G may not actually contain a discrete subgroup. Fortunately, the compact subgroup H can play the role of the fundamental domain, even without an actual lattice. Meanwhile, the discrete group G/H shall act as a group of translation operators, in a manner to be described in Section 3.

The dilation over \mathbb{R}^d is usually an expansive matrix B ; as a linear map, B satisfies $d(\lambda \circ B) = |\det B|d\lambda$, where λ denotes Lebesgue measure on \mathbb{R}^d . In our setting, let A be an automorphism of G ; that is, A is an algebraic isomorphism from G to G such that A is also a topological homeomorphism. Then $\mu \circ A$ is a nontrivial Haar measure on G . Define the *modulus* of A to be $|A| = \mu(AH) > 0$, so that $d\mu(Ax) = |A|d\mu(x)$. It is easy to verify that $|A^{-1}| = |A|^{-1}$. Furthermore, the adjoint A^* of A is an automorphism of \widehat{G} , and $|A^*| = |A|$ in the sense that $\nu(A^*H^\perp) = |A|$ and $d\nu(A^*\gamma) = |A|d\nu(\gamma)$.

If the Fourier transform \widehat{f} of a function $f \in L^2(G)$ has compact support, then as usual, f is smooth. Proposition 2.1 below shall make that notion of smoothness more precise. As a preface to that result, note that for any fixed $r \in \mathbb{Z}$, $A^{-r}H$ is an open subgroup of G , and therefore the sets of the form $c + A^{-r}H$ form a partition of G into open sets. The proposition shall show that if the support of \widehat{f} is contained in the compact set $(A^*)^r H^\perp$, then f is constant on all the sets $c + A^{-r}H$. Thus, f is locally constant.

PROPOSITION 2.1. *Let G be a locally compact abelian group with compact open subgroup H , let A be an automorphism of G , let $r \in \mathbb{Z}$, and let $f \in L^2(G)$.*

Then $\text{Supp } \widehat{f} \subset (A^)^r H^\perp$ if and only if f is constant on every (open) set of the form $c + A^{-r}H$.*

PROOF. To prove the forward implication, pick any $c \in G$ and $x \in c + A^{-r}H$. Note that $x - c \in A^{-r}H$, and therefore $(x - c, \gamma) = 1$ for every $\gamma \in (A^*)^r H^\perp$. Thus,

$$f(x) = \int_{(A^*)^r H^\perp} (x, \gamma) \widehat{f}(\gamma) d\nu(\gamma) = \int_{(A^*)^r H^\perp} (c, \gamma) \widehat{f}(\gamma) d\nu(\gamma) = f(c).$$

For the converse, pick $c \in G$ and $r \in \mathbb{Z}$. We shall show that the desired implication holds for the characteristic function $f = \mathbf{1}_{c+A^{-r}H}$; the general result

shall then follow by linearity. We compute

$$\begin{aligned}
\widehat{f}(\gamma) &= \int_{c+A^{-r}H} \overline{(x, \gamma)} d\mu(x) = \overline{(c, \gamma)} \int_{A^{-r}H} \overline{(x, \gamma)} d\mu(x) \\
&= \overline{(c, \gamma)} |A|^{-r} \int_H \overline{(A^{-r}x, \gamma)} d\mu(x) \\
&= \overline{(c, \gamma)} |A|^{-r} \int_H \overline{(x, (A^*)^{-r}\gamma)} d\mu(x) \\
(2.1) \quad &= \overline{(c, \gamma)} |A|^{-r} \mathbf{1}_{H^\perp}((A^*)^{-r}\gamma) = \overline{(c, \gamma)} |A|^{-r} \mathbf{1}_{(A^*)^r H^\perp}(\gamma),
\end{aligned}$$

which has support contained in the required domain. \square

DEFINITION 2.2. Let G be a locally compact abelian group with compact open subgroup H . Let A be an automorphism of G . We say that A is *expansive* with respect to H if both of the following conditions hold:

- (1) $AH \supsetneq H$, and
- (2) $\bigcap_{n \leq 0} A^n H = \{0\}$.

The idea of Definition 2.2 is that an expansive automorphism acts on G like an expansive integer matrix B acts on \mathbb{R}^d ; the fundamental domain F for the lattice \mathbb{Z}^d is contained in its image BF , and the intersection of all inverse images $B^{-k}F$ is just the origin. Moreover, if A is an expansive automorphism of G , then $|A|$ is an integer greater than 1, just as is true of an expansive integer matrix. Indeed, H is a proper subgroup of AH , so that AH may be covered by (disjoint) cosets $s + H$. Each coset has measure 1, so that AH must have measure equal to the number of cosets. Thus, $|A| = \mu(AH)$ is just the number of elements in the finite quotient group $(AH)/H$.

The following alternate characterization of expansiveness was proven in [2].

PROPOSITION 2.3 (From [2]). *Let G be a locally compact abelian group with compact open subgroup H . Let A be an automorphism of G . Then A is expansive with respect to H if and only if both of the following conditions hold:*

- (1) $A^* H^\perp \supsetneq H^\perp$, and
- (2) $\bigcup_{n \geq 0} (A^*)^n H^\perp = \widehat{G}$.

EXAMPLE 2.4. Let $p \geq 2$ be a prime number. Let $G = \mathbb{Q}_p$ be the set of p -adic rational numbers, and $H = \mathbb{Z}_p$ the set of p -adic integers. That is,

$$\begin{aligned}
G = \mathbb{Q}_p &= \left\{ \sum_{n \geq n_0} c_n p^n : n_0 \in \mathbb{Z}, c_n = 0, 1, \dots, p-1 \right\}, \quad \text{and} \\
H = \mathbb{Z}_p &= \left\{ \sum_{n \geq 0} c_n p^n : c_n = 0, 1, \dots, p-1 \right\}.
\end{aligned}$$

Here, the sums are formal sums, but they indicate that the group law is addition with carrying of digits, so that in \mathbb{Q}_3 , for example, we have

$$\begin{aligned}
(2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + \dots) + (1 + 2 \cdot 3 + 1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + \dots) \\
= (0 + 2 \cdot 3 + 1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + \dots).
\end{aligned}$$

The discrete quotient $G/H = \mathbb{Q}_p/\mathbb{Z}_p$ is naturally isomorphic to μ_{p^∞} , the multiplicative group of all p^n -th roots of unity (as n ranges through all nonnegative integers) in \mathbb{C} . However, \mathbb{Q}_p itself contains no nontrivial discrete subgroups; if a closed subgroup E contains some nonzero element $x \in \mathbb{Q}_p$, then E contains the open subgroup $x\mathbb{Z}_p$.

\mathbb{Q}_p is in fact a field, with multiplication given by multiplication of formal Laurent series together with carrying of digits, though we are considering it only as an additive group. For more of the standard background on \mathbb{Q}_p , see [11, 16, 23]; see [24, 25] for more advanced expositions.

Define a character χ on G by

$$\chi \left(\sum_{n \geq n_0} c_n p^n \right) = \exp \left(2\pi i \sum_{n=n_0}^{-1} c_n p^n \right).$$

Note that $H = \{x \in G : \chi(x) = 1\}$. G is self-dual, with pairing given by $(x, \gamma) = \chi(x\gamma)$, where $x\gamma$ means the product in the field \mathbb{Q}_p . Under this pairing, $H^\perp \subset \mathbb{Q}_p$ is again just \mathbb{Z}_p .

For any nonzero $a \in \mathbb{Q}_p$, the multiplication-by- a map $A : x \mapsto ax$ is an automorphism of G . (In fact, any automorphism must be of this form.) If $a \notin \mathbb{Z}_p$, then A is expansive; in that case, if $a = c_{-m}p^{-m} + c_{-m+1}p^{-m+1} + \dots$ with $c_{-m} \neq 0$, then $|A| = p^m$. (For example, multiplication-by- $1/p$ is an expansive map with modulus p .) The adjoint A^* on $\widehat{G} = \mathbb{Q}_p$ is again multiplication-by- a .

EXAMPLE 2.5. Let $p \geq 2$ be a prime number. Let \mathbb{F}_p denote the field of order p , with elements $\{0, 1, \dots, p-1\}$. Let $G = \mathbb{F}_p((t))$, the set of formal Laurent series in the variable t with coefficients in \mathbb{F}_p , and let $H = \mathbb{F}_p[[t]]$ be the subset consisting of power series (i.e., nonnegative powers of t only). G is a field under addition and multiplication of formal power series. Although G is topologically homeomorphic to \mathbb{Q}_p , the algebraic structure is very different because there is no carrying of digits. In $\mathbb{F}_3((t))$, for example, we have

$$\begin{aligned} (2 + 2 \cdot t + 2 \cdot t^2 + 2 \cdot t^3 + 2 \cdot t^4 + \dots) + (1 + 2 \cdot t + 1 \cdot t^2 + 2 \cdot t^3 + 1 \cdot t^4 + \dots) \\ = (0 + 1 \cdot t + 0 \cdot t^2 + 1 \cdot t^3 + 0 \cdot t^4 + \dots). \end{aligned}$$

Again, we consider G and H as groups under addition; note that for any $x \in G$,

$$\underbrace{x + x + \dots + x}_p = 0.$$

(That is, $\mathbb{F}_p((t))$ is a field of characteristic p .) The discrete quotient G/H is naturally isomorphic to a direct sum of a countable number of copies of \mathbb{F}_p . In fact, G/H is isomorphic to the subgroup $\Gamma \subset G$ consisting of all elements of the form $c_{-m}t^{-m} + \dots + c_{-1}t^{-1}$. See [23] or [25] for more on such fields.

Define a character χ on G by

$$\chi \left(\sum_{n \geq n_0} c_n t^n \right) = \exp(2\pi i c_{-1}/p).$$

Note that the image of χ consists only of the p -th roots of unity, whereas the corresponding character on \mathbb{Q}_p included all p^n -th roots of unity in its image. Another contrast with \mathbb{Q}_p is that this time, $H \subsetneq \{x \in G : \chi(x) = 1\}$. Still, as before, G is

self-dual, with pairing given by $(x, \gamma) = \chi(x\gamma)$, where $x\gamma$ means the product in the field $\mathbb{F}_p((t))$. Under this pairing, $H^\perp \subset \mathbb{F}_p((t))$ is again just $\mathbb{F}_p[[t]]$.

For any nonzero $a \in \mathbb{F}_p((t))$, the multiplication-by- a map $A : x \mapsto ax$ is an automorphism of G . (This time, however, many other automorphisms are possible.) If $a \notin \mathbb{F}_p[[t]]$, then A is expansive; in that case, if $a = c_{-m}t^{-m} + c_{-m+1}t^{-m+1} + \dots$ with $c_{-m} \neq 0$, then $|A| = p^m$. (For example, multiplication-by- $1/t$ is an expansive map with modulus p .) The adjoint A^* on $\widehat{G} = \mathbb{F}_p((t))$ is again multiplication-by- a .

EXAMPLE 2.6. Let K be a field which is a finite extension of \mathbb{Q}_p . In general, \mathbb{Q}_p is equipped with an absolute value function $|\cdot| : \mathbb{Q}_p \rightarrow \mathbb{R}_{\geq 0}$ with $|p| = 1/p$ and which extends uniquely to the algebraic closure of \mathbb{Q}_p , and hence to K . The ring of integers \mathcal{O}_K is precisely the set of elements of K with absolute value at most 1, just as \mathbb{Z}_p is the set of elements in \mathbb{Q}_p with absolute value at most 1. Equivalently, \mathcal{O}_K consists of all elements of K which are roots of monic polynomials with coefficients in \mathbb{Z}_p .

For example, if $K = \mathbb{Q}_3(\sqrt{-1})$, then elements of K may be represented as formal Laurent series $\sum c_n 3^n$, where each c_n is one of the nine numbers $a + b\sqrt{-1}$, with $a, b = 0, 1, 2$. (The addition law for carrying digits requires some extra computation in this case.) Similarly, if $K = \mathbb{Q}_3(\sqrt{3})$, then elements of K may be represented as formal Laurent series $\sum c_n 3^{n/2}$, where each c_n is one of the three numbers $0, 1, 2$, and with carrying from the $3^{n/2}$ term to the $3^{(n+2)/2}$ term. In both of these cases, \mathcal{O}_K consists of those elements of K with no negative power terms.

Let G be the additive group of K , and $H \subset G$ be the additive group of \mathcal{O}_K . Define a character χ_K on G by

$$\chi_K(x) = \chi(\text{Tr } x),$$

where χ was the character defined in Example 2.4 for \mathbb{Q}_p , and Tr is the trace map from K down to \mathbb{Q}_p . G is self-dual, with pairing given by $(x, \gamma) = \chi_K(x\gamma)$, where $x\gamma$ means the product in K . This time, $H^\perp \subset K$ need not be the same as H . Instead, H^\perp is the inverse different of K , which is a (possibly larger) subgroup of K of the form $b\mathcal{O}_K$, for some $b \in K$. (See [23, §B.2] or [25, III] for more on the different and inverse different.) For example, if $K = \mathbb{Q}_3(\sqrt{-1})$, then the inverse different is just $\mathcal{O}_K = \mathbb{Z}_3[\sqrt{-1}]$; but if $K = \mathbb{Q}_3(\sqrt{3})$, then the inverse different is $3^{-1/2}\mathcal{O}_K = 3^{-1/2}\mathbb{Z}_3[\sqrt{3}]$. (The inverse different roughly measures the degree of *ramification* in K , which, speaking informally, means the largest denominator that appears in the powers of p in expansions of elements of K . So $\mathbb{Q}_3(\sqrt{3})$ has ramification degree 2, since its elements involve half-integral powers of 3.)

As before, for any nonzero $a \in K$, the multiplication $A : x \mapsto ax$ is an automorphism of G . (Other automorphisms are possible, incorporating Galois automorphisms of K over \mathbb{Q}_p .) If $a \notin \mathcal{O}_K$, then A is expansive; the modulus $|A|$ is, as usual, the order of the group $(AH)/H$, which is always be a power of p .

One could also choose G to be the additive group of a field which is a finite extension of $\mathbb{F}_p((t))$. Again, G would be self-dual, with pairing given by composing the original χ for $\mathbb{F}_p((t))$ with the trace map, and with H^\perp given by the inverse different. We omit the details here.

EXAMPLE 2.7. Let G_1, \dots, G_N be locally compact abelian groups with compact open subgroups H_1, \dots, H_N , and expansive automorphisms A_1, \dots, A_N , respectively. Then $G = G_1 \times \dots \times G_N$ has open compact subgroup $H = H_1 \times \dots \times H_N$ with expansive automorphism $A = A_1 \times \dots \times A_N$.

As a special case, if $G_1 = \cdots = G_N = K$ is the additive group of one of the fields in the previous three examples, then $G = K^{\times N}$ is a locally compact abelian group with compact open subgroup $H = \mathcal{O}_K^{\times N}$. Moreover, if $A \in GL(N, K)$ is a matrix with all eigenvalues of absolute value greater than 1 and such that $AH \subsetneq H$, then A is expansive. The modulus of A is $|\det A|$, where $|\cdot|$ is a certain appropriately chosen absolute value on K .

EXAMPLE 2.8. The preceding examples are of self-dual σ -compact groups; the following example is of a group which is not self-dual and may not be σ -compact.

Let G_0 be a locally compact abelian group with compact open subgroup H_0 and expansive automorphism A_0 , and let W be any discrete group (possibly uncountable). Let $G = G_0 \times W$, with compact open subgroup $H = H_0 \times \{0\}$. Then \widehat{G} is isomorphic to $\widehat{G}_0 \times \widehat{W}$, and the image of H^\perp under that isomorphism is $H_0^\perp \times \widehat{W}$. If W is uncountable, note that G is not σ -compact.

It is easy to check that the automorphism A of G given by $A = A_0 \times \text{id}_W$ (where id_W is the identity automorphism of W) is expansive, with adjoint $A^* = A_0^* \times \text{id}_{\widehat{W}}$. However, it should be noted that while $\bigcap_{n \leq 0} A^n H = \{0\}$, as required for expansiveness, the union $\bigcup_{n \geq 0} A^n H$ is *not* all of G , but only $G_0 \times \{0\}$. Similarly, on the dual side, the union $\bigcup_{n \geq 0} (A^*)^n H^\perp = \widehat{G}$ is the whole group, but the intersection $\bigcap_{n \leq 0} (A^*)^n H^\perp = \{0\} \times \widehat{W} \neq \{0\}$ includes more than just the identity.

3. Dilation and translation operators

We are now prepared to present the operators on $L^2(G)$ introduced in [2]. Given our group G and automorphism A , we can define a dilation operator δ_A on $L^2(G)$ by

$$\delta_A(f)(x) = |A|^{1/2} f(Ax).$$

It is easy to verify that δ_A is a unitary operator. If A is expansive, then certainly A does not have finite order, and therefore $\{\delta_A^n : n \in \mathbb{Z}\}$ forms a group isomorphic to \mathbb{Z} .

The construction of translation operators requires a little more work. If G contains a discrete subgroup Γ such that $G \cong \Gamma \times H$, then Γ plays the same role that the lattice \mathbb{Z}^d plays in \mathbb{R}^d . That is, for any $s \in \Gamma$, we define τ_s on $L^2(G)$ by $\tau_s(f)(x) = f(x - s)$. For example, if $G = \mathbb{F}_p((t))$, then the choice of Γ described in Example 2.5 is such a lattice. Indeed, Lang [18] used precisely this lattice to define wavelets on $\mathbb{F}_2((t))$, the Cantor dyadic group.

On the other hand, if G contains no such lattice, as is the case for \mathbb{Q}_p , the situation is a little more complicated. Kozyrev's [17] solution for the special case of \mathbb{Q}_p was to let $\mathcal{C} \subset \mathbb{Q}_p$ be the discrete subset (but not subgroup) of elements of the form $c_{-m}p^{-m} + \cdots + c_{-1}p^{-1}$. Then, for each $s \in \mathcal{C}$, the translation operator τ_s is defined in the usual way. He was then able to construct Haar wavelets on \mathbb{Q}_p using dilation-by- $1/p$ and translation by elements of \mathcal{C} .

More generally, one could choose a set \mathcal{C} of coset representatives for the quotient G/H . That is, choose a subset $\mathcal{C} \subset G$ such that the translations $s + H$ of H by elements $s \in \mathcal{C}$ are mutually disjoint, and such that $\bigcup_{s \in \mathcal{C}} (s + H) = G$. (Such a \mathcal{C} always exists if one accepts the Axiom of Choice; but in many cases, it is possible to construct such a set \mathcal{C} without invoking the Axiom of Choice. See, for example, Proposition 3.2.)

Unfortunately, Kozyrev's method has its limitations. For one thing, the resulting group of translation operators does not form a group unless \mathcal{C} itself is a group, which would be impossible for \mathbb{Q}_p . Furthermore, it appears to be difficult or impossible to construct wavelets other than Haar wavelets using such translation operators. More precisely, it appears that any sort of analogue of a multiresolution analysis or of minimally supported frequency wavelets requires the translation operators to behave well on the dual group \widehat{G} , as follows. If $\mathcal{C} = \Gamma$ is indeed a group, then the dual lattice $\Gamma^\perp \subset \widehat{G}$ plays a crucial role in wavelet analysis; but if \mathcal{C} is not a group, then there does not seem to be any reasonable object which can assume the role of the dual lattice.

In [2], the authors solved this problem by choosing a set $\mathcal{D} \subset \widehat{G}$ to play the role of the dual lattice first, and then building the operators to respect \mathcal{D} . Specifically, let \mathcal{D} be a choice of coset representatives for the quotient \widehat{G}/H^\perp . Define functions $\theta = \theta_{\mathcal{D}} : \widehat{G} \rightarrow \mathcal{D}$ and $\eta = \eta_{\mathcal{D}} : \widehat{G} \rightarrow H^\perp$ given by

$$(3.1) \quad \begin{aligned} \theta(\gamma) &= \text{the unique } \sigma \in \mathcal{D} \text{ such that } \gamma - \sigma \in H^\perp, \\ \eta(\gamma) &= \gamma - \theta(\gamma). \end{aligned}$$

Speaking informally, $\eta(\gamma)$ is the unique point in the fundamental domain H^\perp which is congruent to γ modulo the "lattice" \mathcal{D} , and $\theta(\gamma)$ is the unique "lattice" point such that $\gamma = \eta(\gamma) + \theta(\gamma)$.

Any element $[s] \in G/H$ of the discrete quotient is, by definition, a coset of the form $s + H$, with $s \in G$. We wish to define a translation operator associated with $[s]$ which depends only on the coset, not the particular representative s . Note that on the transform side, the translation-by- s operator sends $\widehat{f}(\gamma)$ to $\overline{(s, \gamma)}\widehat{f}(\gamma)$. Thus, we define the translation-by-the-coset operator $\tau_{[s]}$ by defining its action $\tilde{\tau}_{[s]}$ on $L^2(\widehat{G})$ as follows:

$$(3.2) \quad \tilde{\tau}_{[s]}(\widehat{f})(\gamma) = \overline{(s, \eta(\gamma))}\widehat{f}(\gamma).$$

It is easy to check that $\tilde{\tau}_{[s]}$, and hence $\tau_{[s]}$, are unitary operators.

To simplify notation, define

$$w_{[s]}(\gamma) = \overline{(s, \eta(\gamma))},$$

which is a function in $L^\infty(\widehat{G})$, since $|w_{[s]}(\gamma)| = 1$ for all $\gamma \in \widehat{G}$. We compute

$$(3.3) \quad w_{[s+t]}(\gamma) = \overline{(s+t, \eta(\gamma))} = \overline{(s, \eta(\gamma))} \overline{(t, \eta(\gamma))} = w_{[s]}(\gamma)w_{[t]}(\gamma).$$

If we consider $t \in H$, so that $w_{[t]}$ is identically 1, then equation (3.3) shows that $w_{[s+t]} = w_{[s]}$. Thus, as promised, $w_{[s]}$ and hence $\tau_{[s]}$ depend only on $[s]$, not on the particular representative s . Equation (3.3) also shows that the set of operators

$$\Gamma_{G/H} = \{\tau_{[s]} : [s] \in G/H\}$$

forms a group isomorphic to G/H .

On $L^2(G)$, the action of $\tau_{[s]}$ is convolution by the pseudo-measure $\check{w}_{[s]}$, defined to be the inverse transform of $w_{[s]}$. By way of comparison, the usual translation-by- s operator is convolution by a delta measure centered at s . Proposition 3.1 below computes the action of our operator $\tau_{[s]}$, and the result shows that our pseudo-measure $\check{w}_{[s]}$ has support contained in $s + H$, in the sense that it is trivial on any measurable set $U \subset G$ disjoint from $s + H$. Thus, $\check{w}_{[s]}$ may be thought of as a perturbation of the original delta measure. To see this, we consider the action of

$\tau_{[s]}$ on the characteristic function of a set of the form $c + A^{-r}H$ in the following proposition.

PROPOSITION 3.1. *Let G be a locally compact abelian group with compact open subgroup H , and let \mathcal{D} be a set of coset representatives for the quotient \widehat{G}/H^\perp . Let A be an expansive automorphism of G , and let $M = |A| \geq 2$ be the modulus of A . For any $[s] \in G/H$, define the operator $\tau_{[s]}$ as in equation (3.2) above. For any $c \in G$ and any $r \in \mathbb{Z}$, let $\mathbf{1}_{c+A^{-r}H}$ be the characteristic function of the set $c+A^{-r}H$. If $r \leq 0$, let σ_0 be the unique element of \mathcal{D} in H^\perp . If $r > 0$, let $\{\sigma_0, \sigma_1, \dots, \sigma_{M^r-1}\}$ be all the elements of $\mathcal{D} \cap (A^*)^r H^\perp$.*

(1) *If $r \leq 0$, then*

$$\tau_{[s]}\mathbf{1}_{c+A^{-r}H}(x) = (s, \sigma_0)\mathbf{1}_{s+c+A^{-r}H}(x).$$

(2) *If $r > 0$, then*

$$\tau_{[s]}\mathbf{1}_{c+A^{-r}H}(x) = \frac{1}{M^r} \left(\sum_{i=0}^{M^r-1} (x - c, \sigma_i) \right) \mathbf{1}_{s+c+H}(x).$$

Note that in the case $r \leq 0$ (in which case $A^{-r}H$ is large), $\tau_{[s]}$ acts as translation-by- s and multiplication by the constant (s, σ_0) on $\mathbf{1}_{c+A^{-r}H}$. For $r > 0$ (in which case $A^{-r}H$ is small), $\tau_{[s]}$ is similar to translation by s , but it spreads the function out over the full coset $s + c + H$, not the smaller $s + c + A^{-r}H$. Still, the translated function is constant on any region of the form $b + A^{-r}H$. Moreover, when that constant is nonzero (i.e., when $b \in s + c + H$), its value $M^{-r} \sum_i (b, \sigma_i)$ usually is small, due to a lot of cancellation (but usually not complete cancellation) within the sum; see the examples later in this section. Note also that we can easily verify directly from the proposition that $\tau_{[s]}$ is independent of the coset representative s , and that it is the identity operator if $s \in H$.

If a function $f \in L^2(G)$ is constant on all such sets $c + A^{-r}H$, then f is a linear combination of characteristic functions $\mathbf{1}_{c+A^{-r}H}$. Thus, $\tau_{[s]}f$ may be computed using Proposition 3.1 and linearity. In particular, the action of $\tau_{[s]}$ on such a function f is determined not by all of \mathcal{D} but only by a subset of size M^r . Furthermore, the computation of $\tau_{[s]}f$ involves only a finite sum and is therefore easily computable in practice.

PROOF OF PROPOSITION 3.1. Let $f = \mathbf{1}_{c+A^{-r}H}$. By Equation (2.1),

$$\widehat{f}(\gamma) = M^{-r} \overline{(c, \gamma)} \mathbf{1}_{(A^*)^r H^\perp}(\gamma),$$

and therefore

$$\tilde{\tau}_{[s]}\widehat{f}(\gamma) = M^{-r} \overline{(s, \eta(\gamma))} \overline{(c, \gamma)} \mathbf{1}_{(A^*)^r H^\perp}(\gamma).$$

If $r \leq 0$, then $(A^*)^r H^\perp \subset H^\perp$, so that $\eta(\gamma) = \gamma - \sigma_0$ for $\gamma \in (A^*)^r H^\perp$. In that case,

$$\tilde{\tau}_{[s]}\widehat{f}(\gamma) = M^{-r} (s, \sigma_0) \overline{(s + c, \gamma)} \mathbf{1}_{(A^*)^r H^\perp}(\gamma).$$

Taking the inverse transform gives

$$\tau_{[s]}f(x) = M^{-r} (s, \sigma_0) \int_{(A^*)^r} (x - s - c, \gamma) d\nu(\gamma) = (s, \sigma_0) \mathbf{1}_{s+c+A^{-r}H}(x).$$

On the other hand, if $r > 0$, then $(A^*)^r H^\perp \supseteq H^\perp$. To compute $\eta(\gamma)$, then, we partition $(A^*)^r H^\perp$ into the M^r regions $\{\sigma_i + H^\perp : i = 0, 1, \dots, M^r - 1\}$. We have $\eta(\gamma) = \gamma - \sigma_i$ for $\gamma \in \sigma_i + H^\perp$. Thus,

$$\tilde{\tau}_{[s]}\widehat{f}(\gamma) = M^{-r} \sum_{i=0}^{M^r-1} (s, \sigma_i) \overline{(s+c, \gamma)} \mathbf{1}_{\sigma_i + H^\perp}(\gamma).$$

Taking the inverse transform,

$$\begin{aligned} \tau_{[s]}f(x) &= M^{-r} \sum_{i=0}^{M^r-1} (s, \sigma_i) \int_{\sigma_i + H^\perp} (x - s - c, \gamma) d\nu(\gamma) \\ &= M^{-r} \sum_{i=0}^{M^r-1} (x - c, \sigma_i) \mathbf{1}_{s+c+H}(x). \end{aligned} \quad \square$$

Of course, the operator $\tau_{[s]}$ depends on the choice \mathcal{D} of coset representatives. In [2] it was shown that given an expansive automorphism A of G , it is possible to construct such a \mathcal{D} while making only finitely many choices. We restate that result here as a proposition.

PROPOSITION 3.2 (From [2]). *Let G be a locally compact abelian group with compact open subgroup H . Let A be an expansive automorphism of G with modulus $|A| = M \geq 2$, and let $\mathcal{D}_1 = \{\rho_0, \dots, \rho_{M-1}\}$ be a set of coset representatives for the (finite) quotient $H^\perp / ((A^*)^{-1}H^\perp)$, with $\rho_0 = 0$. Define $\mathcal{D} \subset \widehat{G}$ to be the (infinite) set of all elements $\sigma \in \widehat{G}$ of the form*

$$(3.4) \quad \sigma = \sum_{j=1}^n (A^*)^j \rho_{i_j}, \quad \text{where } n \geq 1 \text{ and } i_j \in \{0, 1, \dots, M-1\}.$$

Then \mathcal{D} is a set of coset representatives for the quotient \widehat{G}/H^\perp . Moreover, $A^\mathcal{D} \subsetneq \mathcal{D}$.*

EXAMPLE 3.3. Let $p \geq 2$ be a prime number, $G = \mathbb{Q}_p$, and $H = \mathbb{Z}_p$; identify \widehat{G} as \mathbb{Q}_p and H^\perp as \mathbb{Z}_p , as in Example 2.4. Let the automorphism A be multiplication-by- $1/p$. As mentioned previously, Kozyrev [17] chose a certain standard set of coset representatives in \mathbb{Q}_p for the quotient G/H ; the same set in \mathbb{Q}_p can be used as our choice of coset representatives for \widehat{G}/H^\perp . Specifically, let \mathcal{D} consist of all elements of $\widehat{G} = \mathbb{Q}_p$ of the form

$$\sigma = \sum_{i=1}^n c_i p^{-i}, \quad \text{with } c_i \in \{0, 1, \dots, p-1\}.$$

This is the same set \mathcal{D} that would come from $\mathcal{D}_1 = \{0, 1, 2, \dots, (p-1)\}$ using Proposition 3.2.

The resulting operator $\tau_{[s]}$ takes $f = \mathbf{1}_{c+p^n\mathbb{Z}_p}$ to

$$\tau_{[s]}f(x) = p^{-n} \left[\sum_{j=0}^{p^n-1} (x - c, \sigma_j) \right] \mathbf{1}_{s+c+\mathbb{Z}_p}(x)$$

by Proposition 3.1, with $r = n > 0$. If $s \in \mathbb{Z}_p$ (that is, if $[s] = [0]$), then it is easy to verify from the above formula that $\tau_{[s]}f = f$. On the other hand, if $s \notin \mathbb{Z}_p$, write $x \in s + c + \mathbb{Z}_p$ as $x \in c + m/p^\ell + p^n\mathbb{Z}_p$, for some integer m prime to p and some

$\ell > 1$. (Note that m is only determined modulo $p^{n+\ell}$.) The sum in the formula for $\tau_{[s]}f$ becomes

$$p^{-n} \left[\sum_{j=0}^{p^n-1} (x - c, \sigma_j) \right] = p^{-n} \sum_{j=0}^{p^n-1} (m/p^\ell, j/p^n) = p^{-n} \sum_{j=0}^{p^n-1} e^{2\pi i m j / p^{\ell+n}}.$$

We may assume that $1 \leq m \leq p^{\ell+n} - 1$. For values of m larger than p^ℓ , we should expect a great deal of cancellation in the sum, because the terms must be distributed fairly evenly around the unit circle. Thus, most of the weight of $\tau_{[s]} \mathbf{1}_{c+p^n\mathbb{Z}_p}$ should be concentrated in the (relatively few) regions $b + p^n\mathbb{Z}_p$ corresponding to small values of m .

We note that another standard choice for \mathcal{D} would be to use $\mathcal{D}_1 = \{0\} \cup \{\zeta_{p-1}^i : i = 1, \dots, (p-1)\}$, where $\zeta_{p-1} \in \mathbb{Q}_p$ is a primitive $(p-1)$ st root of unity.

EXAMPLE 3.4. Let $p \geq 2$ be a prime number, $G = \mathbb{F}_p((t))$, and $H = \mathbb{F}_p[[t]]$; identify \widehat{G} as $\mathbb{F}_p((t))$ and H^\perp as $\mathbb{F}_p[[t]]$, as in Example 2.5. Let the automorphism A be multiplication-by- $1/t$. We may choose our set \mathcal{D} of coset representatives for \widehat{G}/H^\perp to be all elements of \widehat{G} of the form

$$\sigma = \sum_{i=1}^n c_i t^{-i}, \quad \text{with } c_i \in \mathbb{F}_p.$$

This is the same set \mathcal{D} that would come from Proposition 3.2 using $\mathcal{D}_1 = \mathbb{F}_p$ (the constants in $\mathbb{F}_p[[t]]$). As a subset of $\mathbb{F}_p((t))$, it is the same as the group Γ mentioned in Example 2.5. This time we are considering it as a subgroup of \widehat{G} , not of G , and so we call the group $\Gamma' = \mathcal{D}$.

In particular, \mathcal{D} is actually a subgroup of \widehat{G} . Thus, the sum

$$p^{-n} \sum_{j=0}^{p^n-1} (x - c, \sigma_j)$$

(where σ_j ranges over the p^n elements of the group $\Gamma' \cap t^{-n}\mathbb{F}_p[[t]]$) from Proposition 3.1 simplifies to zero unless $(x - c, \sigma_j)$ is trivial for all $j = 0, 1, \dots, p^n - 1$. It follows that the operator $\tau_{[s]}$ is the usual translation-by- u operator, where $u \in G$ is the unique element of $(s + H) \cap \Gamma$. In particular, for $p = 2$, in which case $G = \mathbb{F}_2((t))$ is sometimes known as the ‘‘Cantor dyadic group’’, the translation operators induced by \mathcal{D} are the same as the translation operators used by Lang in his study of wavelets on that group [18]. Indeed, for any p , $\mathbb{F}_p((t))$ is a locally compact abelian group with a discrete lattice Γ and dual lattice Γ' , which is the situation studied by Dahlke in [7].

We close this example by noting that one could choose \mathcal{D} in other ways so as not to form a subgroup. In that case, the operators $\tau_{[s]}$ would not be true translation operators, but would behave more like the operators of Example 3.3 above.

EXAMPLE 3.5. Let $G = \mathbb{Q}_3(\sqrt{-1})$ and $H = \mathbb{Z}_3[\sqrt{-1}]$; identify \widehat{G} as $\mathbb{Q}_3(\sqrt{-1})$ and H^\perp as $\mathbb{Z}_3[\sqrt{-1}]$. Let A be multiplication-by- $1/3$, so that $|A| = 9$. We may choose \mathcal{D}_1 to be the set of nine elements

$$\mathcal{D}_1 = \{a + b\sqrt{-1} : a, b = 0, 1, 2\},$$

so that the resulting full set \mathcal{D} of coset representatives for \widehat{G}/H^\perp consists of all elements of the form

$$\sigma = \sum_{i=1}^n (a_i + b_i \sqrt{-1}) 3^{-i}, \quad \text{with } a_i, b_i \in \{0, 1, 2\}.$$

Thus, \mathcal{D} consists of elements of the form $q_1 + q_2 \sqrt{-1}$, with $q_j = m_j / 3^{n_j}$ and $0 \leq m_j \leq 3^{n_j} - 1$.

The resulting operator $\tau_{[s]}$ takes $f = \mathbf{1}_{c+3^n H}$ to

$$\tau_{[s]} f(x) = 9^{-n} \left[\sum_{m_1=0}^{3^n-1} \sum_{m_2=0}^{3^n-1} (x - c, (m_1 + m_2 \sqrt{-1})/3^n) \right] \mathbf{1}_{s+c+H}(x)$$

by Proposition 3.1, with $r = n > 0$. If $s \in H$ (that is, if $[s] = [0]$), then it is easy to verify from the above formula that $\tau_{[s]} f = f$. On the other hand, if $s \notin H$, write $x \in s + c + \mathbb{Z}_p$ as $x \in c + (m_3 + m_4 \sqrt{-1})/3^\ell + 3^n H$, for some $\ell > 1$ and some integers m_3, m_4 not both divisible by 3. (Note that m_3 and m_4 are only determined modulo $p^{n+\ell}$.) The trace map from $\mathbb{Q}_3(\sqrt{-1})$ to \mathbb{Q}_3 takes $r_1 + r_2 \sqrt{-1}$ to $2r_1$, and so the sum in the formula for $\tau_{[s]} f$ becomes

$$\begin{aligned} \tau_{[s]} f(x) &= 9^{-n} \left[\sum_{m_1=0}^{3^n-1} \sum_{m_2=0}^{3^n-1} (x - c, (m_1 + m_2 \sqrt{-1})/3^n) \right] \\ &= 9^{-n} \sum_{m_1=0}^{3^n-1} \sum_{m_2=0}^{3^n-1} e^{4\pi i(m_1 m_3 - m_2 m_4)/3^{\ell+n}}. \end{aligned}$$

We may assume that $1 \leq m_3, m_4 \leq p^{\ell+n} - 1$. As in Example 3.3, we should expect a great deal of cancellation in the sum if either or both of m_3 and m_4 are larger than p^ℓ . Thus, most of the weight of $\tau_{[s]} \mathbf{1}_{c+3^n H}$ should be concentrated in the (relatively few) regions $b + 3^n H$ corresponding to small values of m_3 and m_4 .

4. Haar and Shannon wavelets

Given our dilation operator δ_A and our group of translation operators $\Gamma_{G/H} = \{\tau_{[s]}\}$, we may now define wavelets on G as in [2].

DEFINITION 4.1. Let G be a locally compact abelian group with compact open subgroup $H \subseteq G$. Let A be an automorphism of G , expansive with respect to H . Let \mathcal{D} be a choice of coset representatives in \widehat{G} for $\widehat{H} = \widehat{G}/H^\perp$, and let $\Gamma_{G/H} = \{\tau_{[s]} : [s] \in G/H\}$ be the group of translation operators on $L^2(G)$ determined by \mathcal{D} via equation (3.2).

Let $\Psi = \{\psi_1, \dots, \psi_N\} \subseteq L^2(G)$ be a finite subset of $L^2(G)$. Ψ is a *set of wavelet generators* for $L^2(G)$ with respect to \mathcal{D} and A if

$$\{\delta_A^n \tau_{[s]} \psi_i : 1 \leq i \leq N, n \in \mathbb{Z}, [s] \in G/H\}$$

forms an orthonormal basis for $L^2(G)$. In that case, the resulting basis is called a *wavelet basis* for $L^2(G)$.

If $N = 1$ and $\Psi = \{\psi\}$, then ψ is a *wavelet* for $L^2(G)$.

The following theorems show that any group G of the type we have considered has both Haar and Shannon wavelets, and, perhaps surprisingly, that the Haar wavelets are precisely the same as the Shannon wavelets.

THEOREM 4.2. *Let G be a locally compact abelian group with compact open subgroup H , and let \mathcal{D} be a set of coset representatives for the quotient \widehat{G}/H^\perp . Let A be an expansive automorphism of G , and let $M = |A| \geq 2$ be the modulus of A . Let $\{\sigma_0, \sigma_1, \dots, \sigma_{M-1}\}$ be the M elements of $(A^*H^\perp)/H^\perp$, with $\sigma_0 \in H^\perp$.*

Define

$$\Omega_i = H^\perp + \sigma_i, \quad \text{and} \quad \psi_i = \check{\mathbf{1}}_{\Omega_i}, \quad \text{for all } i = 1, \dots, M-1.$$

That is, ψ_i is the inverse transform of the characteristic function of $H^\perp + \sigma_i$. Then $\{\psi_1, \dots, \psi_{M-1}\}$ is a set of wavelet generators for $L^2(G)$ with respect to \mathcal{D} and A .

THEOREM 4.3. *Let $G, H, \mathcal{D}, A, M, \{\sigma_i : i = 0, \dots, M-1\}$, and $\{\psi_i : i = 1, \dots, M-1\}$ be as in Theorem 4.2. Then for any $i = 1, \dots, M-1$ and any $[s] \in G/H$, the translated function $\tau_{[s]}\psi_i$ is*

$$\tau_{[s]}\psi_i(x) = (x, \sigma_i) \mathbf{1}_{s+H}(x).$$

Moreover, $\tau_{[s]}\psi_i$ is locally constant; specifically, it is constant on every set of the form $c + A^{-1}H$.

PROOF OF THEOREM 4.2. The sets $(A^*)^n(\sigma_i + H^\perp)$ are pairwise disjoint as n ranges over \mathbb{Z} and i ranges over $\{1, \dots, M-1\}$. Moreover, the union of all those sets is $\widehat{G} \setminus \{0\}$. Thus, up to sets of measure zero,

$$\{(A^*)^n \Omega_i : n \in \mathbb{Z}, i = 1, 2, \dots, M-1\}$$

tiles \widehat{G} . At the same time, each Ω_i is the translate of H^\perp by $\sigma_i - \sigma_0$. By the results of [2], then, $\{\psi_1, \dots, \psi_{M-1}\}$ is a set of wavelet generators. \square

PROOF OF THEOREM 4.3. By definition, $\tilde{\tau}_{[s]}\widehat{\psi}_i$ is the function

$$\tilde{\tau}_{[s]}\widehat{\psi}_i(\gamma) = \overline{(s, \eta(\gamma))} \widehat{\psi}_i(\gamma) = \overline{(s, \gamma - \sigma_i)} \mathbf{1}_{\sigma_i + H^\perp}(\gamma).$$

We compute $\tau_{[s]}\psi_i$ by taking the inverse transform:

$$\begin{aligned} \tau_{[s]}\psi_i(x) &= \int_{\sigma_i + H^\perp} \overline{(s, \gamma - \sigma_i)}(x, \gamma) d\nu(\gamma) = \int_{H^\perp} \overline{(s, \gamma)}(x, \gamma) (x, \sigma_i) d\nu(\gamma) \\ &= (x, \sigma_i) \int_{H^\perp} \overline{(x - s, \gamma)} d\nu(\gamma) = (x, \sigma_i) \mathbf{1}_{s+H}(x) \end{aligned}$$

as claimed.

Finally, $\tau_{[s]}\psi_i$ is constant on any set of the form $c + A^{-1}H$, by Proposition 2.1. This local constancy condition may also be proven directly from the above formula for $\tau_{[s]}\psi_i$; we leave the details to the reader. \square

By Theorem 4.2, the transform $\widehat{\psi}_i$ of each ψ_i is the characteristic function of an uncomplicated set $\Omega_i = \sigma_i + H^\perp$ formed by translating the fundamental domain H^\perp . Moreover, these $M-1$ sets, together with $\Omega_0 = H^\perp$ itself, tile A^*H^\perp . Thus, the functions $\{\psi_i\}$ may be considered analogues for G of the standard Shannon wavelets from \mathbb{R}^d .

On the other hand, by Theorem 4.3, each ψ_i is a function supported in H and which is constant on each of the M subsets of H of the form $c + A^{-1}H$. Furthermore, each of those constant values is an M -th root of unity in \mathbb{C} . (After all, $M\sigma_i \in H$, since AH/H is a group of order M .) Therefore, the functions $\{\psi_i\}$ are analogues for G of the standard Haar wavelets for \mathbb{R}^d .

As claimed, then, Shannon wavelets are the same as Haar wavelets for our group G . Moreover, these wavelets ψ_i are locally constant and have compact support; the same is true of their transforms $\widehat{\psi}_i$. Thus, our Haar/Shannon wavelets are simultaneously smooth and of compact support in both the space and transform domains.

According to Theorem 4.3, the Haar wavelet ψ_i satisfies

$$\psi_i(x) = (x, \sigma_i) \mathbf{1}_H(x), \quad \text{and} \quad \tau_{[s]} \psi_i(x) = (x, \sigma_i) \mathbf{1}_{s+H}(x).$$

On the other hand, the usual translation-by- s operator sends ψ_i to

$$\psi_i(x-s) = \overline{(s, \sigma_i)} (x, \sigma_i) \mathbf{1}_{s+H}(x),$$

which is simply the constant $\overline{(s, \sigma_i)}$ multiplied by $\tau_{[s]} \psi_i(x)$. In particular, as Kozyrev [17] showed for the special case of \mathbb{Q}_p , $\{\psi_i : 1 \leq i \leq M-1\}$ is a set of wavelet generators even if the usual translation-by- s operators are used, as s ranges over a set \mathcal{C} of coset representatives for G/H .

Similarly, if we replace our “dual lattice” \mathcal{D} by another choice \mathcal{D}' of coset representatives for \widehat{G}/H^\perp , then the functions $\tau'_{[s]} \psi_i$ are different from $\tau_{[s]} \psi_i$, but only by a constant multiple $(s, \sigma_i - \sigma'_i)$. Thus, $\{\psi_i : 1 \leq i \leq M-1\}$ is still a set of wavelet generators for the translation operators $\tau'_{[s]}$ induced by \mathcal{D}' .

The fact that waveletness is preserved even when the translation operators are changed so arbitrarily is unique to these particular wavelets. It happens mainly because the support of $\widehat{\psi}_i$ is contained in a single coset $\sigma + H^\perp$, and therefore ψ_i is the exponential (x, σ) multiplied by the characteristic function of H . There are simply not very many candidates for a new function $\tau_{[s]} \psi_i$ which could reasonably be considered some kind of translation of ψ_i by the coset $s + H$.

Other wavelets have transforms $\widehat{\psi}$ with support in at least two cosets $\sigma + H^\perp$, and therefore the usual translation operators do not work as well. In particular, the fact that there is no lattice in $G = \mathbb{Q}_p$ (or many other such groups) means that no discrete set of usual translation operators can form a group, making it very difficult for all the translations of a non-Haar ψ to be orthonormal. In the next section we shall consider a few examples of such non-Haar wavelets.

5. Other examples

If our expansive map A has modulus $|A| \geq 3$, then the sets of wavelet generators described in Section 4 have at least two elements. Using wavelet sets, J. Benedetto and the author produced (single) wavelets in the same context [2]. We state a special case of that result here without proof:

THEOREM 5.1 (From [2]). *Let G be a locally compact abelian group with compact open subgroup H , let A be an expansive automorphism of G , and let $M = |A| \geq 2$. Let \mathcal{D} be a set of coset representatives for the quotient \widehat{G}/H^\perp , and let σ_0 be the unique element of $\mathcal{D} \cap H^\perp$. Define*

$$\mathcal{E} = (\mathcal{D} \cap A^* H^\perp) \setminus \{\sigma_0\},$$

and let $\{V_\sigma : \sigma \in \mathcal{E}\}$ be a partition of H^\perp into $M-1$ measurable subsets.

Define a function $T : H^\perp \rightarrow \widehat{G}$ by

$$T(\gamma) = \gamma + \sigma - \sigma_0 \quad \text{for } \gamma \in V_\sigma.$$

Then there is a measurable set $\Omega \subset A^*H^\perp$ such that, up to measure zero subsets,

$$(5.1) \quad \Omega = \left(H^\perp \setminus \bigcup_{n \geq 1} (A^*)^{-n} \Omega \right) \cup T \left(\bigcup_{n \geq 1} (A^*)^{-n} \Omega \right).$$

Let $\psi = \mathbf{1}_\Omega$ be the inverse transform of the characteristic function of Ω . Then ψ is a wavelet for $L^2(G)$.

The set Ω may be constructed from T inductively as follows. Define $\Omega_0 = H^\perp$; for every $n \geq 1$, define

$$\Lambda_n = \Omega_{n-1} \cap \left(\bigcup_{i \geq 1} (A^*)^{-i} \Omega_{n-1} \right) \quad \text{and} \quad \Omega_n = (\Omega_{n-1} \setminus \Lambda_n) \cup T\Lambda_n.$$

Then let

$$(5.2) \quad \Lambda = \bigcup_{n \geq 1} \Lambda_n, \quad \text{and} \quad \Omega = (H^\perp \setminus \Lambda) \cup T\Lambda.$$

We may compute the resulting wavelet ψ , and any translate $\tau_{[s]}\psi$, as follows. With notation as in Theorem 5.1, we have

$$\begin{aligned} \tau_{[s]}\psi(x) &= \int_{\Omega} (x, \gamma) \overline{(s, \eta(\gamma))} d\nu(\gamma) \\ &= \int_{H^\perp \setminus \Lambda} (x, \gamma) \overline{(s, \gamma - \sigma_0)} d\nu(\gamma) + \sum_{\sigma \in \mathcal{E}} \int_{T(\Lambda \cap V_\sigma)} (x, \gamma) \overline{(s, \gamma - \sigma)} d\nu(\gamma) \\ &= (s, \sigma_0) \left[\int_{H^\perp \setminus \Lambda} (x - s, \gamma) d\nu(\gamma) + \sum_{\sigma \in \mathcal{E}} \int_{(\Lambda \cap V_\sigma)} (x, \gamma + \sigma - \sigma_0) \overline{(s, \gamma)} d\nu(\gamma) \right]. \end{aligned}$$

If we add and subtract $(s, \sigma_0) \int_{\Lambda} (x - s, \gamma) d\nu(\gamma)$, we see that the ψ produced by Theorem 5.1 is given by

$$(5.3) \quad \tau_{[s]}\psi(x) = (s, \sigma_0) \left[\mathbf{1}_H(x - s) + \sum_{\sigma \in \mathcal{E}} ((x, \sigma - \sigma_0) - 1) \int_{(\Lambda \cap V_\sigma)} (x - s, \gamma) d\nu(\gamma) \right].$$

Note that ψ has transform $\widehat{\psi} = \mathbf{1}_\Omega$ with support contained in A^*H^\perp . Thus, for any $[s] \in G/H$, $\tau_{[s]}\psi$ is constant on any set of the form $c + A^{-1}H$, by Proposition 2.1.

In the special case that $|A| = 2$, then \mathcal{E} has only one element σ_1 . Therefore there is only one choice for T (namely $T(\gamma) = \gamma + \sigma_1 - \sigma_0$ for all $\gamma \in H^\perp$), which produces $\Omega = \sigma_1 + H^\perp$. The resulting ψ is simply the Haar/Shannon wavelet from Section 4.

On the other hand, if $|A| \geq 3$, then although ψ is locally constant, it does not have compact support. We now consider examples of such wavelets ψ .

EXAMPLE 5.2. Let $p \geq 2$ be a prime number, let $G = \mathbb{Q}_p$, let $H = \mathbb{Z}_p$, and let A be multiplication-by- $1/p^r$, for some integer $r \geq 1$; note that $|A| = p^r$. Identify \widehat{G} as \mathbb{Q}_p and H^\perp as \mathbb{Z}_p , as in Example 2.4. Let \mathcal{D} be the set of coset representatives for \widehat{G}/H^\perp defined in Example 3.3. Define $V_{1/p^r} = H^\perp$, and $V_\sigma = \emptyset$ for all $\sigma \in \mathcal{D} \setminus \{1/p^r\}$; then define T as in Theorem 5.1. That is, for all $\gamma \in H^\perp$, $T(\gamma) = \gamma + 1/p^r$. For every $n \geq 1$, the subset Λ_n of \widehat{G} is

$$\Lambda_n = 1 + p^r + p^{2r} + \cdots + p^{(n-2)r} + p^{nr} \mathbb{Z}_p,$$

which is the closed ball of radius $1/p^{nr}$ about $1 + p^r + \dots + p^{(n-2)r} \in \widehat{G}$, and which has measure $1/p^{nr}$. (Note that $\Lambda_1 = p^r \mathbb{Z}_p$ is the closed ball of radius $1/p^r$ about 0.) Define Λ and Ω as in equation (5.2). Note that $\Lambda = \bigcup \Lambda_n \subset \mathbb{Z}_p$ is a disjoint union of countably many balls, and that $\nu(\Lambda) = 1/(p^r - 1)$. Define $\psi = \check{\mathbf{1}}_\Omega$; then by Theorem 5.1, ψ is a (single) wavelet.

We can describe $\tau_{[s]}\psi(x)$ explicitly using equation (5.3), as follows. Since there is only one nonempty V_σ , namely $V_{1/p^r} = \mathbb{Z}_p$, we have $\Lambda \cap V_{1/p^r} = \Lambda$, and therefore

$$\tau_{[s]}\psi(x) = \mathbf{1}_{\mathbb{Z}_p}(x - s) + ((x, 1/p^r) - 1) \sum_{n \geq 1} \int_{\Lambda_n} (x - s, \gamma) d\gamma.$$

Given $x, s \in G$ and $n \geq 1$, we compute

$$\int_{\Lambda_n} (x - s, \gamma) d\gamma = \frac{1}{p^{nr}} (x - s, 1 + p^r + \dots + p^{(n-2)r}) \mathbf{1}_{p^{-nr} \mathbb{Z}_p}(x - s).$$

Thus, for $x, s \in G$, if we let N be the smallest positive integer such that $x - s \in p^{-Nr} \mathbb{Z}_p$, then

$$\begin{aligned} \tau_{[s]}\psi(x) &= \mathbf{1}_{\mathbb{Z}_p}(x - s) + \frac{[(x, 1/p^r) - 1]}{p^{Nr}} (x - s, 1 + p^r + \dots + p^{(N-2)r}) \\ &\quad + \frac{[(x, 1/p^r) - 1]}{p^{Nr}(p^r - 1)} (x - s, 1 + p^r + \dots + p^{(N-1)r}), \end{aligned}$$

because $(x - s, \gamma) = 1$ for all $\gamma \in p^{-Nr} \mathbb{Z}_p$. Note that the p^{Nr} in the denominator forces $\tau_{[s]}\psi$ to decay as x goes to ∞ in \mathbb{Q}_p .

EXAMPLE 5.3. Let $G = \mathbb{Q}_3(\sqrt{-1})$, let $H = \mathbb{Z}_3[\sqrt{-1}]$, and let A be multiplication-by- $1/3^r$, for some integer $r \geq 1$; note that $|A| = 9^r$. Identify \widehat{G} as $\mathbb{Q}_3(\sqrt{-1})$ and H^\perp as $\mathbb{Z}_3[\sqrt{-1}]$, as in Example 2.6. Let \mathcal{D} be the set of coset representatives for \widehat{G}/H^\perp defined in Example 3.5. Define $V_{1/3^r} = H^\perp$, and $V_\sigma = \emptyset$ for all $\sigma \in \mathcal{D} \setminus \{1/3^r\}$; then define T as in Theorem 5.1. That is, for all $\gamma \in H^\perp$, $T(\gamma) = \gamma + 1/3^r$. For every $n \geq 1$, the subset Λ_n of \widehat{G} is

$$\Lambda_n = 1 + 3^r + 3^{2r} + \dots + 3^{(n-2)r} + 3^{nr} \mathbb{Z}_3[\sqrt{-1}],$$

which is the closed ball of radius $1/3^{nr}$ about $1 + 3^r + \dots + 3^{(n-2)r} \in \widehat{G}$, and which has measure $1/9^{nr}$.

As before, the set Λ from equation (5.2) is a disjoint union of countably many balls, and that $\nu(\Lambda) = 1/(9^r - 1)$. Define $\psi = \check{\mathbf{1}}_\Omega$; then by Theorem 5.1, ψ is a (single) wavelet. By a computation similar to that in Example 5.2, we find that

$$\begin{aligned} \tau_{[s]}\psi(x) &= \mathbf{1}_H(x - s) + \frac{[(x, 1/3^r) - 1]}{9^{Nr}} (x - s, 1 + 3^r + \dots + 3^{(N-2)r}) \\ &\quad + \frac{[(x, 1/3^r) - 1]}{9^{Nr}(9^r - 1)} (x - s, 1 + 3^r + \dots + 3^{(N-1)r}), \end{aligned}$$

where N be the smallest positive integer such that $x - s \in 3^{-Nr} H$.

EXAMPLE 5.4. Let $p \geq 2$ be a prime number, let $G = \mathbb{F}_p((t))$, let $H = \mathbb{F}_p[[t]]$, let A be multiplication-by- $1/t$, and identify \widehat{G} as $\mathbb{F}_p((t))$ and H^\perp as $\mathbb{F}_p[[t]]$, as in Example 2.5. Let \mathcal{D} be the set of coset representatives for \widehat{G}/H^\perp defined in Example 3.4. Define $V_{1/t} = H^\perp$, and $V_\sigma = \emptyset$ for all $\sigma \in \mathcal{D} \setminus \{1/t\}$, and define T as in Theorem 5.1. That is, for all $\gamma \in H^\perp$, $T(\gamma) = \gamma + 1/t$.

The resulting Λ_n , Λ , and Ω are exactly analogous to those in Example 5.2, but with t in place of p . As in that case, we get

$$\tau_{[s]}\psi(x) = \mathbf{1}_{\mathbb{F}_p[[t]]}(x-s) + \left[\frac{(x, 1/t) - 1}{p^N(p-1)} \right] (x-s, 1+t+\dots+t^{N-1}).$$

(We still have p appearing in the denominator because the measure of Λ_n is the *real number* p^n , not an element of G .)

This time, we can simplify the expression even further. Write

$$\begin{aligned} x &= a_{-N}t^{-N} + a_{-N+1}t^{-N+1} + \dots \quad \text{and} \\ x-s &= b_{-N}t^{-N} + b_{-N+1}t^{-N+1} + \dots, \end{aligned}$$

where N is the smallest positive integer for which $b_{-N} \neq 0$ (or $N = 1$ if no such integer exists). Then

$$\tau_{[s]}\psi(x) = \mathbf{1}_{\mathbb{F}_p[[t]]}(x-s) + \left[\frac{(e^{2\pi i a_0/p} - 1)}{p^N(p-1)} \right] \left[e^{2\pi i c/p} + (p-1)e^{2\pi i d/p} \right],$$

where

$$\begin{aligned} c &= b_{-N+1} + b_{-N+2} + \dots + b_{-1} \in \mathbb{F}_p, \quad \text{and} \\ d &= b_{-N} + b_{-N+1} + \dots + b_{-1} \in \mathbb{F}_p. \end{aligned}$$

EXAMPLE 5.5. Let $G = \mathbb{Q}_3$, let $H = \mathbb{Z}_3$, and let A be multiplication-by- $1/3$, so that $|A| = 3$. Identify \widehat{G} as \mathbb{Q}_3 and H^\perp as \mathbb{Z}_3 , as in Example 2.4, for $p = 3$. Let \mathcal{D} be the set of coset representatives for \widehat{G}/H^\perp defined in Example 3.3. Define $V_{1/3} = (3\mathbb{Z}_3) \cup (2 + 3\mathbb{Z}_3)$, $V_{2/3} = 1 + 3\mathbb{Z}_3$, and $V_\sigma = \emptyset$ for all $\sigma \in \mathcal{D} \setminus \{1/3, 2/3\}$, and define T as in Theorem 5.1. That is, for all $\gamma \in 1 + 3\mathbb{Z}_3$, $T(\gamma) = \gamma + 2/3$; and for all $\gamma \in \mathbb{Z}_3 \setminus (1 + 3\mathbb{Z}_3)$, $T(\gamma) = \gamma + 1/3$. Then for $n \geq 1$,

$$\Lambda_n = \begin{cases} 1 + 2 \cdot 3 + 1 \cdot 3^2 + 2 \cdot 3^3 + \dots + 1 \cdot 3^{n-2} + 3^n \mathbb{Z}_3, & \text{if } n \text{ is even;} \\ 2 + 1 \cdot 3 + 2 \cdot 3^2 + 1 \cdot 3^3 + \dots + 1 \cdot 3^{n-2} + 3^n \mathbb{Z}_3, & \text{if } n \text{ is odd} \end{cases}$$

Thus, Λ_n is a closed ball of radius $1/3^n$ (and hence measure $1/3^n$) about a point which is in $1 + 3\mathbb{Z}_3$ if n is even, or in $2 + 3\mathbb{Z}_3$ if $n \geq 3$ is odd, or at 0 if $n = 1$. For Λ and Ω as in equation (5.2), we see as before that $\Lambda = \bigcup \Lambda_n \subset \mathbb{Z}_3$ is a disjoint union of countably many balls, and that $\nu(\Lambda) = 1/(3-1) = 1/2$.

Define $\psi = \mathbf{1}_\Omega$; then by Theorem 5.1, ψ is a (single) wavelet. And again, $\tau_{[s]}\psi$ must be constant on every set of the form $c + 3\mathbb{Z}_3 \subset G$, for any $[s] \in G/H$; but ψ does not have compact support. This time, when we compute $\tau_{[s]}\psi(x)$ by equation (5.3), we get

$$\begin{aligned} \tau_{[s]}\psi(x) &= \mathbf{1}_{\mathbb{Z}_3}(x-s) \\ &+ \frac{[(x, 2/3) - 1]}{3^N} (x-s, 1 + 2 \cdot 3 + 1 \cdot 3^2 + \dots + 1 \cdot 3^{(N-2)}) \\ &+ \frac{[(x, 2/3) - 1]}{8 \cdot 3^N} (x-s, 1 + 2 \cdot 3 + 1 \cdot 3^2 + \dots + 2 \cdot 3^{(N-1)}) \\ &+ \frac{[(x, 1/3) - 1]}{8 \cdot 3^{N-1}} (x-s, 2 + 1 \cdot 3 + 2 \cdot 3^2 + \dots + 1 \cdot 3^{(N-1)}), \end{aligned}$$

if N is even, and

$$\begin{aligned} \tau_{[s]}\psi(x) &= \mathbf{1}_{\mathbb{Z}_3}(x-s) \\ &+ \frac{[(x, 1/3) - 1]}{3^N}(x-s, 2 + 1 \cdot 3 + 2 \cdot 3^2 + \cdots + 1 \cdot 3^{(N-2)}) \\ &+ \frac{[(x, 1/3) - 1]}{8 \cdot 3^N}(x-s, 2 + 1 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{(N-1)}), \\ &+ \frac{[(x, 2/3) - 1]}{8 \cdot 3^{N-1}}(x-s, 1 + 2 \cdot 3 + 1 \cdot 3^2 + \cdots + 1 \cdot 3^{(N-1)}) \end{aligned}$$

if N is odd, where N is the smallest positive integer such that $x-s \in 3^{-N}\mathbb{Z}_3$.

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