Math 410, Fall 2019: Galois Theory  
MTuWF 1–1:50 PM, SMudd 006 

Webpage: https://rlbenedetto.people.amherst.edu/math410/  
(Also accessible from the Math 410 moodle page.)

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Office Hours: Mon 2–4pm; Tue 10:30–11:30am; Thu 2–3pm; or by appointment.

Available at Amherst Books.

Exams:  
• Midterm: TBA, in class  
• Final: Take-home; details TBA  

Calculators, cell phones, ipods, etc. are not permitted in exams.  
The only excuses for missing an exam are incapacitating illness, religious conflict, or the like.

Homework:  
• Reading from the textbook will be assigned each week.  
• Problem sets will be due weekly, usually on Wednesday, at the START of class.  
  See page 3 of this handout for important homework information.

Grading:  
• Problem Sets: 50%  
• Midterm Exam: 15%  
• Final Exam: 35%

For any student who has clearly been devoted and worked hard in the course, week in and week out all semester, I will tweak the above percentages a little for that student to favor the better grades. Final course grades will be curved.

Any student who fails to hand in at least 40 homework problems on time over the course of the semester AUTOMATICALLY gets an F in the class.

About Cell Phones and Mobile Devices  
Cell phones, ipods, tablets, laptops, and other electronic devices have no place in my classroom. Don’t use them. Not for talking, not for texting, not for anything. So at every class:  

Silence your cell phone, put it away, and pay attention.
Prerequisites

The prerequisite for this course is Groups, Rings, and Fields (Math 350). We will use virtually the entire contents of that course: groups, subgroups, normal subgroups, quotients, fields, and polynomials. There will also be some occasional use of commutative rings, ideals, ring quotients, and ring homomorphisms. In addition, we will be doing lots of proofs, so it is very important that you be comfortable with the kinds of proofs that arise in Math 350. If you aren’t sure whether Math 410 is right for you, come talk with me about it.

Course Content

The motivating problem of Galois Theory is to find or understand the roots of a polynomial. One of the many important ideas of the theory is that we should study not just the roots themselves, but the fields that contain them. For example, the polynomial \( f(x) = x^2 + 1 \in \mathbb{Q}[x] \) has rational coefficients, but the smallest field containing both \( \mathbb{Q} \) and a root of \( f \) is the Gaussian rationals, \( \mathbb{Q}(i) = \{ a + bi : a, b \in \mathbb{Q} \} \).

More generally, we have a smaller field \( K \) (in this case, \( \mathbb{Q} \)) contained in a larger field \( L \) (in this case, \( \mathbb{Q}(i) \)). In Math 350, we study how groups and their subgroups relate to one another; similarly, in Galois theory, we study how fields and their subfields interact.

However, the key idea of Galois theory is to associate to an inclusion of fields \( K \subseteq L \) a group \( G \) consisting of ring homomorphisms from \( L \) to itself that fix every element of \( K \). It turns out that this can only be done nicely if the inclusion \( K \subseteq L \) satisfies certain nice properties, in which case it is called a Galois extension of fields. This group \( G \), called the Galois group of the extension of fields, ends up permuting the roots of the original polynomial \( f \).

For example, in the case \( \mathbb{Q} \subseteq \mathbb{Q}(i) \), the associated Galois group turns out to be the two-element cyclic group \( G = \mathbb{Z}/2\mathbb{Z} \). The identity element of \( G \) is just the identity function on all of \( \mathbb{Q}(i) \), while the other element of \( G \) maps \( a + bi \) to \( a - bi \).

From that basic idea, the power of group theory becomes available to attack problems concerning roots of polynomials. Some famous negative results can then be proven: it’s impossible to square the circle with straightedge and compass; it’s impossible to trisect an angle with straightedge and compass; and, using the full power of group theory, it’s impossible (in general) to solve a polynomial of degree \( d \geq 5 \) using only the operations of addition, subtraction, multiplication, division, and the taking of \( n \)-th roots. The theory also gives, as promised, a deeper understand of how the roots of polynomials can behave.

Here’s a more detailed outline of what we’ll cover:

- In Chapters 1–3, we study some fundamentals about polynomials and their roots, including the general solution of the cubic.
- In Chapter 4, we’ll learn the basics of field extensions — that is, inclusions \( K \subseteq L \) of fields.
- In Chapters 5 and 6 we’ll learn about normal, separable, and Galois field extensions; then we’ll define the Galois group of an extension.
- In Chapter 7 we’ll learn the Fundamental Theorem of Galois Theory, which gives a correspondence between the subgroups of the Galois group and the intermediate fields \( F \) satisfying \( K \subseteq F \subseteq L \).
- In Chapters 8–11, depending on how much time we have left, we’ll see applications of the theory, such as to the previously mentioned problems of straightedge and compass constructions and of solving polynomials by radicals. The topics of finite fields and of cyclotomic polynomials are less famous but equally cool.

If by some miracle time permits, we’ll look at topics in the later chapters, such as the computation of Galois groups.
Homework

Your homework consists BOTH of reading the relevant sections of the book AND of doing the weekly problem sets. (Only the written work counts directly in your grade, but I expect you to do both.) Start working on each problem set the same day it is assigned; do not put it off until a day or two before it’s due. Please note the following Important Problem Set Rules:

1. Problem sets are due in class at the start of class.
2. Problems must be in the same order as listed in the assignment.
3. Write legibly, and leave margins on all four edges of your pages.
4. Multiple pages must be clipped or (preferably) stapled together, not merely folded at the corner.
5. Don’t write on any sheet in the corner where the staple/clip is going to go.
6. Your name must be written on all sheets, in case they get separated.
7. If you worked with other students or got help from a source other than me or the book, then say so explicitly on the first page of your problem set. (See the discussion below on the Statement of Intellectual Responsibility.)
8. The Problem Sets grade for any late problem set will be substantially reduced. The later it is, the greater the reduction; see the course webpage under “Problem Set Rules” for details.

I strongly encourage you to work on problem sets together, in pairs or small groups, provided you follow the common-sense guidelines below.

About the Statement of Intellectual Responsibility

Exams: Your work must be entirely your own, so no looking at other people’s papers, no talking to each other or passing signals, and no outside help. For the in-class midterm, no aids like calculators, cell phones, books, notes, or cheat sheets are allowed. For the take-home final, you may use only your own notes and the course textbook; no other books, notes, online or other sources, or communications with other people are allowed.

Problem sets: I urge you to collaborate with each other, under the following ground rules:

1. If you collaborate with, say, Jane and Joe, write a note on the front of your problem set saying, “I worked with Jane and Joe.” (Please make sure your name stands out from Jane’s and Joe’s, so I know that you are the author.) Use similar notation if you get help from a fellow student, a tutor, another professor, another book, the web, etc. If you got help from me or from our textbook, however, you don’t need to write that.
2. Working together does not mean that Joe does the first half of the problem set and Jane does the second half; everyone should work on every problem.
3. Each student must hand in his or her own problem set; you can’t hand in a single packet as the work of multiple people.
4. Each student must write up each problem in his or her own words. Working together means discussing the problems. Copying someone else’s solution (even when the source doesn’t mind) is plagiarism and a violation of intellectual responsibility.

A common question: What if Joe asks Jane about a problem she has already solved? If Joe simply copies Jane’s solution, both Joe and Jane would be guilty of academic dishonesty, leading to an F in the course for both of them and potentially to dismissal from the college. Instead, Jane can explain her solution to Joe (even showing him what she wrote), before Joe writes up his own solution himself, in his own words. Joe would then have to write that he got help from Jane (see rule 1 above), but Jane doesn’t need to write anything unless she also got help in return.

If at any time you aren’t sure about what’s OK and what’s not as far as intellectual responsibility is concerned for this course, talk to me about it.
Extensions, Extra Office Hours, and Class Attendance

Attendance: I do not plan to take formal attendance, but I will easily be able to tell who is absent or late too much, and I will take that into account when deciding the grade of a student close to the borderline between, say, an A- and B+. Of course, if you’re sick, have a religious conflict, or the like, you may be excused; but please let me know (in advance, when possible). Otherwise, however, I expect you to be in class on time for every class meeting.

Extensions: You may request up to two homework extensions over the course of the semester, each one until the start of the next class meeting at the latest. To claim an extension, you must:

1. Not have taken more than one previous extension,
2. Request the extension (by email, by phone, or in person) no later than 7pm the day BEFORE the due date,
3. Have been attending class and handing in homework on time in the recent past, and
4. Attend class on time on the original due date and the following day that the class meets (which is your new due date).

Note: you do not need to provide an excuse or reason for your extension request; just ask.

Office Hours: you are always welcome to attend my regularly scheduled office hours. In addition, IF you have been attending class and doing the homework, you are also welcome to make appointments to see me outside of my regularly scheduled office hours.

Classroom Dynamics

Even though this is a lecture-based course, I VERY STRONGLY encourage class participation. Interrupt me, ask questions, dive in head first. Please try not to worry about saying something stupid; I guarantee you that if you’re confused about something, someone else (and probably most of the class) is, too.

In addition, I ask a lot of questions to the class; and I’m usually expecting a response. If you have an idea (not necessarily a fully formed answer; that’s not what I’m looking for), shout it out; you don’t even need to raise your hand. It’s my feeling that if you never hazard a guess which turns out to be wrong, then you’re probably not speaking up enough.

What to Expect

This course will be a noticeable level up from Math 350, and almost at the first-year graduate level. So this will probably be the most difficult math course you have taken. In particular, I assume that at this point in your college career, you’re pretty comfortable with writing proofs. I also assume that you’re comfortable thinking about abstract objects. If you’re not sure whether you are ready for Math 410, please come talk to me about it.

The homework will usually consist partially of proofs and partially of the computation of examples (say, computing Galois groups of specific field extensions). In both cases, though, unless I specifically comment that you can wave your hands a bit on a given problem, you should assume that you need to justify everything you say. That is, proofs should be fully rigorous, and every step of solutions to computational problems should be clearly justified. Similarly, if you’re asked to give an example of, say, a group with a certain property, you must not only present the group, but also explain or prove that it has the desired property. If you’re ever in doubt about whether some claim you are making requires proof, just ask me.