COMPUTING POINTS OF SMALL HEIGHT FOR CUBIC POLYNOMIALS

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Abstract. Let $\phi \in \mathbb{Q}[z]$ be a polynomial of degree d at least two. The associated canonical height $\hat{h}_\phi$ is a certain real-valued function on $\mathbb{Q}$ that returns zero precisely at preperiodic rational points of ϕ. Morton and Silverman conjectured in 1994 that the number of such points is bounded above by a constant depending only on d. A related conjecture claims that at non-preperiodic rational points, $\hat{h}_\phi$ is bounded below by a positive constant (depending only on d) times some kind of height of ϕ itself. In this paper, we provide support for these conjectures in the case $d = 3$ by computing the set of small height points for several billion cubic polynomials.

Let $\phi(z) \in \mathbb{Q}[z]$ be a polynomial with rational coefficients. Define $\phi^0(z) = z$, and for every $n \geq 1$, let $\phi^n(z) = \phi \circ \phi^{n-1}(z)$; that is, ϕ^n is the n-th iterate of ϕ under composition. A point x is said to be *periodic* under ϕ if there is an integer $n \geq 1$ such that $\phi^n(x) = x$. In that case, we say x is n-periodic; the smallest such positive integer n is called the *period* of x. More generally, x is *preperiodic* under ϕ if there are integers $n > m \geq 0$ such that $\phi^n(x) = \phi^m(x)$; equivalently, $\phi^m(x)$ is periodic for some $m \geq 0$.

In 1950, using the theory of arithmetic heights, Northcott proved that if $\deg \phi \geq 2$, then ϕ has only finitely many preperiodic points in $\mathbb{Q}$. (In fact, his result applied far more generally, to morphisms of N-dimensional projective space over any number field.) In 1994, motivated by Northcott's result and by analogies to torsion points of elliptic curves (for which uniform bounds were proven by Mazur [13] over $\mathbb{Q}$ and by Merel [14] over arbitrary number fields), Morton and Silverman proposed a dynamical Uniform Boundedness Conjecture [17, 18]. Their conjecture applied to the same general setting as Northcott's Theorem, but we state it here only for polynomials over $\mathbb{Q}$.

Conjecture 1 (Morton, Silverman 1994). *For any $d \geq 2$, there is a constant $M = M(d)$ such that no polynomial $\phi \in \mathbb{Q}[z]$ of degree d has more than M rational preperiodic points.*

Thus far only partial results towards Conjecture 1 have been proven. Several authors [17, 18, 19, 21, 24] have bounded the period of a rational periodic point in terms of the smallest prime of good reduction (see Definition 1.3). Others [5, 12, 15, 16, 22] have proven that polynomials of degree two cannot have rational periodic points of certain periods by studying the set of rational points on an associated dynamical modular curve; see also [23, Section 4.2]. A different method, introduced in [3] and generalized and sharpened in [2], gave (still non-uniform) bounds for the number of preperiodic points by taking into account *all* primes, including those of bad reduction.

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In a related vein, the canonical height function $\hat{h}_\phi : \mathbb{Q} \to [0, \infty)$ satisfies the functional equation $\hat{h}_\phi(\phi(z)) = d \cdot \hat{h}_\phi(z)$, where $d = \deg \phi$, and it has the property that $\hat{h}_\phi(x) = 0$ if and only if x is a preperiodic point of ϕ; see Section 1. Meanwhile, if we consider ϕ itself as a point in the appropriate moduli space of all polynomials of degree d, we can also define $h(\phi)$ to be the arithmetic height of that point; see [23, Section 4.11]. For example, the height of the quadratic polynomial $\phi(z) = z^2 + \frac{m}{n}$ is $h(\phi) := h(\frac{m}{n}) = \log \max\{|m|, |n|\}$; a corresponding height for cubic polynomials appears in Definition 4.4. Again by analogy with elliptic curves, we have the following conjecture, stating that the canonical height of a nonpreperiodic rational point cannot be too small in comparison to $h(\phi)$; see [23, Conjecture 4.98] for a more general version.

Conjecture 2. *Let $d \geq 2$. Then there is a positive constant $M' = M'(d) > 0$ such that for any polynomial $\phi \in \mathbb{Q}[z]$ of degree d and any point $x \in \mathbb{Q}$ that is not preperiodic for ϕ, we have $\hat{h}_\phi(x) \geq M' h(\phi)$.*

Just as Conjecture 1 says that any preperiodic rational point must land on a repeated value after a bounded number of iterations, Conjecture 2 essentially says that the size of a non-preperiodic rational point must start to explode within a bounded number of iterations. Some theoretical evidence for Conjecture 2 appears in [1, 10], and computational evidence when $d = 2$ appears in [6]. The smallest known value of $\hat{h}_\phi(x)/h(\phi)$ for $d = 2$ occurs for $x = \frac{7}{12}$ under $\phi(z) = z^2 - \frac{181}{144}$; the first few iterates are

$$\frac{7}{12} \mapsto \frac{-11}{12} \mapsto \frac{-5}{12} \mapsto \frac{-13}{12} \mapsto \frac{-1}{12} \mapsto \frac{-5}{4} \mapsto \frac{11}{36} \mapsto \frac{-377}{324} \mapsto \frac{2445}{26244} \mapsto \cdots.$$

(This example was found in [6] by a computer search.) The small canonical height ratio $\hat{h}_\phi(\frac{7}{12})/\log(181) \approx .0066$ makes precise the observation that although the numerators and denominators of the iterates eventually explode in size, it takes several iterations for the explosion to get underway.

In this paper, we investigate *cubic* polynomials with rational coefficients. We describe an algorithm to find preperiodic and small height rational points of such maps, and we present the resulting data, which supports both conjectures. In particular, after checking the fourteen billion cubics with coefficients of smallest height, we found none with more than eleven rational preperiodic points; those with exactly ten or eleven are listed in Table 2. Meanwhile, as regards Conjecture 2, the smallest height ratio $\mathfrak{h}_\phi(x) := \hat{h}_\phi(x)/h(\phi)$ we found was about .00025, for $\phi(z) = -\frac{25}{24}z^3 + \frac{97}{24}z + 1$ and the point $x = -\frac{7}{5}$, with orbit

$$-\frac{7}{5} \mapsto -\frac{9}{5} \mapsto -\frac{1}{5} \mapsto \frac{1}{5} \mapsto \frac{9}{5} \mapsto \frac{11}{5} \mapsto -\frac{6}{5} \mapsto -\frac{41}{20} \mapsto \frac{4323}{2560} \mapsto \cdots.$$

More importantly, although we found quite a few cubics throughout the search with a nonpreperiodic point of height ratio less than .001, only nine (listed in Table 6) gave $\mathfrak{h}_\phi(x) < .0007$, and the minimal one above was found early in the search. Thus, our data suggests that Conjecture 2 is true for cubic polynomials, with $M'(3) = .00025$.

The outline of the paper is as follows. In Section 1 we review heights, canonical heights, and local canonical heights. In Section 2 we state and prove formulas for estimating local canonical heights accurately in the case of polynomials. In Section 3, we discuss filled Julia sets (both complex and non-archimedean), and in Section 4 we consider cubics specifically. Finally, we describe our search algorithm in Section 5 and present the resulting data in Section 6.

Our exposition does not assume any background in either dynamics or arithmetic heights, but the interested reader is referred to Silverman's text [23]. For more details on non-archimedean filled Julia sets and local canonical heights, see [2, 3, 4].

1. Canonical Heights

Denote by $M_{\mathbb{Q}}$ the usual set $\{|\cdot|_\infty, |\cdot|_2, |\cdot|_3, |\cdot|_5, \dots\}$ of absolute values (also called places) of $\mathbb{Q}$, normalized to satisfy the product formula

$$\prod_{v\in M_{\mathbb{Q}}} |x|_v = 1 \qquad \text{for any nonzero } x \in \mathbb{Q}^\times.$$

(See [7, Chapters 2–3] or [9, Chapter 1], for example, for background on absolute values.) The standard (global) *height function* on $\mathbb{Q}$ is the function $h : \mathbb{Q} \to \mathbb{R}$ given by $h(x) := \log\max\{|m|_\infty, |n|_\infty\}$, if we write $x = m/n$ in lowest terms. Equivalently,

$$h(x) = \sum_{v\in M_{\mathbb{Q}}} \log\max\{1, |x|_v\} \qquad \text{for any } x \in \mathbb{Q}. \tag{1.1}$$

Of course, h extends to the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$; see [11, Section 3.1], [8, Section B.2], or [23, Section 3.1]. The height function satisfies two important properties. First, for any polynomial $\phi(z) \in \mathbb{Q}[z]$, there is a constant $C = C(\phi)$ such that

$$\big|h(\phi(x)) - d\cdot h(x)\big| \le C \qquad \text{for all } x \in \overline{\mathbb{Q}}, \tag{1.2}$$

where $d = \deg\phi$. Second, if we restrict h to $\mathbb{Q}$, then for any bound $B \in \mathbb{R}$,

$$\{x \in \mathbb{Q} : h(x) \le B\} \quad \text{is a finite set.} \tag{1.3}$$

For any fixed polynomial $\phi \in \mathbb{Q}[z]$ (or more generally, rational function) of degree $d \ge 2$, the *canonical height* function $\hat{h}_\phi : \overline{\mathbb{Q}} \to \mathbb{R}$ for ϕ is given by

$$\hat{h}_\phi(x) := \lim_{n\to\infty} d^{-n} h(\phi^n(x)),$$

and it satisfies the functional equation

$$\hat{h}_\phi(\phi(x)) = d\cdot \hat{h}_\phi(x) \qquad \text{for all } x \in \overline{\mathbb{Q}}. \tag{1.4}$$

(The convergence of the limit and the functional equation follow fairly easily from (1.2).) In addition, there is a constant $C' = C'(\phi)$ such that

$$\big|\hat{h}_\phi(x) - h(x)\big| \le C' \qquad \text{for all } x \in \overline{\mathbb{Q}}. \tag{1.5}$$

Northcott's Theorem [20] follows because properties (1.3), (1.4), and (1.5) imply that for any $x \in \mathbb{Q}$ (in fact, for any $x \in \overline{\mathbb{Q}}$), $\hat{h}(x) = 0$ if and only if x is preperiodic under ϕ.

For our computations, we will need to compute $\hat{h}_\phi(x)$ rapidly and accurately. Unfortunately, the constants C and C' in inequalities (1.2) and (1.5) given by the general theory are rather weak and are rarely described explicitly. The goal of Section 2 will be to improve these constants, using local canonical heights.

Definition 1.1. Let K be a field with absolute value v. We denote by $\mathbb{C}_v$ the completion of an algebraic closure of K. The function $\lambda_v : \mathbb{C}_v \to [0, \infty)$ given by

$$\lambda_v(x) := \log \max\{1, |x|_v\}$$

is called the *standard local height* at v. If $\phi(z) \in K[z]$ is a polynomial of degree $d \geq 2$, the associated *local canonical height* is the function $\hat{\lambda}_{v,\phi} : \mathbb{C}_v \to [0, \infty)$ given by

$$\hat{\lambda}_{v,\phi}(x) := \lim_{n\to\infty} d^{-n} \lambda_v\big(\phi^n(x)\big). \tag{1.6}$$

According to [3, Theorem 4.2], the limit in (1.6) converges, so that the definition makes sense. It is immediate that $\hat{\lambda}_{v,\phi}$ satisfies the functional equation $\hat{\lambda}_{v,\phi}(\phi(x)) = d \cdot \hat{\lambda}_{v,\phi}(x)$. In addition, it is well known that $\hat{\lambda}_{v,\phi}(x) - \lambda_v(x)$ is bounded independent of $x \in \mathbb{C}_v$; we shall prove a particular bound in Proposition 2.1 below.

Formula (1.6) of Definition 1.1 is specific to polynomials. For a rational function $\phi = f/g$, where $f, g \in K[z]$ are coprime polynomials and $\max\{\deg f, \deg g\} = d \geq 2$, the correct functional equation for $\hat{\lambda}_{v,\phi}$ is $\hat{\lambda}_{v,\phi}(\phi(x)) = d \cdot \hat{\lambda}_{v,\phi}(x) - \log |g(x)|_v$.

Of course, formula (1.1) may now be writen as $h(x) = \sum_{v \in M_{\mathbb{Q}}} \lambda_v(x)$ for any $x \in \mathbb{Q}$. The local canonical heights provide a similar decomposition for $\hat{h}_\phi$, as follows.

Proposition 1.2. *Let $\phi(z) \in \mathbb{Q}[z]$ be a polynomial of degree $d \geq 2$. Then for all $x \in \mathbb{Q}$,*

$$\hat{h}_\phi(x) = \sum_{v \in M_{\mathbb{Q}}} \hat{\lambda}_{v,\phi}(x).$$

Proof. See [4, Theorem 2.3], which applies to arbitrary number fields, with appropriate modifications. ☐

Often, the local canonical height $\hat{\lambda}_{v,\phi}$ exactly coincides with the standard local height λ_v; this happens precisely at the places of *good reduction* for ϕ. Good reduction of a map ϕ was first defined in [17]; see also [2, Definition 2.1]. For polynomials, it is well known (e.g., see [18, Example 4.2]) that those definitions are equivalent to the following.

Definition 1.3. Let K be a field with absolute value v, and let $\phi(z) = a_d z^d + \cdots + a_0 \in K[z]$ be a polynomial of degree $d \geq 2$. We say that ϕ has *good reduction* at v if

(1) v is non-archimedean,
(2) $|a_i|_v \leq 1$ for all $i = 0, \ldots, d$, and
(3) $|a_d|_v = 1$.

Otherwise, we say ϕ has *bad reduction* at v.

Note that if $K = \mathbb{Q}$ (or more generally, if K is a global field), a polynomial $\phi \in K[z]$ has bad reduction at only finitely many places $v \in M_K$. As claimed above, we have the following result, proven in, for example, [3, Theorem 2.2].

Proposition 1.4. *Let K be a field with absolute value v, and let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$ with good reduction at v. Then $\hat{\lambda}_{v,\phi} = \lambda_v$.*

For more background on heights and canonical heights, see [8, Section B.2], [11, Chapter 3], or [23, Chapter 3]; for local canonical heights, see [3] or [4, Section 2].

2. Computing Local Canonical Heights

Proposition 2.1. *Let K be a field with absolute value v, let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$, and let $\hat{\lambda}_{v,\phi}$ be the associated local canonical height. Write $\phi(z) = a_d z^d + \cdots + a_1 z + a_0 = a_d(z-\alpha_1)\cdots(z-\alpha_d)$, with $a_i \in K$, $a_d \neq 0$, and $\alpha_i \in \mathbb{C}_v$ Let $A = \max\{|\alpha_i|_v : i = 1, \dots, d\}$ and $B = |a_d|_v^{-1/d}$, and define real constants $c_v, C_v \geq 1$ by*

$$c_v = \max\{1, A, B\} \qquad \text{and} \qquad C_v = \max\{1, |a_0|_v, |a_1|_v, \dots, |a_d|_v\}$$

if v is non-archimedean, or

$$c_v = \max\{1, A+B\} \qquad \text{and} \qquad C_v = \max\{1, |a_0|_v + |a_1|_v + \ldots + |a_d|_v\}$$

if v is archimedean. Then for all $x \in \mathbb{C}_v$, $\dfrac{-d\log c_v}{d-1} \leq \hat{\lambda}_{v,\phi}(x) - \lambda_v(x) \leq \dfrac{\log C_v}{d-1}$.

Proof. First, we claim that $\lambda_v(\phi(x)) - d\lambda_v(x) \leq \log C_v$ for any $x \in \mathbb{C}_v$. To see this, if $|x|_v \leq 1$, then $|\phi(x)|_v \leq C_v$, and the desired inequality follows. If $|x|_v > 1$ and $|\phi(x)|_v \leq 1$, the inequality holds because $C_v \geq 1$. Finally, if $|x|_v > 1$ and $|\phi(x)|_v > 1$, then the claim follows from the observation that

$$\left|\frac{\phi(x)}{x^d}\right|_v = \left|a_d + a_{d-1}x^{-1} + \cdots + a_0 x^{-d}\right|_v \leq C_v.$$

Next, we claim that $\lambda_v(\phi(x)) - d\lambda_v(x) \geq -d\log c_v$ for any $x \in \mathbb{C}_v$. If $|x|_v \leq c_v$, then $\lambda_v(x) \leq \log c_v$ because $c_v \geq 1$; the desired inequality is therefore immediate from the fact that $\lambda_v(\phi(x)) \geq 0$. If $|x|_v > c_v$, then

$$\lambda_v(\phi(x)) - d\lambda_v(x) = \lambda_v(\phi(x)) - d\log|x|_v \geq \log|\phi(x)|_v - d\log|x|_v,$$

by definition of λ_v and because $|x|_v > c_v \geq 1$. To prove the claim, then, it suffices to show that $|\phi(x)|_v \geq (|x|_v/c_v)^d$ for $|x|_v > c_v$.

If v is non-archimedean, then $|x-\alpha_i|_v = |x|_v$ for all $i = 1, \dots, d$, since $|x|_v > A \geq |\alpha_i|_v$. Hence, $|\phi(x)|_v = |a_d|_v|x|_v^d = (|x|_v/B)^d \geq (|x|_v/c_v)^d$. If v is archimedean, then

$$\frac{|x-\alpha_i|_v}{|x|_v} \geq 1 - \frac{|\alpha_i|_v}{|x|_v} \geq 1 - \frac{A}{A+B} = \frac{B}{A+B} \qquad \text{for all } i = 1, \dots, d.$$

Thus, $|\phi(x)|_v \geq |a_d|_v(B|x|_v/(A+B))^d = (|x|_v/(A+B))^d \geq (|x|_v/c_v)^d$, as claimed.

To complete the proof, we compute

$$\hat{\lambda}_{v,\phi}(x) - \lambda_v(x) = \lim_{n\to\infty} \frac{1}{d^n}\lambda_v\big(\phi^n(x)\big) - \lambda_v(x) = \lim_{n\to\infty} \sum_{j=0}^{n-1} \frac{1}{d^j}\left[\frac{1}{d}\lambda_v\big(\phi^{j+1}(x)\big) - \lambda_v\big(\phi^j(x)\big)\right]$$

$$\geq \lim_{n\to\infty} \sum_{j=0}^{n-1} -\frac{1}{d^j}\log c_v = -\log c_v \sum_{j=0}^{\infty} \frac{1}{d^j} = \frac{-d\log c_v}{d-1}.$$

Similarly, $\hat{\lambda}_{v,\phi}(x) - \lambda_v(x) \leq (\log C_v)/(d-1)$. □

Remark 2.2. The proof above is just an explicit version of [4, Theorem 5.3], giving good bounds for 1, $1/z^d$, $\phi(z)$, and $\phi(z)/z^d$ in certain cases— e.g. a lower bound for $|\phi(x)/x^d|_v$ when $|x|_v$ is large. These are precisely the four functions $\{s_{ij}\}_{i,j\in\{0,1\}}$ in [4].

Remark 2.3. If v is non-archimedean, the quantity $A = \max\{|\alpha_i|_v\}$ can be computed directly from the coefficients of ϕ. Specifically,

$$A = \max\{|a_j/a_d|_v^{1/(d-j)} : 0 \le j \le d-1\}.$$

This identity is easy to verify by recognizing $(-1)^{d-j}a_j/a_d$ as the $(d-j)$-th symmetric polynomial in the roots $\{\alpha_i\}$; see also [3, Lemma 5.1].

On the other hand, if v is archimedean and $|x|_v > \sum_{j=0}^{d-1} |a_j/a_d|_v^{1/(d-j)}$, then

$$|a_d x^d|_v = |x|_v \cdot |a_d x^{d-1}|_v > \sum_{j=0}^{d-1} \left|\frac{a_j}{a_d}\right|_v^{1/(d-j)} \cdot |x^{d-j-1}|_v \cdot |a_d x^j|_v \ge \sum_{j=0}^{d-1} \left|\frac{a_j}{a_d}\right|_v \cdot |a_d x^j|_v$$
$$= \sum_{j=0}^{d-1} |a_j x^j|_v \ge |a_0 + a_1 x + \cdots + a_{d-1}x^{d-1}|_v,$$

and hence $\phi(x) \ne 0$. Thus, $A \le \sum_{j=0}^{d-1} |a_j/a_d|_v^{1/(d-j)}$ if v is archimedean.

Remark 2.4. Proposition 1.4 can be proven as a corollary of Proposition 2.1, because the constants c_v and C_v are both clearly zero if ϕ has good reduction.

The constants c_v and C_v of Proposition 2.1 can sometimes be improved (i.e., made smaller) by changing coordinates, and perhaps even leaving the original base field K. The following Proposition shows how local canonical heights change under scaling; but it actually applies to any linear fractional coordinate change.

Proposition 2.5. *Let K be a field with absolute value v, let $\phi(z) \in K[z]$ be a polynomial of degree $d \ge 2$, and let $\gamma \in \mathbb{C}_v^\times$. Define $\psi(z) = \gamma\phi(\gamma^{-1}z) \in \mathbb{C}_v[z]$. Then*

$$\hat{\lambda}_{v,\phi}(x) = \hat{\lambda}_{v,\psi}(\gamma x) \qquad \text{for all } x \in \mathbb{C}_v.$$

Proof. By exchanging ϕ and ψ if necessary, we may assume that $|\gamma|_v \ge 1$. For any $x \in \mathbb{C}_v$ and $n \ge 0$, let $y = \phi^n(x)$. Then $0 \le \lambda_v(\gamma y) - \lambda_v(y) \le \log|\gamma|_v$, because $\max\{|y|_v, 1\} \le \max\{|\gamma y|_v, 1\} \le |\gamma|_v \max\{|y|_v, 1\}$. Thus,

$$\hat{\lambda}_{v,\psi}(\gamma x) - \hat{\lambda}_{v,\phi}(x) = \lim_{n\to\infty} d^{-n}[\lambda_v(\psi^n(\gamma x)) - \lambda_v(\phi^n(x))]$$
$$= \lim_{n\to\infty} d^{-n}[\lambda_v(\gamma\phi^n(x)) - \lambda_v(\phi^n(x))] = 0 \qquad \square$$

Corollary 2.6. *Let K be a field with absolute value v, let $\phi(z) \in K[z]$ be a polynomial of degree $d \ge 2$, and let $\hat{\lambda}_{v,\phi}$ be the associated local canonical height. Let $\gamma \in \mathbb{C}_v^\times$, and define $\psi(z) = \gamma\phi(\gamma^{-1}z) \in \mathbb{C}_v[z]$. Let c_v and C_v be the constants from Proposition 2.1 for ψ. Then for all $x \in \mathbb{C}_v$, $\dfrac{-d\log c_v}{d-1} \le \hat{\lambda}_{v,\phi}(x) - \lambda_v(\gamma x) \le \dfrac{\log C_v}{d-1}$.*

We can now prove the main result of this section.

Theorem 2.7. *Let $\phi(z) \in \mathbb{Q}[z]$ be a polynomial of degree $d \ge 2$ with lead coefficient $a \in \mathbb{Q}^\times$. Let $e \ge 1$ be a positive integer, let $\gamma = \sqrt[e]{a} \in \overline{\mathbb{Q}}$ be an e-th root of a, and define $\psi(z) = \gamma\phi(\gamma^{-1}z)$. For each $v \in M_{\mathbb{Q}}$ at which ϕ has bad reduction, let c_v and C_v be the associated constants in Proposition 2.1 for $\psi \in \mathbb{C}_v[z]$. Then*

$$-\frac{1}{d^n}\tilde{c}(\phi,e) \le \hat{h}_\phi(x) - \frac{1}{ed^n} h\Big(a\big(\phi^n(x)\big)^e\Big) \le \frac{1}{d^n}\tilde{C}(\phi,e),$$

for all $x \in \mathbb{Q}$ and all integers $n \geq 0$, where

$$\tilde{c}(\phi, e) = \frac{d}{d-1} \sum_{v \text{ bad}} \log c_v, \qquad \text{and} \qquad \tilde{C}(\phi, e) = \frac{1}{d-1} \sum_{v \text{ bad}} \log C_v.$$

Proof. For any prime v of good reduction for ϕ, we have $|a|_v = 1$; therefore $|\gamma|_v = 1$, and $\lambda_v(\gamma y) = \lambda_v(y)$ for all $y \in \mathbb{C}_v$. Hence, by equation (1.4), Propositions 1.2 and 1.4, and Corollary 2.6, we compute

$$\begin{aligned} d^n \hat{h}_\phi(x) = \hat{h}_\phi\big(\phi^n(x)\big) &= \sum_{v \in M_\mathbb{Q}} \hat{\lambda}_{v,\phi}\big(\phi^n(x)\big) = \sum_{v \text{ good}} \lambda_v\big(\phi^n(x)\big) + \sum_{v \text{ bad}} \hat{\lambda}_{v,\phi}\big(\phi^n(x)\big) \\ &\geq -\tilde{c}(\phi, e) + \sum_{v \in M_\mathbb{Q}} \lambda_v\big(\gamma \phi^n(x)\big) = -\tilde{c}(\phi, e) + \frac{1}{e} \sum_{v \in M_\mathbb{Q}} \lambda_v\Big(a\big(\phi^n(x)\big)^e\Big), \end{aligned}$$

since $e\lambda_v(y) = \lambda_v(y^e)$ for all $y \in \mathbb{C}_v$. The lower bound is now immediate from the summation formula (1.1). The proof of the upper bound is similar. $\square$

Remark 2.8. The point of Theorem 2.7 is to approximate $\hat{h}_\phi(x)$ even more accurately than the naive estimate $d^{-n}h(\phi^n(x))$, by first changing coordinates to make ϕ monic. Of course, that coordinate change may not be defined over $\mathbb{Q}$; fortunately, the expression $a(\phi^n(x))^e$ at the heart of the Theorem still lies in $\mathbb{Q}$, and hence its height is easy to compute quickly.

Remark 2.9. By essentially the same proof, Theorem 2.7 also holds (with appropriate modifications) for any global field K in place of $\mathbb{Q}$.

3. Filled Julia sets

The following definition is standard in both complex and non-archimedean dynamics.

Definition 3.1. Let K be a field with absolute value v, and let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$. The *filled Julia set* $\mathfrak{K}_v$ of ϕ at v is

$$\mathfrak{K}_v := \big\{x \in \mathbb{C}_v : \{|\phi^n(x)|_v : n \geq 0\} \text{ is bounded}\big\}.$$

Note that $\phi^{-1}(\mathfrak{K}_v) = \mathfrak{K}_v$. Also note that $\mathfrak{K}_v$ can be defined equivalently as the set of $x \in \mathbb{C}_v$ such that $|\phi^n(x)|_v \not\to \infty$ as $n \to \infty$. In addition, the following well known result relates $\mathfrak{K}_v$ to $\hat{\lambda}_{v,\phi}$; the (easy) proof can be found in [3, Theorem 6.2].

Proposition 3.2. *Let K be a field with absolute value v, and let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$. For any $x \in \mathbb{C}_v$, we have $\hat{\lambda}_{v,\phi}(x) = 0$ if and only if $x \in \mathfrak{K}_v$.*

Because the local canonical height of a polynomial takes on only nonnegative values, Propositions 1.2 and 3.2 imply that any rational preperiodic points must lie in $\mathfrak{K}_v$ at every place v. However, $\mathfrak{K}_v$ is often a complicated fractal set. Thus, the following Lemmas, which specify disks containing $\mathfrak{K}_v$, will be useful. We set some notation: for any $x \in \mathbb{C}_v$ and $r > 0$, we denote the open and closed disks of radius r about x by

$$D(x, r) = \{y \in \mathbb{C}_v : |y - x|_v < r\} \qquad \text{and} \qquad \overline{D}(x, r) = \{y \in \mathbb{C}_v : |y - x|_v \leq r\}.$$

Lemma 3.3. *Let K be a field with non-archimedean absolute value v, let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$ and lead coefficient a_d, and let $\mathfrak{K}_v \subseteq \mathbb{C}_v$ be the filled Julia set of ϕ. Define $s_v = \max\{A, |a_d|_v^{-1/(d-1)}\}$, where $A = \max\{|\alpha|_v : \phi(\alpha) = 0\}$ as in Proposition 2.1. Then $\mathfrak{K}_v \subseteq \overline{D}(0, s_v)$.*

Proof. See [3, Lemma 5.1]. Alternately, it is easy to check directly that if $|x|_v > s_v$, then $|\phi(x)|_v = |a_d x^d|_v > |x|_v$; it follows that $|\phi^n(x)|_v \to \infty$. $\square$

Lemma 3.4. *Let K be a field with non-archimedean absolute value v, and let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$ with lead coefficient a_d. Let $\mathfrak{K}_v \subseteq \mathbb{C}_v$ be the filled Julia set of ϕ at v, let $r_v = \sup\{|x-y|_v : x, y \in \mathfrak{K}_v\}$ be the diameter of $\mathfrak{K}_v$, and let $U_0 \subseteq \mathbb{C}_v$ be the intersection of all disks containing $\mathfrak{K}_v$. Then:*

(1) *$U_0 = \overline{D}(x, r_v)$ for any $x \in \mathfrak{K}_v$.*
(2) *There exists $x \in \mathbb{C}_v$ such that $|x|_v = r_v$.*
(3) *$r_v \geq |a_d|_v^{-1/(d-1)}$, with equality if and only if $\mathfrak{K}_v = U_0$.*
(4) *If $r_v > |a_d|_v^{-1/(d-1)}$, let $\alpha \in U_0$, and let $\beta_1, \ldots, \beta_d \in \mathbb{C}_v$ be the roots of $\phi(z) = \alpha$. Then $\mathfrak{K}_v \subseteq U_1$, where $U_1 = \bigcup_{i=1}^{d} \overline{D}(\beta_i, |a_d|_v^{-1/d-1})$.*

Proof. Parts (1–3) are simply a rephrasing of [2, Lemma 2.5].

As for part (4), if $r_v = |a_d|_v^{-1/(d-1)}$, then $\mathfrak{K}_v = U_0$ by part (3), and hence also $\phi^{-1}(U_0) = \phi^{-1}(\mathfrak{K}_v) = \mathfrak{K}_v = U_0$. In particular, $\beta_i \in U_0$ for all i, and the result follows.

If $r_v > |a_d|_v^{-1/(d-1)}$, [2, Lemma 2.7] says that $\phi^{-1}(U_0)$ is a disjoint union of ℓ strictly smaller disks $V_1, \ldots, V_\ell$, each contained in U_0, and each of which maps onto U_0 under ϕ, for some integer $2 \leq \ell \leq d$.

Suppose there is some $x \in \mathfrak{K}_v$ such that $|x - \beta_i|_v > |a_d|_v^{-1/(d-1)}$ for all $i = 1, \ldots, d$. By part (1), there is some $y \in \mathfrak{K}_v$ such that $|x - y|_v = r_v$. Without loss, $x \in V_1$ and $y \in V_2$; V_1 and V_2 are distinct and in fact disjoint, because each has radius strictly smaller than r_v, and v is non-archimedean. The disk V_2 must also contain some β_j (without loss, β_d), since $\phi(V_2) = U_0$ by the previous paragraph; hence $|x - \beta_d|_v = r_v$. Thus,

$$|\phi(x) - \alpha|_v = |a_d|_v \cdot |x - \beta_d|_v \prod_{i=1}^{d-1} |x - \beta_i|_v > |a_d|_v \cdot r_v \cdot (|a_d|_v^{-1/(d-1)})^{d-1} = r_v.$$

However, $\phi(x) \in \mathfrak{K}_v \subseteq U_0$ and $\alpha \in U_0$, and therefore $|\phi(x) - \alpha|_v \leq r_v$. Contradiction. $\square$

Remark 3.5. Lemma 3.4(4) says that $\mathfrak{K}_v$ is contained in a union of at most d disks of radius $|a_d|_v^{-1/(d-1)}$. However, if $d \geq 3$, then at most one of the disks needs to be that large; the rest can be strictly smaller. Still, the weaker statement of Lemma 3.4 above suffices for our purposes.

4. Cubic Polynomials

In the study of quadratic polynomial dynamics, it is useful to note that (except in characteristic 2) any such polynomial is conjugate over the base field to a unique one of the form z^2+c. For cubics, it might appear at first glance that a good corresponding form would be $z^3 + az + b$. However, this form is not unique, since $z^3 + az + b$ is conjugate to $z^3 + az - b$ by $z \mapsto -z$. In addition, it is not even possible to make most cubic polynomials monic by conjugation over $\mathbb{Q}$. More precisely, if ϕ is a cubic with leading coefficient a, and if $\eta(z) = \alpha z + \beta$, then $\eta^{-1} \circ \phi \circ \eta$ has leading coefficient $\alpha^{-2} a$, which can only be 1 if a is a perfect square. Instead of $z^3 + az + b$, then, we propose the following two forms as normal forms when conjugating over a (not necessarily algebraically closed) field of characteristic not equal to three.

Definition 4.1. Let K be a field, and let $\phi \in K[z]$ be a cubic polynomial. We will say that ϕ is *in normal form* if either

$$\phi(z) = az^3 + bz + 1 \tag{4.1}$$

or

$$\phi(z) = az^3 + bz. \tag{4.2}$$

Proposition 4.2. *Let K be a field of characteristic not equal to 3, and let $\phi(z) \in K[z]$ be a cubic polynomial. Then there is a degree one polynomial $\eta \in K[z]$ such that $\psi = \eta^{-1} \circ \phi \circ \eta$ is in normal form. Moreover, if another conjugacy $\tilde{\eta}(z)$ also gives a normal form $\tilde{\psi} = \tilde{\eta}^{-1} \circ \phi \circ \tilde{\eta}$, then either $\tilde{\eta} = \eta$ and $\tilde{\psi} = \psi$, or else both normal forms $\psi(z) = az^3 + bz$ and $\tilde{\psi}(z) = \tilde{a}z^3 + bz$ are of the type in (4.2) with the same linear term, and the quotient $\tilde{a}/a$ of their lead coefficients is the square of an element of K.*

Proof. Write $\phi(z) = az^3 + bz^2 + cz + d \in K[z]$, with $a \neq 0$. Conjugating by $\eta_1(z) = z - b/(3a)$ gives

$$\psi_1(z) := \eta_1^{-1} \circ \phi \circ \eta_1(z) = az^3 + b'z + d'.$$

(Note that $b', d' \in K$ can be computed explicitly in terms of a, b, c, d, but their precise values are not important here.) If $d' = 0$, then we have a normal form of the type in (4.2). Otherwise, conjugating ψ_1 by $\eta_2(z) = d'z$ gives the normal form

$$\eta_2^{-1} \circ \psi_1 \circ \eta_2(z) = a'z^3 + b'z + 1,$$

where $a' = a/(d')^2$.

For the uniqueness, suppose $\phi_1 = \eta^{-1} \circ \phi_2 \circ \eta$, where $\eta(z) = \alpha z + \beta$, $\phi_1(z) = a_1 z^3 + b_1 z + c_1$ and $\phi_2(z) = a_2 z^3 + b_2 z + c_2$, with $c_1, c_2 \in \{0, 1\}$ and $\alpha a_1 a_2 \neq 0$. Because the z^2-coefficient of $\eta^{-1} \circ \phi_2 \circ \eta(z)$ is $\alpha\beta a_1$, we must have $\beta = 0$. Thus, $\phi_2(z) = \alpha^{-1}\phi_1(\alpha z)$, which means that $c_2 = \alpha c_1$ and $a_2/a_1 = \alpha^2$. If either c_1 or c_2 is 1, then $\alpha = 1$ and $\phi_1 = \phi_2$. Otherwise, we have $c_1 = c_2 = 0$, $b_1 = b_2$, and $a_2/a_1 \in (K^\times)^2$, as claimed. $\square$

Remark 4.3. The cubic $\phi(z) = az^3 + bz$ is self-conjugate under $z \mapsto -z$; that is, $\phi(-z) = -\phi(z)$. (It is not a coincidence that those cubic polynomials admitting non-trivial self-conjugacies are precisely those with the more complicated "$\tilde{a}/a$ is a square" condition in Proposition 4.2; see [23, Example 4.75 and Theorem 4.79].) As a result, $\hat{h}_\phi(-x) = \hat{h}_\phi(x)$ for all $x \in \mathbb{Q}$; and if x is a preperiodic point of ϕ, then so is $-x$.

In addition, the function $-\phi(z) = -az^3 - bz$ satisfies $(-\phi) \circ (-\phi) = \phi \circ \phi$. Thus, $\hat{h}_\phi(x) = \hat{h}_{-\phi}(x)$ for all $x \in \mathbb{Q}$. Moreover, ϕ and $-\phi$ have the same set of preperiodic points, albeit with slightly different arrangements of points into cycles.

The normal forms of Definition 4.1 have two key uses. The first is that they allow us to list a unique (or, in the case of form (4.2), essentially unique) element of each conjugacy class of cubic polynomials over $\mathbb{Q}$ in a systematic way, which is helpful for having a computer algorithm test them one at a time. The second is that the forms provide a description of the moduli space $\mathcal{M}_3$ of all cubic polynomials up to conjugation. This second use is crucial to the very statement of Conjecture 2, because the quantity $h(\phi)$ is defined to be the height of the conjugacy class of ϕ viewed as a point on $\mathcal{M}_3$.

In particular, Proposition 4.2 says that $\mathcal{M}_3$ can be partitioned into two pieces: the first piece is an affine subvariety of $\mathbb{P}^2$, and the second is an affine line. More specifically, the conjugacy class of the polynomial $\phi(z) = az^3 + bz + 1$ corresponds to the point (a, b)

in $\{(a,b) \in \mathbb{A}^2 : a \neq 0\}$. To compute heights, then, we should view $\mathbb{A}^2$ as an affine subvariety of $\mathbb{P}^2$, thus declaring $h(\phi)$ to be the height $h([a:b:1])$ of the point $[a:b:1]$ in $\mathbb{P}^2$. Meanwhile, the conjugacy class of az^3+bz over $\overline{\mathbb{Q}}$ is determined solely by b, because az^3+bz is conjugate to $a'z^3+bz$ over $\overline{\mathbb{Q}}$. (As noted in [23, Section 4.4 and Remark 4.39], $\mathcal{M}_3$ is the moduli space of $\overline{\mathbb{Q}}$-conjugacy classes of cubic polynomials, not $\mathbb{Q}$-conjugacy classes.) Thus, the $\overline{\mathbb{Q}}$-conjugacy class of $\phi_0(z) = az^3+bz$ corresponds to the point b in $\mathbb{A}^1$, and the corresponding height is $h(\phi_0) = h([b:1])$, the height of the point $[b:1]$ in $\mathbb{P}^1$. We phrase these assignments formally in the following definition.

Definition 4.4. Given $a,b \in \mathbb{Q}$ with $a \neq 0$, define $\phi(z) = az^3+bz+1$ and $\phi_0(z) = az^3+bz$. Write $a = k/m$ and $b = \ell/m$ with $k,\ell,m \in \mathbb{Z}$ and $\gcd(k,\ell,m) = 1$; also write $b = \ell_0/m_0$ with $\gcd(\ell_0,m_0) = 1$. Then we define the *heights* $h(\phi)$, $h(\phi_0)$ of the maps ϕ and ϕ_0 to be

$$h(\phi) := \log\max\{|k|_\infty, |\ell|_\infty, |m|_\infty\} \qquad \text{and} \qquad h(\phi_0) := \log\max\{|\ell_0|_\infty, |m_0|_\infty\}.$$

Note that $h(\phi_0) = h(b) = \sum_v \log\max\{1,|b|_v\}$, and $h(\phi) = \sum_v \log\max\{1,|a|_v,|b|_v\}$.

Proposition 4.5. *Given a,b,ϕ,ϕ_0 as in Definition 4.4, let $\gamma = \sqrt{a} \in \overline{\mathbb{Q}}$ be a square root of a, and define*

$$\psi(z) = \gamma\phi(\gamma^{-1}z) = z^3+bz+\sqrt{a}, \qquad \textit{and} \qquad \psi_0(z) = \gamma\phi_0(\gamma^{-1}z) = z^3+bz.$$

Let $\tilde{c}(\phi,2)$, $\tilde{C}(\phi,2)$, $\tilde{c}(\phi_0,2)$, and $\tilde{C}(\phi_0,2)$ be the corresponding constants from Theorem 2.7. Then

$$\tilde{c}(\phi,2) \leq 1.84 \cdot \max\{h(\phi),1\}, \qquad \tilde{C}(\phi,2) \leq .75 \cdot \max\{h(\phi),1\},$$
$$\tilde{c}(\phi_0,2) \leq 1.57 \cdot \max\{h(\phi_0),1\}, \qquad \textit{and} \qquad \tilde{C}(\phi_0,2) \leq .75 \cdot h(\phi_0).$$

Proof. Note that

$$\log(1+|a|_\infty^{1/6}+|b|_\infty^{1/2}) \leq \log\left(3\max\{1,|a|_\infty^{1/6},|b|_\infty^{1/2}\}\right) \leq \log 3 + \frac{1}{2}\log\max\{1,|a|_\infty,|b|_\infty\}.$$

Thus, if $h(\phi) \geq \log 9$, then by Remark 2.3 and the definition of $\tilde{c}(\phi,2)$,

$$\begin{aligned}\frac{2}{3}\tilde{c}(\phi,2) &\leq \log(1+|a|_\infty^{1/6}+|b|_\infty^{1/2}) + \sum_{v\neq\infty}\log\max\{1,|a|_v^{1/6},|b|_v^{1/2}\} \\ &\leq \log 3 + \frac{1}{2}\sum_v \log\max\{1,|a|_v,|b|_v\} = \log 3 + \frac{1}{2}h(\phi) \leq h(\phi).\end{aligned}$$

Hence, $\tilde{c}(\phi,2) < 1.5 \cdot h(\phi)$ if $h(\phi) > \log 9$.

Similarly, $\log(1+|a|_\infty+|b|_\infty) \leq \log 3 + \log\max\{1,|a|_\infty,|b|_\infty\}$, and therefore

$$2\tilde{C}(\phi,2) \leq \log 3 + h(\phi) \leq 1.5 \cdot h(\phi)$$

if $h(\phi) \geq \log 9$. The bounds for ϕ_0 can be proven in the same fashion in the case that $h(\phi_0) \geq \log 4$. (That is, $h(b) \geq 4$.)

Finally, there are fifteen choices of $b \in \mathbb{Q}$ for which $h(b) < \log 4$, and 1842 pairs $(a,b) \in \mathbb{Q}$ for which $h(\phi) < \log 9$. By a simple computer computation (working directly from the definitions in Proposition 2.1 and Theorem 2.7, not the estimates of Remark 2.3), one can check that the desired inequalities hold in all cases. $\square$

Remark 4.6. In fact, $\tilde{c}(\phi_0, 2) \leq 1.5 \cdot \max\{h(b), 1\}$ in all but four cases: $b = \pm 2/3$ and $b = \pm 3/2$, which give $h(b) = \log 3$ and $\tilde{c}(\phi_0, 2) = 1.5 \cdot \log(\sqrt{2} + \sqrt{3})$. Similarly, $\tilde{c}(\phi, 2) \leq 1.5 \cdot \max\{h(\phi), 1\}$ in all but 80 cases. The maximum ratio of $1.838\ldots$ is attained twice, when (a, b) is $(-1, 2/3)$ or $(1, -2/3)$. In both cases, $h(\phi) = \log 3$ and $\tilde{c}(\phi_0, 2) = 1.5 \cdot \log((\alpha + 1)\sqrt{3})$, where $\alpha \approx 1.22$ is the unique real root of $3z^3 - 2z - 3$.

The next Lemma says that for cubic polynomials in normal form, and for v a p-adic absolute value with $p \neq 3$, the radius s_v from Lemma 3.3 coincides with the radius r_v from Lemma 3.4. Thus, when we search for rational preperiodic points, we are losing no efficiency by searching in $\overline{D}(0, s_v)$ instead of the ostensibly smaller disk U_0.

Lemma 4.7. *Let K be a field with non-archimedean absolute value v such that $|3|_v = 1$. Let $\phi(z) \in K[z]$ be a cubic polynomial in normal form, and let $\mathfrak{K}_v \subseteq \mathbb{C}_v$ be the filled Julia set of ϕ at v. Let $r_v = \sup\{|x - y|_v : x, y \in \mathfrak{K}_v\}$ be the diameter of $\mathfrak{K}_v$. Then $|x|_v \leq r_v$ for all $x \in \mathfrak{K}_v$.*

Proof. If $\phi(z) = az^3 + bz$, then $\phi(0) = 0$, and therefore $0 \in \mathfrak{K}_v$. The desired conclusion is immediate. Thus, we consider $\phi(z) = az^3 + bz + 1$. Note that the three roots $\alpha, \beta, \gamma \in \mathbb{C}_v$ of the equation $\phi(z) - z = 0$ are fixed by ϕ and hence lie in $\mathfrak{K}_v$.

Without loss, assume $|\alpha|_v \geq |\beta|_v \geq |\gamma|_v$. It suffices to show that $|\alpha - \gamma|_v = |\alpha|_v$; if $x \in \mathfrak{K}_v$, then $|x|_v \leq \max\{|x - \alpha|_v, |\alpha - \gamma|_v\} \leq r_v$, as desired. If $|\alpha|_v > |\gamma|_v$, then $|\alpha - \gamma|_v = |\alpha|_v$, and we are done. Thus, we may assume $|\alpha|_v = |\beta|_v = |\gamma|_v = |a|_v^{-1/3}$, which implies that $|a|_v \geq |b - 1|_v^3$. We may also assume that $|\alpha - \gamma|_v \geq |\alpha - \beta|_v$.

The polynomial $Q(z) = \phi(z + \alpha) - (z + \alpha)$ has roots 0, $\beta - \alpha$, and $\gamma - \alpha$; on the other hand, $Q(z) = az[z^2 + 3\alpha z + (a^{-1}(b - 1) + 3\alpha^2)]$ by direct computation. Thus,

$$\big(z - (\beta - \alpha)\big)\big(z - (\gamma - \alpha)\big) = z^2 + 3\alpha z + \big(a^{-1}(b - 1) + 3\alpha^2\big). \tag{4.3}$$

Since $|a^{-1}(b - 1)|_v \leq |a|_v^{-2/3} = |\alpha|_v^2$, the constant term of (4.3) has absolute value at most $|\alpha|_v^2$; meanwhile, the linear coefficient satisfies $|3\alpha|_v = |\alpha|_v$. Thus, either from the Newton polygon or simply by inspection of (4.3), it follows that $|\alpha - \gamma|_v = |\alpha|_v$. □

Remark 4.8. Lemma 4.7 can be false in non-archimedean fields in which $|3|_v < 1$. For example, if $K = \mathbb{Q}_3$ (in which $|3|_3 = 1/3 < 1$) and $\phi(z) = -(1/27)z^3 + z + 1$, then it is not difficult to show that the diameter of the filled Julia set is $3^{-3/2}$. However, $\alpha = 3$ is a fixed point, and $|\alpha|_3 = 1/3 > 3^{-3/2}$.

At the archimedean place $v = \infty$, we will study not $\mathfrak{K}_\infty$ itself, but rather the simpler set $\mathfrak{K}_\infty \cap \mathbb{R}$, which we will describe in terms of the real fixed points. Note, of course, that any cubic with real coefficients has at least one real fixed point; and if there are exactly two real fixed points, then one must appear with multiplicity two.

Lemma 4.9. *Let $\phi(z) \in \mathbb{R}[z]$ be a cubic polynomial with positive lead coefficient. If ϕ has precisely one real fixed point $\gamma \in \mathbb{R}$, then $\mathfrak{K}_\infty \cap \mathbb{R} = \{\gamma\}$ is a single point.*

Proof. We can write $\phi(z) = z + (z - \gamma)^j \psi(z)$, where $1 \leq j \leq 3$, and where $\psi \in \mathbb{R}[z]$ has positive lead coefficient and no real roots. Thus, there is a positive constant $c > 0$ such that $\psi(x) \geq c$ for all $x \in \mathbb{R}$. Given any $x \in \mathbb{R}$ with $x > \gamma$, then, $\phi(x) > x + c(x - \gamma)^j$. It follows that $\phi^n(x) > x + nc(x - \gamma)^j$, and hence $\phi^n(x) \to \infty$ as $n \to \infty$. Similarly, for $x < \gamma$, $\phi^n(x) \to -\infty$ as $n \to \infty$. □

Lemma 4.10. *Let $\phi(z) \in \mathbb{R}[z]$ be a cubic polynomial with positive lead coefficient $a > 0$ and at least two distinct fixed points. Denote the fixed points by $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, with $\gamma_1 \leq \gamma_2 \leq \gamma_3$. Then $\mathfrak{K}_\infty \cap \mathbb{R} \subseteq [\gamma_1, \gamma_3]$, and*

$$\phi^{-1}([\gamma_1, \gamma_3]) \subseteq [\gamma_1, \gamma_1 + a^{-1/2}] \cup [\gamma_2 - a^{-1/2}, \gamma_2 + a^{-1/2}] \cup [\gamma_3 - a^{-1/2}, \gamma_3].$$

Proof. Let $\alpha = \inf(\mathfrak{K}_\infty \cap \mathbb{R})$; then $\alpha \in \mathfrak{K}_\infty \cap \mathbb{R}$, since this set is closed. Therefore, $\phi(\alpha) \geq \alpha$, because $\phi(\mathfrak{K}_\infty \cap \mathbb{R}) \subseteq \mathfrak{K}_\infty \cap \mathbb{R}$. On the other hand, if $\phi(\alpha) > \alpha$, then by continuity (and because ϕ has positive lead coefficient), there is some $\alpha' < \alpha$ such that $\phi(\alpha') = \alpha$, contradicting the minimality of α. Thus $\phi(\alpha) = \alpha$, giving $\alpha = \gamma_1$. Similarly, $\sup(\mathfrak{K}_\infty) = \gamma_3$, proving the first statement.

For the second statement, note that $\phi(z) = a(z-\gamma_1)(z-\gamma_2)(z-\gamma_3) + z$, and consider $x \in \mathbb{R}$ outside all three desired intervals. We will show that $\phi(x) \notin [\gamma_1, \gamma_3]$.

If $x > \gamma_3$, then $\phi(x) > x > \gamma_3$. Similarly, if $x < \gamma_1$, then $\phi(x) < x < \gamma_1$.

If $\gamma_1 + a^{-1/2} < x < \gamma_2 - a^{-1/2}$, then, noting that $\gamma_3 > x$, we have

$$\phi(x) - \gamma_3 = \big[a(x-\gamma_1)(\gamma_2 - x) - 1\big](\gamma_3 - x) > (a(a^{-1/2})^2 - 1)(\gamma_3 - x) \geq 0.$$

Similarly, if $\gamma_2 + a^{-1/2} < x < \gamma_3 - a^{-1/2}$, we obtain $\phi(x) < \gamma_1$. $\square$

Lemma 4.11. *Let $\phi(z) \in \mathbb{R}[z]$ be a cubic polynomial with negative lead coefficient. If $\mathfrak{K}_\infty \cap \mathbb{R}$ consists of more than one point, then ϕ has at least two distinct real periodic points of period two. Moreover, if $\alpha \in \mathbb{R}$ is the smallest such periodic point, then $\phi(\alpha)$ is the largest, and $\mathfrak{K}_\infty \cap \mathbb{R} \subseteq [\alpha, \phi(\alpha)]$.*

Proof. Let $\alpha = \inf(\mathfrak{K}_\infty \cap \mathbb{R})$ and $\beta = \sup(\mathfrak{K}_\infty \cap \mathbb{R})$, so that $\mathfrak{K}_\infty \cap \mathbb{R} \subseteq [\alpha, \beta]$. By hypothesis, $\alpha < \beta$. It suffices to show that $\phi(\alpha) = \beta$ and $\phi(\beta) = \alpha$.

Note that $\phi(\alpha) \in \mathfrak{K}_\infty \cap \mathbb{R}$, and therefore $\phi(\alpha) \leq \beta$. If $\phi(\alpha) < \beta$, then by continuity, there is some $\alpha' < \alpha$ such that $\phi(\alpha') = \beta$, contradicting the minimality of α. Thus, $\phi(\alpha) = \beta$; similarly, $\phi(\beta) = \alpha$. $\square$

5. The search algorithm

We are now ready to describe our algorithm to search for preperiodic points and points of small height for cubic polynomials over $\mathbb{Q}$.

Algorithm 5.1. Given $a \in \mathbb{Q}^\times$ and $b \in \mathbb{Q}$, set $\phi(z) = az^3 + bz + 1$ or $\phi(z) = az^3 + bz$, define $h(\phi)$ as in Definition 4.4, and set $h_+(\phi) = \max\{h(\phi), 1\}$.

1. Let S be the set of all (bad) prime factors p of the numerator of a, denominator of a, and denominator of b. Compute each radius s_p from Lemma 3.3; by Remark 2.3,

$$s_p = \begin{cases} \max\{|b/a|_p^{1/2}, |1/a|_p^{1/2}\} & \text{for } \phi(z) = az^3 + bz, \\ \max\{|b/a|_p^{1/2}, |1/a|_p^{1/3}, |1/a|_p^{1/2}\} & \text{for } \phi(z) = az^3 + bz + 1. \end{cases}$$

Shrink s_p if necessary to be an integer power of p. Let $M = \prod_{p \in S} s_p \in \mathbb{Q}^\times$. Thus, for any preperiodic rational point $x \in \mathbb{Q}$, we have $Mx \in \mathbb{Z}$.

2. If $a > 0$ and ϕ has only one real fixed point, or if $a < 0$ and ϕ has no real two-periodic points, then (by Lemma 4.9 or Lemma 4.11) $\mathfrak{K}_\infty \cap \mathbb{R}$ consists of a single point $\gamma \in \mathbb{R}$, which must be fixed. In that case, check whether γ is rational by seeing whether $M\gamma$ is an integer; report either the one or zero preperiodic points, and end.

3. Let S' be the set of all $p \in S$ for which $|a|_p^{-1/2} < s_p$. Motivated by Lemma 3.4(4), for each such p consider the (zero, one, two, or three) disks of radius $|a|_p^{-1/2}$ that contain both an element of $\phi^{-1}(0)$ and a $\mathbb{Q}_p$-rational point. If for at least one $p \in S'$ there are no such disks, then report zero preperiodic points, and end.

4. Otherwise, use the Chinese Remainder Theorem to list all rational numbers that lie in the real interval(s) given by Lemma 4.10 or 4.11, are integer multiples of the rational number M from Step 1, and lie in the disks from Step 3 at each $p \in S'$.

5. For each point x in Step 4, compute $\phi^i(x)$ for $i = 0, \ldots, 6$. If any are repeats, record a preperiodic point. Otherwise, compute $h(a(\phi^6(x))^2)/(2 \cdot 3^6 \cdot h_+(\phi))$. If the value is less than .03, record $h(a(\phi^{12}(x))^2)/(2 \cdot 3^{12})$ as $\hat{h}_\phi(x)$, and

$$\mathfrak{h}(x) = \hat{h}_\phi(x)/h_+(\phi) \tag{5.1}$$

as the *scaled height* of x.

Remark 5.2. The definition of $h_+(\phi)$ is designed to avoid dividing by zero when computing $\mathfrak{h}(x)$. In particular, the choice of 1 as a minimum value is arbitrary. Of course, the height $h(\phi)$ already depends on our choice of normal forms in Definition 4.1; moreover, without reference to some kind of canonical structure, Weil heights on varieties are only natural objects up to bounded differences. In other words, $h_+(\phi)$ is no more arbitrary than $h(\phi)$ as a height on the moduli space $\mathcal{M}_3$.

In addition, none of the polynomials we found with points of particularly small scaled height $\mathfrak{h}(x)$ had $h(\phi) \leq 1$, even though the change from $h(\phi)$ to $h_+(\phi)$ could only make $\mathfrak{h}(x)$ smaller. Thus, our use of h_+ had no significant effect on the data.

Remark 5.3. Algorithm 5.1 tests only points that, at all places, are in regions where the filled Julia set might be. At non-archimedean places, that means the region U_1 in Lemma 3.4(4); and at the archimedean place, that means the regions described in Lemma 4.10 or Lemma 4.11. Thus, as mentioned in the discussion following Proposition 3.2, the algorithm is guaranteed to test all preperiodic points, but there is a possibility it may miss a point of small positive height that happens to lie outside the search region at some place. However, such a point must have a non-negligible positive contribution to its canonical height, coming from the local canonical height at that place.

For example, any point x lying outside the region U_1 at a non-archimedean place v must satisfy $\phi(x) \notin U_0$. If $p_v \neq 3$, then by Lemma 4.7, $U_0 = \overline{D}(0, s_v)$, outside of which it is easy to show that $\hat{\lambda}_{\phi,v}(x) = \lambda_v(x) + \frac{1}{2}\log|a|_v$. Since $\hat{\lambda}_{\phi,v}(\phi(x)) > 0$ and $\lambda_v(x)$ takes values in $(\log p_v)\mathbb{Z}$, it follows that $\hat{\lambda}_{\phi,v}(\phi(x)) \geq (\log p_v)/2$, and therefore

$$\hat{h}_\phi(x) \geq \hat{\lambda}_{\phi,v}(x) \geq \frac{\log p_v}{6} \geq \frac{\log 2}{6} = .1155\ldots.$$

Thus, we are not missing points of height smaller than .11 by restricting to U_1.

Admittedly, at the archimedean place we have no such lower bound, and the possibility exists of missing a point of small height just outside the search region. However, because the denominators of such points (and all their forward iterates!) must divide M, there still cannot be many omitted points of small height unless $h(\phi)$ is very large.

Remark 5.4. The bounds of 6 (for preperiodic repeats) and .03 (for $\mathfrak{h}(x)$), and the decision to test $\hat{h}_\phi(x)$ first at 6 iterations and then again at 12 were chosen by trial

n	number of form az^3+bz		number of form az^3+bz+1	
$h(a),h(b)<$	$\log 200$	$\log 300$	$\log 200$	$\log 300$
11	10	10	0	0
10	0	0	0	3
9	30	36	20	28
8	0	0	36	52
7	196	318	144	193
6	0	0	257	358
5	524	774	533	751
4	0	0	1,533	2,314
3	132,352	297,826	52,402	115,954
2	0	0	42,447	92,221
1	422,358,932	2,072,790,448	187,391	432,131
0	0	0	2,362,307,079	11,939,398,165
total	422,492,044	2,073,089,412	2,362,591,842	11,940,042,170

TABLE 1. Number of distinct cubic polynomials az^3+bz and az^3+bz+1 with $h(a),h(b) < \log 200, \log 300$ and n rational points of scaled height smaller than .03.

and error. For example, there seem to be many cubic polynomials with points of scaled height smaller than .03, suggesting that our choice of that cutoff is safely large.

Meanwhile, if there happened to be a preperiodic chain of length 7 or longer, our algorithm would not identify the starting point as preperiodic. However, the first point in such a chain would still have shown up in our data as a point of extraordinarily small scaled height; but we found no such points in our entire search. That is, *none* of the maps we tested have preperiodic chains of length greater than six.

Finally, by Proposition 4.5 and Theorem 2.7, our preliminary estimate (after six iterations) for $\mathfrak{h}$ is accurate to within $3^{-6}\cdot 1.84 < .0026$, and our sharper estimate (after twelve iterations) is accurate to within $3^{-12}\cdot 1.84 < .0000035$. Thus, the points we test with $\mathfrak{h} < .027$ or $\mathfrak{h} > .033$ cannot be misclassified; and our recorded computations of $\mathfrak{h}$ are accurate to at least the first five places after the decimal point.

6. Data Collected

We ran Algorithm 5.1 on every cubic polynomial az^3+bz+1 and az^3+bz for which $a\in\mathbb{Q}^\times$, $b\in\mathbb{Q}$, and both numerators and both denominators are smaller than 300 in absolute value. That means $109,270$ choices for a and (because $b=0$ is allowed) $109,271$ choices for b, giving almost 12 billion pairs (a,b). (Not coincidentally, $109,271$ is approximately $(12/\pi^2)\cdot 300^2$; see [23, Exercise 3.2(b)].) Of course, in light of Proposition 4.2, we skipped polynomials of the form γ^2az^3+bz for $\gamma\in\mathbb{Q}$ if we had already tested az^3+bz. That meant only $18,972$ choices for a, but the same $109,271$ choices for b; as a result, there were only about 2 billion truly different cubics of the second type. Combining the two families, then, we tested over 14 billion truly different cubic polynomials. We summarize our key observations here; the complete data may be found online at `http://www.cs.amherst.edu/~rlb/cubicdata/`

Table 1 lists the number of such polynomials with a prescribed number of points $x\in\mathbb{Q}$ of small height (that is, with $\mathfrak{h} < .03$, where $\mathfrak{h}(x) = \hat{h}_\phi(x)/\max\{h(\phi),1\}$ is the scaled height of equation (5.1)); it also lists the totals for $h(a),h(b) < \log 200$, for

a, b	periodic cycles	strictly preperiodic	small height > 0 point(s)	$\mathfrak{h}$
$-\frac{3}{2}, \frac{19}{6}, 0$	$\{\frac{5}{3}, -\frac{5}{3}\}$, $\{0\}$	$\pm\frac{4}{3}$, $\pm\frac{2}{3}$, $\pm\frac{1}{3}$, ± 1	—	
$\frac{3}{2}, -\frac{19}{6}, 0$	$\{\frac{5}{3}\}$, $\{-\frac{5}{3}\}$, $\{0\}$	$\pm\frac{4}{3}$, $\pm\frac{2}{3}$, $\pm\frac{1}{3}$, ± 1	—	
$-3, \frac{37}{12}, 0$	$\{\frac{7}{6}, -\frac{7}{6}\}$, $\{\frac{5}{6}\}$, $\{-\frac{5}{6}\}$, $\{0\}$	$\pm\frac{1}{6}$, $\pm\frac{1}{2}$, $\pm\frac{2}{3}$,	—	
$3, -\frac{37}{12}, 0$	$\{\frac{5}{6}, -\frac{5}{6}\}$, $\{\frac{7}{6}\}$, $\{-\frac{7}{6}\}$, $\{0\}$	$\pm\frac{1}{6}$, $\pm\frac{1}{2}$, $\pm\frac{2}{3}$,	—	
$-\frac{3}{2}, \frac{73}{24}, 0$	$\{\frac{1}{2}, \frac{4}{3}\}$, $\{-\frac{1}{2}, -\frac{4}{3}\}$, $\{\frac{7}{6}\}$, $\{-\frac{7}{6}\}$, $\{0\}$	$\pm\frac{1}{6}, \pm\frac{3}{2}$	—	
$\frac{3}{2}, -\frac{73}{24}, 0$	$\{\frac{1}{2}, -\frac{4}{3}\}$, $\{-\frac{1}{2}, \frac{4}{3}\}$, $\{\frac{7}{6}, -\frac{7}{6}\}$, $\{0\}$	$\pm\frac{1}{6}, \pm\frac{3}{2}$	—	
$-\frac{5}{3}, \frac{109}{60}, 0$	$\{\frac{13}{10}, -\frac{13}{10}\}$, $\{\frac{7}{10}\}$, $-\{\frac{7}{10}\}$, $\{0\}$	$\pm\frac{3}{10}$, $\pm\frac{6}{5}$, $\pm\frac{1}{2}$	—	
$\frac{5}{3}, -\frac{109}{60}, 0$	$\{\frac{7}{10}, -\frac{7}{10}\}$, $\{\frac{13}{10}\}$, $\{-\frac{13}{10}\}$, $\{0\}$	$\pm\frac{3}{10}$, $\pm\frac{6}{5}$, $\pm\frac{1}{2}$	—	
$-\frac{6}{5}, \frac{169}{120}, 0$	$\{\frac{17}{12}, -\frac{17}{12}\}$, $\{\frac{7}{12}\}$, $\{-\frac{7}{12}\}$, $\{0\}$	$\pm\frac{13}{12}$, $\pm\frac{2}{3}$, $\pm\frac{5}{4}$	—	
$\frac{6}{5}, -\frac{169}{120}, 0$	$\{\frac{7}{12}, -\frac{7}{12}\}$, $\{\frac{17}{12}\}$, $\{-\frac{17}{12}\}$, $\{0\}$	$\pm\frac{13}{12}$, $\pm\frac{2}{3}$, $\pm\frac{5}{4}$	—	
$\frac{1}{240}, -\frac{151}{60}, 1$	$\{-10, 22\}, \{12, -22\}, \{18, -20\}$	10, -18, ± 28	—	
$-\frac{1}{240}, \frac{151}{60}, 1$	—	—	$-12, 20$: 10, 18, -22, -28: -10, -18, 22, 28:	.00244 .00733 .02198
$-\frac{169}{240}, \frac{259}{60}, 1$	$\{-2\}, \{-\frac{4}{13}\}, \{\frac{30}{13}\}$	$\frac{4}{13}, -\frac{10}{13}, -\frac{30}{13}, \pm\frac{34}{13}, \frac{36}{13}$	$-\frac{14}{13}$:	.02947

TABLE 2. Cubic polynomials $az^3 + bz + c$ with ten or more points of small height

comparison. Of course, every polynomial of the form $az^3 + bz$ has an odd number of small height points, by Remark 4.3 and because $x = 0$ is fixed. Meanwhile, there are more polynomials of the form az^3+bz+1 with three small points than with two, because there are several ways to have three preperiodic points (three fixed points, a fixed point with two extra pre-images, or a 3-cycle), but essentially only one way to have two: a 2-cycle. After all, a cubic ϕ with two rational fixed points has a third, except in the rare case of multiple roots of $\phi(z) - z$; and if ϕ has a fixed point $\alpha \in \mathbb{Q}$ with a distinct preimage $\beta \in \mathbb{Q}$, then the third preimage is also rational.

According to our data, no cubic polynomial with $h(a), h(b) < \log(300)$ has more than 11 rational points of small height. In fact, there only ten such polynomials with 11 small points; see Table 2. All ten have 11 preperiodic points and no other points of small height; all have $h(a), h(b) < 200$; and all are in the $az^3 + bz$ family. (Five are negatives of the other five, and similarly the negative of any preperiodic point is also preperiodic, as discussed in Remark 4.3.)

Table 2 also lists the only three polynomials in our search with exactly ten points of small height; a complete list (ordered by $h(\phi)$) of those with exactly nine points of small height can be found in Tables 3 and 4. To save space, Table 3 only lists polynomials $az^3 + bz$ with $a > 0$; to obtain those with $a < 0$, simply replace each pair (a, b) by $(-a, -b)$ and adjust the cycle structure of the periodic points according to Remark 4.3.

Remark 6.1. Most of the points sharing the same canonical height in Tables 3 and 4 do so simply because one or two iterates later, they coincide. For example, consider the fourth map in Table 4, namely $\phi(z) = \frac{3}{8}z^3 - \frac{49}{24}z + 1$. The three points $0, \pm\frac{7}{3}$ all satisfy $\phi(x) = 1$, and hence all three have the same canonical height. Meanwhile, $\phi(-\frac{1}{3}) = \frac{5}{3} \neq 1$, but $\phi(1) = \phi(\frac{5}{3}) = -\frac{2}{3}$, and hence $-\frac{1}{3}$ also has the same common canonical height.

The map $\phi(z) = -\frac{27}{80}z^3 + \frac{151}{60}z + 1$, near the bottom of Table 4, is an exception to this trend. The points -2, $-\frac{10}{9}$, and $\frac{28}{9}$ all satisfy $\phi(x) = -\frac{4}{3}$, but all iterates of $-\frac{22}{9}$ appear to be distinct from those of -2. Nonetheless, all four points share the same

a, b	periodic cycles	strictly preperiodic	small height > 0 point(s)	$\mathfrak{h}$
$\frac{3}{2}, -\frac{13}{6}$	$\{0\}$, $\{\frac{2}{3}, -1\}$, $\{-\frac{2}{3}, 1\}$	$\pm\frac{1}{3}$, $\pm\frac{4}{3}$	—	
$3, -\frac{13}{12}$	$\{0\}$, $\{\frac{1}{6}, -\frac{1}{6}\}$, $\{\frac{5}{6}\}$, $\{-\frac{5}{6}\}$	$\pm\frac{1}{2}$, $\pm\frac{2}{3}$	—	
$\frac{5}{3}, -\frac{49}{15}$	$\{0\}$, $\{\frac{8}{5}\}$, $\{-\frac{8}{5}\}$	± 1, $\pm\frac{3}{5}$, $\pm\frac{7}{5}$	—	
$\frac{3}{2}, -\frac{49}{24}$	$\{0\}$, $\{\frac{5}{6}, -\frac{5}{6}\}$	$\pm\frac{1}{2}$, $\pm\frac{4}{3}$, $\pm\frac{7}{6}$	—	
$\frac{5}{2}, -\frac{49}{40}$	$\{0\}$, $\{\frac{3}{10}, -\frac{3}{10}\}$	$\pm\frac{1}{2}$, $\pm\frac{4}{5}$, $\pm\frac{7}{10}$	—	
$\frac{6}{5}, -\frac{61}{30}$	$\{0\}$, $\{1, -\frac{5}{6}\}$, $\{-1, \frac{5}{6}\}$	$\pm\frac{2}{3}$, $\pm\frac{3}{2}$	—	
$\frac{6}{5}, -\frac{79}{30}$	$\{0\}$, $\{\frac{7}{6}, -\frac{7}{6}\}$	$\pm\frac{1}{2}$, $\pm\frac{5}{3}$	$\pm\frac{1}{3}$:	.02046
$\frac{6}{5}, -\frac{91}{30}$	$\{0\}$, $\{\frac{11}{6}\}$, $\{-\frac{11}{6}\}$	± 1, $\pm\frac{5}{6}$	$\pm\frac{2}{3}$:	.01982
$\frac{6}{5}, -\frac{139}{30}$	$\{0\}$, $\{\frac{13}{6}\}$, $\{-\frac{13}{6}\}$	$\pm\frac{1}{2}$, $\pm\frac{5}{3}$	± 2:	.02525
$\frac{7}{6}, -\frac{163}{42}$	$\{0\}$, $\{\frac{11}{7}, -\frac{11}{7}\}$	± 2, $\pm\frac{3}{7}$, $\pm\frac{4}{7}$	—	
$\frac{7}{2}, -\frac{169}{56}$	$\{0\}$, $\{\frac{15}{14}\}$, $\{-\frac{15}{14}\}$	$\pm\frac{1}{2}$, $\pm\frac{4}{7}$, $\pm\frac{13}{14}$	—	
$\frac{5}{3}, -\frac{169}{60}$	$\{0\}$, $\{\frac{1}{2}, -\frac{6}{5}\}$, $\{-\frac{1}{2}, \frac{6}{5}\}$	$\pm\frac{13}{10}$	$\pm\frac{3}{10}$:	.02744
$\frac{15}{7}, -\frac{169}{105}$	$\{0\}$, $\{\frac{8}{15}, -\frac{8}{15}\}$	± 1, $\pm\frac{7}{15}$, $\pm\frac{13}{15}$	—	
$\frac{5}{3}, -\frac{181}{60}$	$\{0\}$, $\{\frac{11}{10}, -\frac{11}{10}\}$	$\pm\frac{3}{2}$, $\pm\frac{2}{5}$, $\pm\frac{9}{10}$	—	
$\frac{7}{3}, -\frac{193}{84}$	$\{0\}$, $\{\frac{1}{2}, -\frac{6}{7}\}$, $\{-\frac{1}{2}, \frac{6}{7}\}$	$\pm\frac{8}{7}$, $\pm\frac{9}{14}$	—	
$\frac{5}{3}, -\frac{229}{60}$	$\{0\}, \{\frac{13}{10}, -\frac{13}{10}\}, \{\frac{17}{10}\}, \{-\frac{17}{10}\}$	$\pm\frac{6}{5}, \pm\frac{1}{2}$	—	
$\frac{5}{3}, -\frac{241}{60}$	$\{0\}, \{\frac{3}{2}, -\frac{2}{5}\}, \{-\frac{3}{2}, \frac{2}{5}\}$	$\pm\frac{8}{5}, \pm\frac{1}{10}$	—	
$\frac{6}{5}, -\frac{289}{120}$	$\{0\}, \{\frac{2}{3}, -\frac{5}{4}\}, \{-\frac{2}{3}, \frac{5}{4}\}, \{\frac{13}{12}, -\frac{13}{12}\}$	$\pm\frac{17}{12}$	—	

TABLE 3. Cubic polynomials $az^3 + bz$ with $a > 0$ and nine points of small height

canonical height $\hat{h}_\phi(-\frac{22}{9}) = \hat{h}_\phi(-2) = \frac{1}{18}\log 5 \approx .08941$. (The scaled height .01396 is of course .08941 divided by $h(\phi) = \log(604)$.) We can compute this explicit value as follows. The bad primes are $v = 2, 3, 5, \infty$. In $\mathbb{R}$, the iterates of all four points approach the fixed point at -1.639. At $v = 3$, ϕ maps the set $\{x \in \mathbb{Q}_3 : |x|_3 \le 9\}$ into itself, since $9\phi(z/9) = \frac{1}{3}(z^3 - z) - \frac{27}{80}z^3 + \frac{57}{20}z + 9$ maps 3-adic integers to 3-adic integers. At $v = 2$, one can show that ϕ maps $\overline{D}(4, \frac{1}{16})$ into $\overline{D}(2, \frac{1}{4})$, $\overline{D}(2, \frac{1}{8})$ into $\overline{D}(-2, \frac{1}{8})$, $\overline{D}(-2, \frac{1}{16})$ into $\overline{D}(4, \frac{1}{16})$, and $\overline{D}(6, \frac{1}{16})$ into $\overline{D}(4, \frac{1}{16})$; hence the orbit any point $x \in \mathbb{Q}_2$ in these disks stays in the same disks. Thus, $\hat{\lambda}_{\phi,\infty}(x) = \hat{\lambda}_{\phi,3}(x) = \hat{\lambda}_{\phi,2}(x) = 0$ for all four points x; by Propositions 1.2 and 1.4, then, $\hat{h}_\phi(x) = \hat{\lambda}_{\phi,5}(x)$. Finally, all four points satisfy $|\phi^3(x)|_5 = 5$, and therefore $|\phi^n(x)|_5 = 5^{e_n}$ for all $n \ge 3$, where $e_n = 1 + 3 + \cdots + 3^{n-3} = (3^{n-2} - 1)/2$. Thus, $\hat{\lambda}_{\phi,5}(x) = \lim_{n\to\infty} \dfrac{e_n}{3^n} \log 5 = \dfrac{1}{18}\log 5$, as claimed. Incidentally, the same argument almost applies to the fifth point $x = \frac{22}{9}$ as well, except that $\phi^7(\frac{22}{9}) \approx 36.19$. As a result, $\hat{\lambda}_{\phi,\infty}(\frac{22}{9}) \approx .0014$ is positive; dividing by $\log(604)$ gives the extra contibution of .00022 to the scaled height.

A similar phenomenon occurs for the map $\phi(z) = -\frac{1}{240}z^3 + \frac{259}{60}z + 1$, listed near the bottom of Table 2. For that map, -12 and 20 have the same small height but apparently disjoint orbits. The point 20 maps to 18, and all three of 10, 18, and -28 map to 22, which then maps to 12. Meanwhile, -12 maps to -22, which maps to -10; and all three of -10, -18, and 28 map to -20.

As mentioned in Remark 5.4, no preperiodic point in our data took more than six iterations to produce a repeated value. In fact, all but one function required only five. The one exception is $\phi(z) = \frac{1}{12}z^3 - \frac{25}{12}z + 1$, for which the preperiodic point 0 lands on the 5-periodic point 1 after one iteration. (There are a total of 8 small height points for

a, b	periodic cycles	strictly preperiodic	small height > 0 point(s)	$\mathfrak{h}$
$\frac{1}{6}, -\frac{13}{6}$	$\{3, -1\}$	$0,\ 1,\ \pm 2,\ -3,\ \pm 4$	—	
$-\frac{1}{6}, \frac{13}{6}$	$\{3\}, \{-1\}, \{-2\}$	$0,\ 1,\ 2,\ -3,\ \pm 4$	—	
$\frac{3}{4}, -\frac{49}{12}$	$\{1, -\frac{7}{3}\}$	$0,\ \pm\frac{1}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{7}{3},\ -\frac{8}{3}$	—	
$\frac{3}{8}, -\frac{49}{24}$	$\{-3\}, \{\frac{1}{3}\}, \{\frac{8}{3}\}$	$-1,\ -\frac{5}{3}$	$0,\ -\frac{1}{3},\ \pm\frac{7}{3}$:	.02309
$\frac{25}{24}, -\frac{49}{24}$	$\{0, 1\}$	$\pm\frac{1}{5},\ \frac{3}{5},\ \pm\frac{7}{5},\ -\frac{8}{5},\ -\frac{9}{5}$	—	
$\frac{5}{12}, -\frac{49}{60}$	$\{\frac{3}{5}\}$	$0,\ \pm 1,\ -\frac{3}{5},\ \pm\frac{7}{5},\ \pm\frac{8}{5}$	—	
$-\frac{5}{12}, \frac{49}{60}$	$\{1, \frac{7}{5}\}$	$0,\ -1,\ -\frac{7}{5},\ \pm\frac{3}{5},\ \pm\frac{8}{5}$	—	
$-\frac{49}{48}, \frac{19}{12}$	$\{\frac{12}{7}, -\frac{10}{7}\}, \{\frac{2}{7}, \frac{10}{7}\}$	$-\frac{2}{7},\ \pm\frac{4}{7},\ \pm\frac{6}{7}$	—	
$\frac{2}{15}, -\frac{91}{30}$	$\{\frac{5}{2}, -\frac{9}{2}\}$	$3,\ \pm 5,\ \pm\frac{1}{2},\ \frac{9}{2},\ -\frac{11}{2}$	—	
$-\frac{1}{30}, \frac{91}{30}$	$\{4, 11, -10\}$	$0,\ 1,\ 9,\ -5,\ -6$	-4:	.01983
$\frac{1}{48}, -\frac{31}{12}$	$\{-4, 10\}, \{-10, 6\}$	$\pm 2,\ -6,\ \pm 12$	—	
$-\frac{1}{48}, \frac{31}{12}$	—	—	4: $2,\ 10,\ -12$: ± 6: $-2,\ -10,\ 12$:	.00039 .00118 .00355 .01065
$\frac{49}{48}, -\frac{31}{12}$	$\{-2\}, \{\frac{2}{7}\}, \{\frac{12}{7}\}$	$-\frac{2}{7},\ \frac{4}{7},\ \pm\frac{10}{7},\ -\frac{12}{7}$	$\frac{6}{7}$	.02587
$\frac{3}{16}, -\frac{43}{12}$	$\{-4, \frac{10}{3}\}$	$-\frac{2}{3},\ \frac{14}{3}$	2: $\frac{4}{3}$: $4,\ \frac{2}{3},\ -\frac{14}{3}$:	.00677 .01653 .02030
$-\frac{3}{16}, \frac{43}{12}$	$\{-4, -\frac{4}{3}, -\frac{10}{3}\}$	$\frac{14}{3},\ -\frac{2}{3}$	-2: $4,\ \frac{2}{3},\ -\frac{14}{3}$:	.00488 .01463
$\frac{3}{40}, -\frac{241}{120}$	$\{\frac{1}{3}\}$	$\pm 3, 5, -\frac{16}{3}, -\frac{19}{3}$	$-\frac{1}{3}, -5, \frac{16}{3}$:	.02182
$-\frac{3}{40}, \frac{241}{120}$	$\{-3\}$	—	$3, -5, -\frac{1}{3}, \frac{16}{3}, \frac{19}{3}$: $5, \frac{1}{3}, -\frac{16}{3}$:	.00933 .02800
$\frac{27}{80}, -\frac{91}{60}$	$\{\frac{4}{3}, -\frac{2}{9}\}$	$-2,\ -\frac{4}{3},\ \pm\frac{10}{9},\ \frac{20}{9},\ \pm\frac{22}{9}$	—	
$-\frac{27}{80}, \frac{91}{60}$	—	—	$2, \frac{2}{9}, -\frac{20}{9}$: $\pm\frac{4}{3}, \pm\frac{10}{9}, \pm\frac{22}{9}$:	.00505 .01516
$\frac{121}{80}, -\frac{91}{20}$	$\{-2\}, \{\frac{2}{11}\}, \{\frac{20}{11}\}$	$-\frac{2}{11},\ \frac{10}{11},\ \frac{12}{11},\ \pm\frac{18}{11},\ -\frac{20}{11}$	—	
$\frac{1}{240}, -\frac{91}{60}$	$\{-10, 12\}$	$10, -12, \pm 22$	$2, 18, -20$:	.01806
$-\frac{1}{240}, \frac{91}{60}$	$\{-2\}, \{-10\}, \{12\}$	$10, -12, -18, 20, \pm 22$	—	
$\frac{169}{96}, -\frac{133}{24}$	$\{-2\}, \{\frac{2}{13}\}, \{\frac{24}{13}\}$	$-\frac{2}{13},\ \frac{8}{13},\ \frac{18}{13},\ \pm\frac{22}{13},\ -\frac{24}{13}$	—	
$-\frac{289}{240}, \frac{139}{60}$	$\{\frac{4}{17}, \frac{26}{17}\}, \{-\frac{26}{17}, \frac{30}{17}\}$	$\pm\frac{6}{17}, \pm\frac{20}{17}$	$-\frac{14}{17}$	.02485
$-\frac{27}{80}, \frac{151}{60}$	$\{\frac{10}{3}, -\frac{28}{9}\}$	$2,\ \frac{10}{9}$	$-2, -\frac{10}{9}, \frac{28}{9}, -\frac{22}{9}$: $\frac{22}{9}$	.01396 .01418
$\frac{3}{112}, -\frac{247}{84}$	$\{12\}$	$2, -\frac{14}{3}, -\frac{22}{3}, \frac{28}{3}, -\frac{34}{3}$	$-2, -\frac{28}{3}\frac{34}{3}$:	.02313
$-\frac{3}{112}, \frac{247}{84}$	—	—	$-2, -12, \frac{14}{3}$ $\frac{22}{3}, -\frac{28}{3}, \frac{34}{3}$: $2, \frac{28}{3}, -\frac{34}{3}$:	 .01568 .01995
$\frac{3}{80}, -\frac{259}{60}$	$\{-12\}$	$\frac{10}{3},\ \frac{26}{3}$	$-\frac{4}{3}, -10, \frac{34}{3}$: $\frac{4}{3}, 10, -\frac{34}{3}$:	.02012 .02974

TABLE 4. Cubic polynomials $az^3 + bz + 1$ with nine points of small height

ϕ, because -5 also maps to 1, and because -4 has scaled height $.01595\ldots$.) This map was also the only cubic polynomial in our search with a rational 5-periodic point; all other periods were at most 4. Table 5 lists all those cubic polynomials in our search for which some rational preperiodic point required 5 or more iterations to reach a repeat; note that all are of the form $az^3 + bz + 1$.

Our data supports Conjecture 1 for cubic polynomials inasmuch as the number of rational preperiodic points does not grow as $h(\phi)$ increases. For example, even though

a,b,c	$x,\phi(x),\phi^2(x),\dots$	other preperiodic	other small height point $\mathfrak{h}$
$\frac{1}{12},-\frac{25}{12},1$	$0,\{1,-1,3,-3,5\}$	-5	-4: .01595
$-\frac{3}{2},\frac{7}{6},1$	$0,\ 1,\ \frac{2}{3},\ \{\frac{4}{3},-1\}$	$\pm\frac{1}{3},\ -\frac{2}{3}$	—
$\frac{1}{6},-\frac{13}{6},1$	$2,\ -2,\ 4,\ \{3,-1\}$	$0,\ 1,\ -3,\ -4$	—
$\frac{2}{3},-\frac{13}{6},1$	$\frac{3}{2},\ 0,\ 1,\ -\frac{1}{2},\ \{2\}$	$-2,\ \frac{1}{2},\ -\frac{3}{2}$	—
$-\frac{4}{3},\frac{13}{12},1$	$0,\ 1,\ \{\frac{3}{4},\frac{5}{4},-\frac{1}{4}\}$	$-1,\ \frac{1}{4},\ -\frac{3}{4}$	—
$\frac{3}{4},-\frac{25}{12},1$	$\frac{4}{3},\ 0,\ \{1,-\frac{1}{3},\frac{5}{3}\}$	$\frac{1}{3},\ -\frac{5}{3}$	—
$-\frac{1}{3},\frac{37}{12},1$	$\frac{1}{2},\ \frac{5}{2},\ \frac{7}{2},\ \{-\frac{5}{2},-\frac{3}{2}\}$	$-2,\ -\frac{1}{2}$	—
$-\frac{4}{3},\frac{37}{12},1$	$-\frac{5}{4},\ -\frac{1}{4},\ \frac{1}{4},\ \frac{7}{4},\ \{-\frac{3}{4}\}$	-1	—
$\frac{3}{4},-\frac{49}{12},1$	$\frac{1}{3},\ -\frac{1}{3},\ \frac{7}{3},\ \{1,-\frac{7}{3}\}$	$0,\ \frac{4}{3},\ \frac{5}{3},\ -\frac{8}{3}$	—
$\frac{6}{5},-\frac{61}{30},1$	$-\frac{3}{2},\ \{0,1,\frac{1}{6},\frac{2}{3}\}$	$\frac{5}{6}$	—
$-\frac{32}{3},\frac{37}{6},1$	$-\frac{5}{8},\ -\frac{1}{4},\ -\frac{3}{8},\ \{-\frac{3}{4},\frac{7}{8}\}$	$-\frac{1}{2}$	—
$\frac{1}{48},-\frac{19}{12},1$	$2,\ -2,\ 4,\ \{-4,6\}$	$-6,\ \pm 10$	—
$-\frac{2}{15},\frac{79}{30},1$	$-4,\ -1,\ -\frac{3}{2},\ \{-\frac{5}{2},-\frac{7}{2}\}$	5	0: .00688 1: .02065
$-\frac{1}{30},\frac{91}{30},1$	$0,\ 1,\ \{4,11,-10\}$	$-5,\ -6,\ 9$	-4: .01983
$\frac{8}{15},-\frac{121}{30},1$	$\frac{1}{4},\ 0,\ \{1,-\frac{5}{2},\frac{11}{4}\}$	$-\frac{11}{4}$	—
$-\frac{49}{48},\frac{31}{12},1$	$-\frac{6}{7},\ -\frac{4}{7},\ -\frac{2}{7},\ \{\frac{2}{7},\frac{12}{7}\}$	$\pm\frac{10}{7},\ -\frac{12}{7}$	—
$\frac{5}{48},-\frac{211}{60},1$	$\frac{22}{5},\ -\frac{28}{5},\ \{\frac{12}{5},-6,-\frac{2}{5}\}$	$6,\ \frac{28}{5},\ \frac{2}{5}$	—

TABLE 5. Cubic polynomials az^3+bz+c with a rational preperiodic chain of length ≥ 5

a,b,c	$h(\phi)$	$x,\phi(x),\phi^2(x),\dots$	$\mathfrak{h}(x)$
$-\frac{25}{24},\frac{97}{24},1$	3.3672	$-\frac{7}{5},\ -\frac{9}{5},\ -\frac{1}{5},\ \frac{1}{5},\ \frac{9}{5},\ \frac{11}{5},\ -\frac{6}{5},\ -\frac{41}{20},\ \frac{4323}{2560},\dots$	.00025
$\frac{8}{15},-\frac{289}{120},1$	5.6664	$\frac{5}{8},\ -\frac{3}{8},\ \frac{15}{8},\ 0,\ 1,\ -\frac{7}{8},\ \frac{11}{4},\ \frac{175}{32},\ \frac{307441}{4096},\dots$	.00030
$-27,\frac{85}{12},1$	5.7807	$-\frac{2}{9},\ -\frac{5}{18},\ -\frac{7}{18},\ -\frac{1}{6},\ -\frac{1}{18},\ \frac{11}{18},\ -\frac{5}{6},\ \frac{193}{18},\ -\frac{597703}{18},\dots$	.00032
$-\frac{1}{48},\frac{31}{12},1$	4.8202	$4,\ 10,\ 6,\ 12,\ -4,\ -8,\ -9,\ -\frac{113}{16},\ -\frac{649189}{65536},\dots$	.00039
$-\frac{3}{4},\frac{25}{12},1$	3.2188	$-1,\ -\frac{1}{3},\ \frac{1}{3},\ \frac{5}{3},\ 1,\ \frac{7}{3},\ -\frac{11}{3},\ \frac{91}{3},\ -\frac{62605}{3},\dots$	.00046
$\frac{21}{128},-\frac{295}{168},1$	8.4596	$-4,\ -\frac{52}{21},\ \frac{20}{7},\ -\frac{4}{21},\ \frac{4}{3},\ -\frac{20}{21},\ \frac{124}{49},\ -\frac{39572}{50421},\dots$	.00047
$-\frac{243}{224},\frac{85}{168},1$	6.5917	$-\frac{2}{27},\ \frac{26}{27},\ \frac{14}{27},\ \frac{10}{9},\ \frac{2}{27},\ \frac{28}{27},\ \frac{17}{54},\ \frac{2593}{2304},\ \frac{2336653975}{101468602368},\dots$	.00057
$\frac{4}{21},-\frac{205}{84},1$	5.3230	$\frac{3}{4},\ -\frac{3}{4},\ \frac{11}{4},\ -\frac{7}{4},\ \frac{17}{4},\ \frac{21}{4},\ \frac{63}{4},\ \frac{2827}{4},\ \frac{1882717007}{28},\dots$	.00058
$\frac{15}{8},-\frac{289}{120},1$	5.6664	$\frac{1}{5},\ \frac{8}{15},\ 0,\ 1,\ \frac{7}{15},\ \frac{1}{15},\ \frac{21}{25},\ \frac{276}{3125},\ \frac{9626315307}{12207031250},\dots$	.00063

TABLE 6. Cubic polynomials az^3+bz+c with rational points of scaled height less than .0007

Table 2 shows a number of maps az^3+bz with eleven preperiodic points, it is important to note that the first such map had height as small as $h(\phi)=\log(19)$. Similarly, every preperiodic structure appearing anywhere in Tables 2, 3, and 4 appeared already for some map of relatively small height. That is, the data suggests that all the phenomena that can occur have already occurred among the small height maps.

In the same way, the data also supports Conjecture 2 for cubic polynomials. Table 6 lists the only nine points of scaled height smaller than .0007 in our entire search. (There were only twenty points with scaled height smaller than .001; three of the extra eleven are iterates of the first three points listed in Table 6.) Once again, even though there are two maps of fairly large height ($\log(289)\approx 5.67$ and $\log(27\cdot 12)\approx 5.78$) with a point of small scaled height, there was already a map of substantially smaller height ($\log(97) \approx 3.37$) with an even smaller point.

Moreover, the intuition (mentioned in the introduction) that the scaled height measures the number of iterates required to start the "explosion" is on clear display in Table 6. For these points, it takes seven applications of ϕ to get to an iterate with noticeably larger numerator or denominator than its predecessors. To get a point of smaller scaled height than Table 6's record of .00025, then, it seems one would need a point and map with *eight* iterations required to start the explosion.

Also of note is that, just as in Table 5, all the maps in Table 6 are of the form az^3+bz+1. In fact, the smallest scaled height for a map az^3+bz occurs for $\pm\frac{5}{3}z^3 \mp \frac{77}{30}z$, at $x = \pm 4/5$. (Once again, see Remark 4.3 to explain the four-way tie.) The scaled height is .00591, more than twenty times as large as the current record for az^3+bz+1; indeed, it takes a mere four iterations to land on 43/40, at which point the numerator and denominator both start to explode.

This phenomenon supports the heuristic behind Conjectures 1 and 2, that it is hard to have a lot of points of small height, as follows. If x were a small height point for az^3+bz, then $-x$ would have the same small height; their iterates would also have (not quite as) small heights, too. Together with the fixed point at 0, then, there would be more small height points than the heuristic would say are allowed. This idea is further supported by Tables 2, 3, and 4: while it is possible to have eleven preperiodic points or ten points of small height, or even some of each, it does not seem possible to have more than eleven total such points. Thus, there seems to be an upper bound for the total number of points of small height, as predicted by Conjectures 1 and 2.

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