Math 410, Spring 2026: Galois Theory

MWF 9:00am – 9:50am, SCCE C101

Webpage: `http://rlbenedetto.people.amherst.edu/math410`
(Also linked from the Math 410 moodle page.)

Instructor: Rob Benedetto **Office**: SMUD 406 **Email**: `rlbenedetto@amherst.edu`

Office Hours: Mon, 2:00–3:30pm; Tue, 1:30–3:15pm; Fri, 1:00-2:00pm; or by appointment.

Text: David Cox, *Galois Theory*, 2nd ed. Wiley, Hoboken, 2012
For Amherst students: available through the Amherst College Textbook program.
(Five-College students must obtain the book on own.)

Exams:

- **Midterm**: **Wednesday, March 25**, in class.
- **Final**: **Take-home; details TBA**

The only excuses for missing an exam are incapacitating illness, religious conflict, or the like.

Using cell phones, etc. during an in-class exam is CHEATING.
Using outside sources on homework or exams — AI tools, other websites, outside textbooks, etc. — without advance permission is CHEATING.
All cheating will be prosecuted, potentially leading to an F in the course.

Homework:

- Reading from the textbook will be assigned each week, along with short videos to watch.
- Problem sets will be due once a week, usually on Wednesday nights, **on Gradescope by 11:59pm ET**.

See Homework 0, and see page 3 of this syllabus, for important homework information.

Grading:

- **Effort**: 5%
- **Problem Sets**: 45%
- **Midterm Exam**: 15%
- **Final Exam**: 35%

"Effort" is a combination of class attendance (including being on time), class participation, and handing in problem sets. It is not computed linearly; a student deficient in any one of those areas will get a very low Effort grade. (See page 4 of this syllabus for more on attendance and participation.)

"Problem Sets" means actual grades on the problem sets. Late problem sets will be marked down substantially in the Problem Sets portion of your grade (see the webpage for details); but all problem sets submitted by the last day of classes count towards Effort.

If one component of an individual student's course grade is substantially higher or lower than their other grades, and if the student's Effort grade is strong, I will tweak the above percentages a little for that student to favor the better grades. Overall course grades will be curved.

About Cell Phones and Mobile Devices

Cell phones and similar devices have no place in my classroom. Don't use them. Not for talking, not for texting, not for anything. So at every class:

Silence your cell phone, put it away, and pay attention.

Necessary Background

The prerequisite for this course is Groups, Rings, and Fields (Math 350). We will use virtually the entire contents of that course: groups, subgroups, normal subgroups, quotients, fields, and polynomials. There will also be some occasional use of commutative rings, ideals, ring quotients, and ring homomorphisms. In addition, we will be doing lots of proofs, so it is very important that you be comfortable with the kinds of proofs that arise in Math 350. If you aren't sure whether Math 410 is right for you, please feel free to talk to me about it.

Course Content

The motivating problem of Galois Theory is to find or understand the roots of a polynomial. One of the many important ideas of the theory is that we should study not just the roots themselves, but the fields that contain them. For example, the polynomial $f(x) = x^2 + 1 \in \mathbb{Q}[x]$ has rational coefficients, but the smallest field containing both $\mathbb{Q}$ and a root of f is the field of **Gaussian rationals**,

$$\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}.$$

More generally, we have a smaller field K (in this case, $\mathbb{Q}$) contained in a larger field L (in this case, $\mathbb{Q}(i)$). In Math 350, we study how groups and their subgroups relate to one another; similarly, in Galois theory, we study how fields and their subfields interact.

However, the key idea of Galois theory is to associate to an inclusion of fields $K \subseteq L$ a **group** G consisting of ring homomorphisms from L to itself that fix every element of K. It turns out that this can only be done nicely if the inclusion $K \subseteq L$ satisfies certain nice properties, in which case it is called a **Galois extension** of fields. This group G, called the **Galois group** of the extension of fields, ends up permuting the roots of the original polynomial f.

For example, in the case $\mathbb{Q} \subseteq \mathbb{Q}(i)$, the associated Galois group turns out to be the two-element cyclic group $G = \mathbb{Z}/2\mathbb{Z}$. The identity element of G is just the identity function on all of $\mathbb{Q}(i)$, while the other element of G is the **complex conjugation** function that maps $a + bi$ to $a - bi$.

From that basic idea, the power of group theory becomes available to attack problems concerning roots of polynomials. Some famous negative results can then be proven: it's impossible to square the circle with straightedge and compass; it's impossible to trisect an angle with straightedge and compass; and, using the full power of group theory, it's impossible (in general) to solve a polynomial of degree $d \geq 5$ using only the operations of addition, subtraction, multiplication, division, and the taking of n-th roots. The theory also gives, as promised, a deeper understand of how the roots of polynomials can behave.

Here's a more detailed outline of what we'll cover:

- In Chapters 1–3, we will study some fundamentals about polynomials and their roots, including the general solution of the cubic.
- In Chapter 4, we will learn the basics of field extensions — that is, inclusions $K \subseteq L$ of fields.
- In Chapters 5 and 6, we will learn about normal, separable, and Galois field extensions. Then we will define the Galois group of a field extension.
- In Chapter 7, we will learn the Fundamental Theorem of Galois Theory, which gives a correspondence between the subgroups of the Galois group and the intermediate fields F satisfying $K \subseteq F \subseteq L$.
- In Chapters 8–11, depending on how much time we have left, we will see applications of the theory, such as to the previously mentioned problems of straightedge and compass constructions and of solving polynomials by radicals. The topics of finite fields and of cyclotomic polynomials are less famous but equally cool.
- If by some miracle time permits, we'll look at topics in the later chapters, such as the explicit computation of Galois groups.

Homework

Your homework consists of **ALL THREE** of reading the book, watching the assigned videos, **AND** doing problem sets. **Start working on each problem set the same day it is assigned**; do *not* put it off until a day or two before it's due.

Please note the following **Important Problem Set Rules:**

1. Problem sets are due **on Gradescope**, each by its specified deadline.
2. You must **assign pages** in Gradescope, matching problems to pages of your submission.
3. **Write legibly**, and organize your work clearly. **Make it a pleasure to read!**
4. If you worked with other students in the class, then say so explicitly on the first page of your problem set. (See the discussion below on the Statement of Intellectual Responsibility.)
5. You **may** also get help from me, the book, and all of the handouts and videos from the course and the course websites. You may **not** use AI tools at all. If you want to use any **other** resources or aides, including outside websites, people, or textbooks,
   you must first get advance permission from me for each such outside source.
6. The Problem Sets grade for any late assignment will be substantially reduced. The later it is, the greater the reduction; see the course webpage under "Problem Set Rules" for details.

I am often willing to grant penalty-free extensions on problem sets; but see "Attendance, Extensions, and Extra Office Hours" on page 4 of this syllabus. I also encourage you to work on problem sets together, in pairs or small groups, provided you follow the common-sense guidelines below.

About the Statement of Intellectual Responsibility

Exams: Your work must be entirely your own, so no looking at other people's papers, no communication, and no outside help. For the in-class midterm, no books, notes, phones, smartwatches, or other resources are allowed, either. For the take-home final, you may use only course materials, **your own** notes, and the textbook; you may also consult me (Prof. Benedetto), but no one else. No other books, notes, online resources, AI tools, or communications with other people are allowed.

Problem sets: As in Rule 5 above, no AI or unapproved outside resources are allowed on homework, but as in Rule 4, I urge you to collaborate with each other, under the following ground rules:

1. If you collaborate with, say, Jane and Joe, write a note on the front of your problem set saying, "I worked with Jane and Joe." Use similar notation if you got help from a tutor, fellow student, another professor, or an outside source that I approved. However, you do **not** need to write about help you got from me, the textbook, or the other course materials.
2. Working together does not mean that Joe does the first half of the problem set and Jane does the second half; everyone should work on every problem.
3. Each student must submit their own problem set; you can't submit a single document as the work of multiple people.
4. Each student must write up each problem **in their own words**. Working together means discussing the problems. Using outside sources without approval, copying someone else's solution, or knowingly allowing someone else to copy your solution, is a violation of intellectual responsibility.

A common question: What if Joe asks Jane about a homework problem she has already solved? If Joe copies Jane's solution, both Joe and Jane would be guilty of academic dishonesty, potentially leading to an F in the course for both of them, among other bad consequences. Instead, Jane can explain her solution to Joe (even showing him what she wrote), before Joe writes up his own solution himself, **in his own words**. Joe would then have to write that he got help from Jane (see rule 1 above), but Jane doesn't need to write anything unless she also got help in return.

If at any time you aren't sure about what's OK and what's not as far as intellectual responsibility is concerned for this course, please talk to me about it.

Class Attendance, Extensions, and Extra Office Hours

Attendance: You should be at every class meeting, and you should be **on time**. Of course, if you're sick, have a religious conflict, or the like, just let me know (in advance, when possible). One or two accidental misses are OK, too; oversleeping can happen, but it should be **rare**. Otherwise, however,

I expect you to be in class, and on time, for every class meeting.

Extensions: You may request **up to two** homework extensions over the course of the semester, each for up to 48 hours. To claim an extension, you must:

1. Request the extension (by email, or in person) **no later than 8pm ET on the due date**,
2. Not have used both your extensions yet, and
3. Have been attending class devotedly and meeting homework deadlines.

Note: you do **not** need to provide an excuse or reason for your extension request; just ask.

Office Hours: You are always welcome to visit my regularly scheduled office hours. In addition, **IF you have been attending class and doing the homework**, you are also welcome to make appointments to see me **outside** of my regularly scheduled office hours.

Class Participation and Classroom Dynamics

Class participation is part of the Effort portion of your grade. If you are quiet by nature, don't worry; as long as you attend class devotedly, pay close attention, and do the homework, you will get full Effort credit. That said, finding your voice in class helps you learn better. So for your own benefit, **speak up, ask questions, and even try to answer questions I may ask in class**.

On the flip side,

respect your fellow students

both in and out of the classroom. Treat every person's ideas the same way you would if it had been me or the college president sharing them.

What to Expect

This course will be a noticeable level up from Math 350, and almost at the first-year graduate level. So this will probably be the most difficult math course you have taken. In particular, I assume that at this point in your college career, you're pretty comfortable with writing proofs. I also assume that you're comfortable thinking about abstract objects. If you're not sure whether you are ready for Math 410, please come talk to me about it.

The homework will usually consist partially of proofs and partially of the computation of examples (say, computing Galois groups of specific field extensions). In both cases, though, unless I explicitly say that you can wave your hands a bit on a given problem, you should assume that you need to

justify every claim you make.

That is, proofs should be fully rigorous, and every step of solutions to computational problems should be clearly justified. Similarly, if you're asked to give an example of, say, a group with a certain property, you must not only present the group, but also explain or prove that it has the desired property. (Often, these justifications are very quick, but they can't be absent.) If you're ever in doubt about whether some claim you are making requires proof, just ask me.