HEIGHTS AND PREPERIODIC POINTS OF POLYNOMIALS OVER FUNCTION FIELDS

ROBERT L. BENEDETTO

ABSTRACT. Let K be a function field in one variable over an arbitrary field $\mathbb{F}$. Given a rational function $\phi \in K(z)$ of degree at least two, the associated canonical height on the projective line was defined by Call and Silverman. The preperiodic points of ϕ all have canonical height zero; conversely, if $\mathbb{F}$ is a finite field, then every point of canonical height zero is preperiodic. However, if $\mathbb{F}$ is an infinite field, then there may be non-preperiodic points of canonical height zero. In this paper, we show that for polynomial ϕ, such points exist only if ϕ is isotrivial. In fact, such K-rational points exist only if ϕ is defined over the constant field of K after a K-rational change of coordinates.

1. INTRODUCTION

Let K be a field with algebraic closure $\hat{K}$, and let $\phi : \mathbb{P}^1(\hat{K}) \to \mathbb{P}^1(\hat{K})$ be a morphism defined over K. We may write ϕ as a rational function $\phi \in K(z)$. Denote the n^{th} iterate of ϕ under composition by ϕ^n. That is, ϕ^0 is the identity function, and for $n \geq 1$, $\phi^n = \phi \circ \phi^{n-1}$. A point $x \in \mathbb{P}^1(\hat{K})$ is said to be *preperiodic* under ϕ if there are integers $n > m \geq 0$ such that $\phi^m(x) = \phi^n(x)$. Note that x is preperiodic if and only if its forward orbit $\{\phi^n(x) : n \geq 0\}$ is finite.

If K is a number field or a function field in one variable, and if $\deg \phi \geq 2$, there is a *canonical height* function $\hat{h}_\phi : \mathbb{P}^1(\hat{K}) \to \mathbb{R}$ associated to ϕ. (In this context, the degree $\deg \phi$ of ϕ is the maximum of the the degrees of its numerator and denominator.) The canonical height gives a rough measure of the arithmetic complexity of a given point, and it also satisfies the functional equation $\hat{h}_\phi(\phi(x)) = d \cdot \hat{h}_\phi(x)$, where $d = \deg \phi$. Canonical heights will be discussed further in Section 4; for more details, we refer the reader to the original paper [6] of Call and Silverman, or to the exposition in Part B of [9].

All preperiodic points of ϕ clearly have canonical height zero. Conversely, if K is a global field (i.e., a number field or a function field in one variable over a finite field), then $\hat{h}_\phi(x) = 0$ if and only if x is preperiodic. This equivalence is very useful in the study of rational preperiodic points over such fields; see, for example, [4, 5].

However, if K is a function field over an infinite field, there may be points of canonical height zero which are not preperiodic. For example, if $K = \mathbb{Q}(T)$ and $\phi(z) = z^2$, then the preperiodic points in $\mathbb{P}^1(\hat{K})$ are 0, ∞, and the roots of unity in $\hat{\mathbb{Q}}$; however, *all* points in $\mathbb{P}^1(\hat{\mathbb{Q}})$ have canonical height zero. Similarly, for the same field K, consider the function $\psi(z) = Tz^3$. In this case, the only K-rational points of canonical height zero are 0 and ∞,

Date: October 20, 2005; revised December 12, 2005.

2000 *Mathematics Subject Classification.* Primary: 11G50. Secondary: 11D45, 37F10.

Key words and phrases. filled Julia set, canonical height.

The author gratefully acknowledges the support of a Miner D. Crary Research Fellowship from Amherst College and NSA Young Investigator Grant H98230-05-1-0057.

both of which are preperiodic. Nevertheless, in $\mathbb{P}^1(\hat{K})$, any point of the form $aT^{-1/2}$ with $a \in \hat{\mathbb{Q}}$ has canonical height zero, but such a point is preperiodic if and only if a is either 0, ∞, or a root of unity.

It is important to note that both of the examples in the previous paragraph are of isotrivial maps (see Section 3). Specifically, $K = \mathbb{Q}(T)$ has constant field $\mathbb{Q}$, and ϕ is defined over $\mathbb{Q}$ as written. Similarly, although the coefficients of ψ are not constants of K, the $K(T^{1/2})$-rational change of coordinates $\gamma(z) = T^{-1/2}z$ makes ψ into the map $\gamma^{-1} \circ \psi \circ \gamma(z) = z^3$, which is defined over $\mathbb{Q}$.

In the theory of elliptic curves over function fields, it is well known that certain finiteness results for heights hold when the curve is not isotrivial; see, for example, Theorem 5.4 of [20]. In the same vein, the main result of this paper is that in the dynamical setting, once isotrivial maps are excluded, canonical heights on one-variable function fields have the same relation to preperiodic points as their analogues on global fields.

Theorem A. *Let $\mathbb{F}$ be an arbitrary field, let K be a finite extension of $\mathbb{F}(T)$, and let $\phi \in K[z]$ be a polynomial of degree at least two. Suppose that there is no K-rational affine change of coordinates $\gamma(z) = az + b$ for which $\gamma^{-1} \circ \phi \circ \gamma$ is defined over the constant field of K. Then for every point $x \in \mathbb{P}^1(K)$, x is preperiodic under ϕ if and only if $\hat{h}_\phi(x) = 0$.*

The substance of Theorem A is the statement that set $\mathcal{Z}$ of K-rational points of canonical height zero is finite. In fact, as we will observe in Remark 6.2, under the hypotheses of Theorem A, we can bound the size of $\mathcal{Z}$ in terms of the number s of places of bad reduction. Specifically, by invoking the arguments in [4], we can obtain a bound of $O(s \log s)$. Such statements are related to Northcott's finiteness theorem [16] and Morton and Silverman's uniform boundedness conjecture [14], both originally stated for global fields.

In addition, as an immediate consequence of Theorem A, we have the following result.

Theorem B. *Let $\mathbb{F}$ and K be as in Theorem A, and let $\phi \in K[z]$ be a polynomial of degree at least two. Suppose that ϕ is not isotrivial. Then for every point $x \in \mathbb{P}^1(\hat{K})$, x is preperiodic under ϕ if and only if $\hat{h}_\phi(x) = 0$.*

To prove the theorems, we will recall some basic background on function fields and their associated local fields in Section 2. Our main tools will be the standard notions of good and bad reduction and of filled Julia sets, defined in Section 3. In Section 4, we will recall the theory of canonical heights, which we will relate to reduction and filled Julia sets. Finally, after stating some lemmas in Section 5, we will prove our theorems in Section 6.

The author would like to thank Lucien Szpiro and Thomas Tucker for introducing him to the questions considered in this paper. Many thanks to Matthew Baker, Joseph Silverman, and again to Thomas Tucker, as well as to the referee, for a number of helpful suggestions to improve the exposition.

2. Background on Function Fields and Local Fields

In this section we recall the necessary fundamentals from the theory of function fields and their associated local fields. We also set some notational conventions for this paper. For more details on function fields and their absolute values, we refer the reader to Chapter 5 of [19], as well as Chapters 1–2 of [12], Section B.1 of [9], and Section 4.4 of [17]. See [8, 10] for expositions concerning the local fields $\mathbb{C}_v$.

2.1. **Function Fields, Places, and Absolute Values.** We set the following notation throughout this paper.

$\mathbb{F}$	an arbitrary field
K	a function field in one variable over $\mathbb{F}$; i.e., a finite extension of $\mathbb{F}(T)$
$\hat{K}$	an algebraic closure of K
M_K	a proper set of absolute values on K, satisfying a product formula.

We will explain this terminology momentarily. Here, $\mathbb{F}(T)$ is the field of rational functions in one variable with coefficients in $\mathbb{F}$. Recall that the *constant field* $\mathbb{F}_K$ of K is the set of all elements of K that are algebraic over $\mathbb{F}$. It is a finite extension of $\mathbb{F}$.

Recall that an *absolute value* on K is a function $|\cdot|_v : K \to \mathbb{R}$ satisfying $|x|_v \geq 0$ (with equality if and only if $x = 0$), $|xy|_v = |x|_v|y|_v$, and $|x + y|_v \leq |x|_v + |y|_v$, for all $x, y \in K$. The *trivial* absolute value is given by $|0|_v = 0$ and $|x|_v = 1$ for $x \in K^\times$. Two absolute values on K are said to be *equivalent* if they induce the same topology on K.

In this setting, to say that the set M_K is *proper* is to say that each $v \in M_K$ is nontrivial; that $v, w \in M_K$ are equivalent if and only if $v = w$; and that for any $x \in K^\times$, there are only finitely many $v \in M_K$ for which $|x|_v \neq 1$. That M_K satisfies a *product formula* is to say that there are positive real numbers $\{n_v : v \in M_K\}$ such that for any $x \in K^\times$,

$$\prod_{v \in M_K} |x|_v^{n_v} = 1. \tag{1}$$

(Note that the properness of M_K means that for any given x, the product on the left-hand side of (1) is really a finite product.) The absolute values $v \in M_K$ are called the *places* of K, though we note that some authors consider a place to be an equivalence class of absolute values on K.

The existence of the set M_K in the function field setting is guaranteed by the association between primes and equivalence classes of nontrivial absolute values on K. (Here, a prime is the maximal ideal P of a discrete valuation ring $R \subseteq K$ such that $\mathbb{F}_K \subseteq R$ and K is the field of fractions of R.)

Because K is a function field, we also have the following additional facts. First, all absolute values v in M_K are *non-archimedean*, which is to say that they satisfy the ultrametric triangle inequality

$$|x + y|_v \leq \max\{|x|_v, |y|_v\}.$$

Second, for any $x \in K^\times$, we have $x \in \mathbb{F}_K^\times$ (i.e., x is a nonzero constant) if and only if $|x|_v = 1$ for all $v \in M_K$. Third, we may normalize our absolute values so that $n_v = 1$ for all $v \in M_K$ in (1), giving the simplified product formula

$$\prod_{v \in M_K} |x|_v = 1 \qquad \text{for all } x \in K^\times$$

2.2. **Local fields.** For each $v \in M_K$, we can form the local field K_v, which is the completion of K with respect to $|\cdot|_v$. We write $\mathbb{C}_v$ for the completion of an algebraic closure $\hat{K}_v$ of K_v. (The absolute value v extends in a unique way to $\hat{K}_v$ and hence to $\mathbb{C}_v$.) The field $\mathbb{C}_v$ is then a complete and algebraically closed field.

Because v is non-archimedean, the disk $\mathcal{O}_v = \{c \in \mathbb{C}_v : |c|_v \leq 1\}$ forms a ring, called the *ring of integers* of $\mathbb{C}_v$, which has a unique maximal ideal $\mathcal{M}_v = \{c \in \mathbb{C}_v : |c|_v < 1\}$. The quotient $k_v = \mathcal{O}_v/\mathcal{M}_v$ is called the *residue field* of $\mathbb{C}_v$. The natural reduction map from $\mathcal{O}_v$

to k_v, sending $a \in \mathcal{O}$ to $\overline{a} = a + \mathcal{M}_v \in k_v$, will be used to define good and bad reduction of a polynomial in Definition 3.1 below; but after invoking a few simple lemmas about good and bad reduction, we will not need to refer to $\mathcal{O}_v$, $\mathcal{M}_v$, or k_v again.

2.3. **Disks.** Let $\mathbb{C}_v$ be a complete and algebraically closed field with absolute value v. Given $a \in \mathbb{C}_v$ and $r > 0$, we write

$$\overline{D}(a,r) = \{x \in \mathbb{C}_v : |x-a|_v \le r\} \quad \text{and} \quad D(a,r) = \{x \in \mathbb{C}_v : |x-a|_v < r\}$$

for the closed and open disks, respectively, of radius r centered at a. Note our convention that all disks have positive radius.

If v is non-archimedean and $U \subseteq \mathbb{C}_v$ is a disk, then the radius of U is unique; it is the same as the diameter of the set U viewed as a metric space. However, any point $b \in U$ is a center. That is, if $|b-a|_v \le r$, then $\overline{D}(a,r) = \overline{D}(b,r)$, and similarly for open disks. It follows that two disks intersect if and only if one contains the other.

3. Bad Reduction and Filled Julia Sets

The following definition originally appeared in [14]. We have modified it slightly so that "bad reduction" now means not potentially good, as opposed to not good.

Definition 3.1. *Let $\mathbb{C}_v$ be a complete, algebraically closed non-archimedean field with absolute value $|\cdot|_v$, ring of integers $\mathcal{O}_v = \{c \in \mathbb{C}_v : |c|_v \le 1\}$, and residue field k_v. Let $\phi(z) \in \mathbb{C}_v(z)$ be a rational function with homogenous presentation*

$$\phi([x,y]) = [f(x,y), g(x,y)],$$

where $f, g \in \mathcal{O}_v[x,y]$ are relatively prime homogeneous polynomials of degree $d = \deg \phi$, and at least one coefficient of f or g has absolute value 1. We say that ϕ has good reduction *at v if $\overline{f}$ and $\overline{g}$ have no common zeros in $k_v \times k_v$ besides $(x,y) = (0,0)$. We say that ϕ has* potentially good reduction *at v if there is some linear fractional transformation $h \in \mathrm{PGL}(2,\mathbb{C}_v)$ such that $h^{-1} \circ \phi \circ h$ has good reduction. If ϕ does not have potentially good reduction, we say it has* bad reduction *at v.*

Naturally, for $f(x,y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$, the reduction $\overline{f}(x,y)$ in Definition 3.1 means $\sum_{i=0}^{d} \overline{a}_i x^i y^{d-i}$.

In this paper, we will consider only polynomial ϕ of degree at least 2; that is, $\phi(z) = a_d z^d + \cdots + a_0$, where $d \ge 2$, $a_i \in \mathbb{C}_v$, and $a_d \ne 0$. It is easy to check that such ϕ has good reduction if and only if $|a_i|_v \le 1$ for all i and $|a_d|_v = 1$. In particular, by the properness of M_K, there are only finitely many places $v \in M_K$ at which ϕ has bad reduction. (The same finiteness result also holds for $\phi \in K(z)$, but we do not need that fact here.)

We recall the following standard definition from polynomial dynamics.

Definition 3.2. *Let $\mathbb{C}_v$ be a complete, algebraically closed field with absolute value $|\cdot|_v$, and let $\phi(z) \in \mathbb{C}_v[z]$ be a polynomial of degree $d \ge 2$. The* filled Julia set *of ϕ at v is*

$$\mathfrak{K}_{\phi,v} = \{x \in \mathbb{C}_v : \{|\phi^n(x)|_v\}_{n\ge 0} \text{ is bounded}\}.$$

We recall the following three fundamental facts. First, $\mathfrak{K}_{\phi,v}$ is invariant under ϕ; that is, $\phi^{-1}(\mathfrak{K}_{\phi,v}) = \phi(\mathfrak{K}_{\phi,v}) = \mathfrak{K}_{\phi,v}$. Second, all the finite preperiodic points of ϕ (that is, all the preperiodic points in $\mathbb{P}^1(\mathbb{C}_v)$ other than the fixed point at ∞) are contained in $\mathfrak{K}_{\phi,v}$. Finally, the polynomial $\phi \in \mathbb{C}_v[z]$ has good reduction if and only if $\mathfrak{K}_{\phi,v} = \overline{D}(0,1)$. (However, if ϕ has bad reduction, then $\mathfrak{K}_{\phi,v}$ can be a very complicated fractal set.)

Filled Julia sets have been studied extensively in the archimedean case $\mathbb{C}_v = \mathbb{C}$; see, for example, [1, 7, 13]. For the non-archimedean setting, see [3, 4, 18].

Finally, we recall that if $\phi \in K(z)$ is a rational function over the function field K, we say that ϕ is *isotrivial* if there is a finite extension L/K and an L-rational change of coordinates $\gamma \in \mathrm{PGL}(2, L)$ such that $\gamma^{-1} \circ \phi \circ \gamma$ is defined over a the constant field $\mathbb{F}_L$ of L. Note that if ϕ is a polynomial and such a γ exists, then there is a map β of the form $\beta(z) = az + b$ with $a \in L^\times$ and $b \in L$ such that $\beta^{-1} \circ \phi \circ \beta$ is defined over $\mathbb{F}_L$.

We comment for the interested reader that the term "isotrivial" is borrowed from the theory of moduli spaces. In that context, a trivial family has no dependence on the parameter(s), and an isotrivial family is one where all smooth fibers are isomorphic. See [2], for example, for more information.

4. Canonical Heights

We recall the following facts from the theory of heights and canonical heights presented in [6, 9, 12]. The standard *height function* $h : \mathbb{P}^1(\hat{K}) \to [0, \infty)$ is given by the formula

$$h(x) = \frac{1}{[L:K]} \sum_{v \in M_K} \sum_{w \in M_L,\, w|v} n_w \log \max\{|x_0|_w, |x_1|_w\}, \tag{2}$$

where L/K is a finite extension over which $x = [x_0, x_1]$ is defined, $n_w = [L_w : K_v]$ is the corresponding local field extension degree at w, and $w \mid v$ means that the restriction of w to K is v. It is well known that (2) is independent of the choices of homogeneous coordinates x_0, x_1 for x and of the extension L. In the special case that $x = f/g \in \mathbb{F}(T)$, where $f, g \in \mathbb{F}[T]$ are relatively prime polynomials, (2) can be rewritten as

$$h(f/g) = \max\{\deg f, \deg g\},$$

assuming a certain standard normalization is chosen for the absolute values.

In addition, if $\mathbb{F}$ is finite, then h is *nondegenerate* in the sense that for any real constants B, D, the set of $x \in \mathbb{P}^1(\hat{K})$ with $h(x) \le B$ and $[K(x):K] \le D$ is a finite set. Conversely, if $\mathbb{F}$ is infinite, then h fails to be nondegenerate, because all (infinitely many) points of $\mathbb{P}^1(\mathbb{F})$ have height zero. The essential point of this paper is that certain finiteness results can still be proven under weak hypotheses even in the case that $\mathbb{F}$ is infinite.

One crucial property of the height function is that it satisfies an approximate functional equation for any morphism. Indeed, if $\phi \in K(z)$ is a rational function of degree $d \ge 1$, there is a constant $C = C_\phi \ge 0$ such that

$$\text{for all } x \in \mathbb{P}^1(\hat{K}), \quad |h(\phi(x)) - d \cdot h(x)| \le C.$$

For a *fixed* $\phi(z)$ of degree $d \ge 2$, Call and Silverman [6] introduced a related *canonical* height function $\hat{h}_\phi : \mathbb{P}^1(\hat{k}) \to [0, \infty)$, with the property that there is a constant $C' = C'_\phi \ge 0$ such that for all $x \in \mathbb{P}^1(\hat{K})$,

$$\hat{h}_\phi(\phi(x)) = d \cdot \hat{h}(x) \quad \text{and} \quad |\hat{h}_\phi(x) - h(x)| \le C'. \tag{3}$$

(The height $\hat{h}_\phi$ generalized a construction of Néron and Tate; see [11, 15].) Note that by (3), a preperiodic point x of ϕ must have canonical height zero.

Call and Silverman proved that the canonical height has a decomposition

$$\hat{h}_\phi(x) = \sum_{v \in M_K} \hat{\lambda}_{\phi,v}(x) \qquad \text{for all } x \in K = \mathbb{P}^1(K) \setminus \{\infty\}, \tag{4}$$

where $\hat{\lambda}_{\phi,v}$ is the *local canonical height* for ϕ at v. For all places v of good reduction, we have simply $\hat{\lambda}_{\phi,v}(x) = \log \max\{|x|_v, 1\}$, as in the corresponding term of (2). More generally, if ϕ is a polynomial (still of degree $d \geq 2$), we have

$$\hat{\lambda}_{\phi,v}(x) = \lim_{n\to\infty} d^{-n} \log \max\{|\phi^n(x)|_v, 1\}. \tag{5}$$

It is easy to show that the limit converges and is nonnegative.

We will be able to avoid direct use of local canonical heights, because of the crucial fact that if ϕ is a polynomial, then $\hat{\lambda}_{\phi,v}(x) \geq 0$ for all $x \in K$, with equality if and only if $x \in \mathfrak{K}_{\phi,v}$. (In fact, $\hat{\lambda}_{\phi,v}$ satisfies the functional equation $\hat{\lambda}_{\phi,v}(\phi(x)) = d \cdot \hat{\lambda}_{\phi,v}(x)$ in the polynomial case.) Thus, a point $x \in K \subseteq \mathbb{P}^1(\hat{K})$ has (global) canonical height zero if and only if $x \in \mathfrak{K}_{\phi,v}$ for all $v \in M_K$.

For more information on local canonical heights, we refer the reader to [6] and to simpler results for polynomials in [5]. (The results of [5] assume that K is a number field, but all of the relevant proofs go through verbatim in the function field case.) In particular, the above description of $\hat{\lambda}_{\phi,v}$ comes from the algorithm in Theorem 5.3 of [6], a simplified form of which appears as Theorem 3.1 of [5]. Indeed, the description (5) for polynomials is the same as that given in Theorem 4.2 of [5], and the connections between local canonical heights and filled Julia sets are given in Corollary 5.3 of the same paper.

Remark 4.1. It is easy to show, using the defining properties (3) for $\hat{h}_\phi$, that a point $x \in \mathbb{P}^1(\hat{K})$ has canonical height zero if and only if the forward orbit $\{\phi^n(x) : n \geq 0\}$ is a set of bounded (standard) height. Thus, our main theorems can be rephrased without reference to canonical heights by saying that once isotrivial maps are excluded, preperiodic points in $\mathbb{P}^1(\hat{K})$ are precisely those points whose forward iterates have bounded height.

5. Some lemmas

We will need three lemmas, two of which appeared in [4].

Lemma 5.1. *Let $\mathbb{C}_v$ be a complete, algebraically closed field with non-archimedean absolute value $|\cdot|_v$. Let $\phi \in \mathbb{C}_v[z]$ be a polynomial of degree $d \geq 2$ with lead coefficient $a_d \in \mathbb{C}_v^\times$. Denote by $\mathfrak{K}_{\phi,v}$ the filled Julia set of ϕ in $\mathbb{C}_v$, and write $\rho_v = |a_d|_v^{-1/(d-1)}$. Then:*

a. *There is a unique smallest disk $U_0 \subseteq \mathbb{C}_v$ which contains $\mathfrak{K}_{\phi,v}$.*
b. *U_0 is a closed disk of some radius $r_v \geq \rho_v$.*
c. *ϕ has potentially good reduction if and only if $r_v = \rho_v$. In this case, $\mathfrak{K}_{\phi,v} = U_0$.*

Proof. This is the non-archimedean case of Lemma 2.5 of [4]. □

Lemma 5.2. *With notation as in Lemma 5.1, suppose that $r_v > \rho_v$. Then $\phi^{-1}(U_0)$ is a disjoint union of closed disks $V_1, \ldots, V_m \subseteq U_0$, where $2 \leq m \leq d$. Moreover, for each $i = 1, \ldots, m$, $\phi(V_i) = U_0$.*

Proof. This is a part of Lemma 2.7 of [4]. □

Lemma 5.3. *With notation as in Lemma 5.1, suppose that $r_v > \rho_v$. Let z_0 be any point in $\mathfrak{K}_{\phi,v}$, and let $X = \phi^{-2}(z_0)$, which is a set of at most d^2 points. Then*

$$\mathfrak{K}_{\phi,v} \subseteq \bigcup_{x\in X} D(x,\rho_v).$$

Proof. Denote by $y_1,\ldots,y_d$ the elements (repeated according to multiplicity) of $\phi^{-1}(z_0)$. For each $i = 1,\ldots,d$, denote by $x_{i,1},\ldots,x_{i,d}$ the elements (again, with multiplicity) of $\phi^{-1}(y_i)$. Thus, $X = \{x_{i,j} : 1 \le i,j \le d\}$.

Let $V_1,\ldots,V_m$ be the disks of Lemma 5.2. There must be two distinct disks V_k and V_ℓ of distance exactly r_v apart, by Lemma 5.1.a and the ultrametric triangle inequality, or else all m disks (and hence $\mathfrak{K}_{\phi,v}$) would fit inside a disk of radius smaller than r_v. In addition, by Lemma 5.2, for each $i = 1,\ldots,d$, each of V_k and V_ℓ must contain at least one of $x_{i,1},\ldots,x_{i,d}$. Again by the ultrametric property, then, for every $w \in \mathbb{C}_v$ and every $i = 1,\ldots,d$, there is some $j = 1,\ldots,d$ such that $|w - x_{i,j}|_v \ge r_v$.

Thus, given any point

$$w \in \mathbb{C}_v \setminus \bigcup_{x\in X} D(x,\rho_v),$$

and any index $i = 1,\ldots,d$, we have

$$|\phi(w) - y_i|_v = |a_d|_v \prod_{j=1}^{d} |w - x_{i,j}|_v \ge |a_d|_v \rho_v^{d-1} r_v = r_v,$$

where the inequality is by the previous paragraph, and the final equality is by the definition of ρ_v. It follows that for any such w,

$$|\phi^2(w) - z_0| = |a_d|_v \prod_{i=1}^{d} |w - y_i|_v \ge |a_d|_v r_v^d > |a_d|_v \rho_v^{d-1} r_v = r_v.$$

However, $U_0 = \overline{D}(z_0, r_v)$, so that $\phi^2(w) \notin U_0$, and hence $w \notin \mathfrak{K}_{\phi,v}$, as desired. □

6. Proofs

The following proposition will allow us to rephrase isotriviality in terms of reduction.

Proposition 6.1. *Let $\mathbb{F}$ and K be as in Theorem A, and let $\phi \in K[z]$ be a polynomial of degree at least two. Suppose that there are at least three points in $\mathbb{P}^1(K)$ of canonical height zero. Then the following are equivalent:*

a. *There is a K-rational affine change of coordinates $\gamma(z) = az+b$ for which $\gamma^{-1}\circ\phi\circ\gamma$ is defined over $\mathbb{F}_K$.*
b. *ϕ has potentially good reduction at every $v \in M_K$.*

Proof. The forward implication is immediate from the comments following Definition 3.1. For the converse, note that $\hat{h}_\phi(\infty) = 0$. By hypothesis, there are at least two other (distinct) points $z_0, z_1 \in K$ with $\hat{h}_\phi(z_0) = \hat{h}_\phi(z_1) = 0$. Let γ be the affine automorphism

$$\gamma(z) = (z_1 - z_0)z + z_0,$$

which is defined over K. Let $\psi = \gamma^{-1}\circ\phi\circ\gamma$, which is a polynomial in $K[z]$ of the same degree as ϕ, and which satisfies $0, 1 \in \mathfrak{K}_{\psi,v}$ for all $v \in M_K$. Write $\psi(z) = b_d z^d + \cdots + b_1 z + b_0$.

By Lemma 5.1, for each $v \in M_K$, the filled Julia set $\mathfrak{K}_{\psi,v}$ is contained in a disk of radius $\sigma_v = |b_d|_v^{-1/(d-1)}$. In fact, since $0, 1 \in \mathfrak{K}_{\psi,v}$, we have $\mathfrak{K}_{\psi,v} \subseteq \overline{D}(0, \sigma_v)$ and $\sigma_v \geq 1$. By the product formula applied to b_d, it follows that $\sigma_v = |b_d|_v = 1$ for all v. Thus, by Lemma 5.1.c, $\mathfrak{K}_{\psi,v} = \overline{D}(0,1)$, and therefore, by the comments following Definition 3.2, ψ has good reduction at each $v \in M_K$. By the comments following Definition 3.1, then, $|b_d|_v = 1$ and $|b_i|_v \leq 1$ for all $v \in M_K$ and all $0 \leq i \leq d-1$. By the product formula, this means for each i either that $b_i = 0$ or that $|b_i|_v = 1$ for all $v \in M_K$. Hence, by the second property listed near the end of Section 2.1, all coefficients of ψ lie in $\mathbb{F}_K$. □

We are now prepared to prove Theorem A.

Proof. Let $\mathcal{Z}$ denote the set of K-rational points points of canonical height zero. It suffices to show that $\mathcal{Z}$ is a finite set. Indeed, we have already seen that all preperiodic points have canonical height zero. Conversely, if $\mathcal{Z}$ is finite, then since $\phi(\mathcal{Z}) \subseteq \mathcal{Z}$, every point of $\mathcal{Z}$ has finite forward orbit under ϕ and is therefore preperiodic.

If $\mathcal{Z}$ has fewer than three elements, we are done. Otherwise, let S be the set of places $v \in M_K$ at which ϕ has bad (i.e., not potentially good) reduction, and let $s = \#S$. As remarked in Section 3, $s < \infty$. By the hypotheses and by Proposition 6.1, we have $s \geq 1$.

Write $\phi(z) = a_d z^d + \cdots + a_1 z + a_0$, and for each $v \in M_K$, let $\rho_v = |a_d|_v^{-1/(d-1)}$. For each bad place $v \in S$, $\mathfrak{K}_{\phi,v}$ is contained in a union of at most d^2 open disks of radius ρ_v, by Lemma 5.3. In addition, by Lemma 5.1, for each potentially good place $v \in M_K \setminus S$, $\mathfrak{K}_{\phi,v}$ is a single closed disk of radius ρ_v. For each choice $\{V_v : v \in S\}$ of one of the d^2 disks at each bad place, then, there is at most one K-rational point which lies in V_v at each $v \in S$ and in $\mathfrak{K}_{\phi,v}$ at each $v \in M_K \setminus S$. Indeed, if there were two such points x and y, then by the product formula,

$$1 = \prod_{v \in S} |x-y|_v \cdot \prod_{v \in M_K \setminus S} |x-y|_v < \prod_{v \in M_K} |a_d|_v^{-1/(d-1)} = 1,$$

which is a contradiction. (Here, the strict inequality is because $S \neq \emptyset$.)

There are only finitely many choices of one disk V_v at each $v \in S$. Thus, there are only finitely many K-rational points which lie in every filled Julia set, and hence only finitely many of canonical height zero. □

Remark 6.2. In the case $s \geq 1$ of Theorem A, the above proof shows that ϕ has at most $1 + d^{2s}$ K-rational preperiodic points, including the point at infinity. However, we can prove a far stronger bound by invoking verbatim the (longer and more complicated) argument in Case 1 of the proof of Theorem 7.1 in [4]. That proof, which involves bounds for certain products of differences of points in the various filled Julia sets, shows that the number of K-rational preperiodic points of ϕ is no more than

$$1 + (d^2 - 2d + 2)(s \log_d s + 3s)$$

if $1 \leq s \leq d-1$, and no more than

$$1 + (d^2 - 2d + 2)(s \log_d s + s \log_d \log_d s + 3s)$$

if $s \geq d$. Of course, if $s = 0$ but ϕ still satisfies the hypotheses of Theorem A, then the above proof gives an upper bound of 2.

Theorem B is now a direct consequence of Theorem A, as follows.

Proof. If $x \in \mathbb{P}^1(\hat{K})$ is preperiodic, we already know that $\hat{h}_\phi(x) = 0$. Conversely, if $\hat{h}_\phi(x) = 0$, let $L = K(x)$ be the field generated by x. By hypothesis, since L/K is a finite extension, ϕ is not L-rationally conjugate to a morphism defined over $\mathbb{F}_L$. Thus, by Theorem A, all points of $\mathbb{P}^1(L)$ of canonical height zero are preperiodic. In particular, x is preperiodic. □

References

1. Alan F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991, Complex analytic dynamical systems. MR MR1128089 (92j:30026)
2. Arnaud Beauville, *Le nombre minimum de fibres singulières d'une courbe stable sur* $\mathbf{P}^1$, Astérisque (1981), no. 86, 97–108, Séminaire sur les Pinceaux de Courbes de Genre au Moins Deux. MR MR642675 (83c:14020)
3. Robert L. Benedetto, *Components and periodic points in non-Archimedean dynamics*, Proc. London Math. Soc. (3) **84** (2002), no. 1, 231–256. MR MR1863402 (2002k:11215)
4. ———, *Preperiodic points of polynomials over global fields*, preprint. Available at `arxiv:math.NT/0506480`, 2005.
5. Gregory S. Call and Susan W. Goldstine, *Canonical heights on projective space*, J. Number Theory **63** (1997), no. 2, 211–243. MR MR1443758 (98c:11060)
6. Gregory S. Call and Joseph H. Silverman, *Canonical heights on varieties with morphisms*, Compositio Math. **89** (1993), no. 2, 163–205. MR MR1255693 (95d:11077)
7. Lennart Carleson and Theodore W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. MR MR1230383 (94h:30033)
8. Fernando Q. Gouvêa, *p-adic numbers*, second ed., Universitext, Springer-Verlag, Berlin, 1997, An introduction. MR MR1488696 (98h:11155)
9. Marc Hindry and Joseph H. Silverman, *Diophantine geometry*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000, An introduction. MR MR1745599 (2001e:11058)
10. Neal Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions*, second ed., Graduate Texts in Mathematics, vol. 58, Springer-Verlag, New York, 1984. MR MR754003 (86c:11086)
11. Serge Lang, *Les formes bilinéaires de Néron et Tate*, Séminaire Bourbaki, 1963/64, Fasc. 3, Exposé 274, Secrétariat mathématique, Paris, 1963/1964, p. 11. MR MR0176984 (31 #1252)
12. ———, *Fundamentals of Diophantine geometry*, Springer-Verlag, New York, 1983. MR MR715605 (85j:11005)
13. John Milnor, *Dynamics in one complex variable*, Friedr. Vieweg & Sohn, Braunschweig, 1999, Introductory lectures. MR MR1721240 (2002i:37057)
14. Patrick Morton and Joseph H. Silverman, *Rational periodic points of rational functions*, Internat. Math. Res. Notices (1994), no. 2, 97–110. MR MR1264933 (95b:11066)
15. André Néron, *Quasi-fonctions et hauteurs sur les variétés abéliennes*, Ann. of Math. (2) **82** (1965), 249–331. MR MR0179173 (31 #3424)
16. Douglas G. Northcott, *Periodic points on an algebraic variety*, Ann. of Math. (2) **51** (1950), 167–177. MR MR0034607 (11,615c)
17. Dinakar Ramakrishnan and Robert J. Valenza, *Fourier analysis on number fields*, Graduate Texts in Mathematics, vol. 186, Springer-Verlag, New York, 1999. MR MR1680912 (2000d:11002)
18. Juan Rivera-Letelier, *Dynamique des fonctions rationnelles sur des corps locaux*, Astérisque (2003), no. 287, xv, 147–230, Geometric methods in dynamics. II. MR MR2040006 (2005f:37100)
19. Michael Rosen, *Number theory in function fields*, Graduate Texts in Mathematics, vol. 210, Springer-Verlag, New York, 2002. MR MR1876657 (2003d:11171)
20. Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994.

Department of Mathematics and Computer Science, Amherst College, Amherst, MA 01002, USA

E-mail address: `rlb@cs.amherst.edu`

URL: `http://www.cs.amherst.edu/~rlb`