1. (25 points) Prove the first statement of Theorem 3.12 from the book:
Let $E_1, E_2, \ldots \subseteq \mathbb{R}^n$ be measurable sets, and let $E = \bigcup_{k=1}^{\infty} E_k$. Then $E$ is also measurable.

**Proof.** Given $\varepsilon > 0$, for each $k \geq 1$, since $E_k$ is measurable, there is an open set $G_k$ such that $E_k \subseteq G_k$ and $|G_k \setminus E_k| < \varepsilon/2^k$.
Let $G = \bigcup_{k \geq 1} G_k$, which is open because it is a union of opens. Then

$$G \setminus E = \left( \bigcup_{k \geq 1} G_k \right) \setminus \left( \bigcup_{k \geq 1} E_k \right) \subseteq \bigcup_{k \geq 1} (G_k \setminus E_k).$$

Thus, $|G \setminus E|_e \leq \sum_{k \geq 1} |G_k \setminus E_k|_e < \sum_{k \geq 1} \frac{\varepsilon}{2^k} = \varepsilon$. QED

2. (20 points) Let $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ be a countable decreasing sequence of subsets of $\mathbb{R}^n$. Suppose that

$$\lim_{n \to \infty} \text{diam} (E_n) = 0.$$ 

Prove that $\bigcap_{n \geq 1} E_n$ contains at most one point.

**Proof.** Given $x, y \in \bigcap_{n \geq 1} E_n$. [Our goal is to show $x = y$.]

Given $\varepsilon > 0$, the limit equation says that we may choose $N \geq 1$ so that for all $k \geq N$, we have diam $(E_k) < \varepsilon$.

Since $x, y \in E_N$, we have $0 \leq d(x, y) \leq \text{diam} (E_n) < \varepsilon$.

Because we have proven that $0 \leq d(x, y) < \varepsilon$ for every $\varepsilon > 0$, we must have $d(x, y) = 0$.

Hence, $x = y$. QED

3. (30 points) Let

$$\mathcal{A} = \{ E \subseteq \mathbb{R}^1 : \text{either } E \text{ or } \mathbb{R}^1 \setminus E \text{ is countable} \}.$$ 

Prove that $\mathcal{A}$ is a $\sigma$-algebra.

**Proof.** First, we show that $\mathcal{A}$ is closed under complements. If $E \in \mathcal{A}$, then either $E$ or $\mathbb{R} \setminus E$ is countable. If $E$ is countable, then $\mathbb{R} \setminus E \in \mathcal{A}$, because $\mathbb{R} \setminus (\mathbb{R} \setminus E) = E$ is countable.

If $\mathbb{R} \setminus E$ is countable, then by definition $\mathbb{R} \setminus E \in \mathcal{A}$.

Second, we show that $\mathcal{A}$ is closed under countable unions. Let $E_1, E_2, \ldots \in \mathcal{A}$, and let $E = \bigcup_{k \geq 1} E_k$. We consider two cases:

**Case 1:** if there is some $k$ such that $\mathbb{R} \setminus E_k$ is countable, then $\mathbb{R} \setminus E \subseteq \mathbb{R} \setminus E_k$ is countable, so $E \in \mathcal{A}$.

**Case 2:** if no such $k$ exists, then by definition of $\mathcal{A}$, every $E_k$ must be countable. So $E$ is a countable union of countable sets and is therefore countable. So $E \in \mathcal{A}$. QED
4. **(25 points)** Let $E \subseteq \mathbb{R}^n$. Show that $E$ is measurable if and only if there are Borel sets $A$ and $B$ such that $A \subseteq E \subseteq B$ and $|B \setminus A| = 0$.

**Proof.** ($\Rightarrow$): Because $E$ is measurable, a Theorem tells us that there is a set $B$ of type $G_\delta$ and a set $Z_1$ of measure zero such that $E = B \setminus Z_1$. replacing $Z_1$ by $Z_1 \cap B$ if necessary, we may assume that $Z_1 \subseteq B$.

A related theorem tells us that there is a set $A$ of type $F_\sigma$ and a set $Z_2$ of measure zero such that $E = A \cup Z_2$.

Both $A$ and $B$ are Borel sets, since $A$ is a countable union of closeds, and $B$ is a countable intersection of opens.

By construction we have $A \subseteq E \subseteq B$. In addition, we have

$$B \setminus A = (B \setminus E) \cup (E \setminus A) = Z_1 \cup Z_2.$$ 

Thus,

$$0 \leq |B \setminus A| \leq |B \setminus E| + |E \setminus A| = 0,$$

so $|B \setminus A| = 0$. QED ($\Rightarrow$)

($\Leftarrow$): Let $Z = E \setminus A$. Since $E \subseteq B$, we have $Z \subseteq B \setminus A$, and hence

$$0 \leq |Z|_e \leq |B \setminus A| = 0.$$ 

Thus, $Z$ is a set of measure zero and hence is measurable. Note that $A$ is also measurable, since it is a Borel set. Therefore, $E = Z \cup A$ is a union of two measurable sets and hence is measurable. QED