Midterm Exam, In-Class, Wednesday, October 23, 2019

Instructions: Do all three problems. (Total: 100 points)
Write all solutions and scratch work in your blue book(s). You do not need to solve the problems in order, but please number the problems clearly in your blue book(s). Please try to write legibly.
You must fully justify your answers. Unless otherwise noted, you may use theorems from class, the book, or homework to do so, but unless it is utterly clear from context what result you are using, please say enough about the theorem (e.g., its name, if it has one, or a brief description) that I know what you’re talking about. If you are not sure whether or not some argument or statement requires further justification, please ask me about it.

1. (30 points) Let \( f(x) = x^4 + 9x^3 + 15x^2 - 6 \in \mathbb{Q}[x] \). Let \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C} \) be the roots of \( f \), so that \( f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4) \).

1(a) Prove that \( f \) is irreducible over \( \mathbb{Q} \).
1(b) Prove that \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are all distinct.
1(c) Compute \( \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 \).

2. (35 points) Let \( M/F \) be an extension of fields, and let \( K, L \) be intermediate fields. (That is, \( F \subseteq K \subseteq M \) and \( F \subseteq L \subseteq M \).) The compositum \( KL \) of \( K \) and \( L \) (in \( M \)) is defined to be the intersection of all subfields \( N \subseteq M \) that contain both \( K \) and \( L \), i.e.,

\[
KL = \bigcap_{N \subseteq M \text{ subfield}} N \cap K, L \subseteq N.
\]

2(a) If \( K = F(\alpha_1, \ldots, \alpha_m) \) for some \( \alpha_1, \ldots, \alpha_m \in M \), prove that \( KL = L(\alpha_1, \ldots, \alpha_m) \).
(Suggestion: For \( (\subseteq) \), prove that each field \( N \) in the intersection contains \( L(\alpha_1, \ldots, \alpha_m) \).
For \( (\supseteq) \), use one very specific field \( N \).)
2(b) If \([K : F] < \infty\), prove that \([KL : L] < \infty\).
2(c) If \([K : F] < \infty\) and \([L : F] < \infty\), prove that \([KL : F] < \infty\).

3. (35 points) Recall that \( \mathbb{F}_2 \) denotes the field with 2 elements. Let \( h(x) = x^4 + x + 1 \in \mathbb{F}_2[x] \). You may take my word for it that \( h \) is irreducible over \( \mathbb{F}_2 \). Let \( L/\mathbb{F}_2 \) be a splitting field of \( h \) over \( \mathbb{F}_2 \), and let \( \beta \in L \) be a root of \( h \).

3(a) Prove that \( h(\beta^2) = 0 \), and that \( h(\beta + 1) = 0 \).
3(b) Find four distinct roots of \( h \) in \( \mathbb{F}_2(\beta) \). That is, find roots \( \beta_1, \beta_2, \beta_3, \beta_4 \) of \( h \), each written as an explicit expression in terms of \( \beta \), and verify that they are all distinct.
(Suggestion: Part (a) might provide some useful inspiration.)
3(c) Prove that \( L = \mathbb{F}_2(\beta) \).
3(d) Prove that \( \mathbb{F}_2(\beta) \) is normal and separable over \( \mathbb{F}_2 \).