

Homework #9Due **Wednesday, April 8** in Gradescope by **11:59 pm ET****READ** Sections 6.4, 7.1, 7.2 in Cox

- WATCH** 1. Video 22: The Universal Extension (10:53)
 2. Video 23: More Fixed Field Results (11:32)
 3. Optional Video 24: Galois Closure (9:07)

WRITE AND SUBMIT solutions to the following problems.**Problem 1.** (15 points) Cox, Section 6.4, Exercise 1, variant:Let F be a field. For any $a, b \in F$, define $\gamma_{a,b} : F \rightarrow F$ by $\gamma_{a,b}(u) = au + b$.

- (a) (3 points) For any $a, b, c, d \in F$, prove that $\gamma_{a,b} \circ \gamma_{c,d} = \gamma_{s,t}$, where $s = ac$ and $t = ad + b$.
 (b) (5 points) For any $a, b \in F$, prove that $\gamma_{a,b}$ is bijective if and only if $a \neq 0$, and in that case, $\gamma_{a,b}^{-1} = \gamma_{c,d}$, where $c = a^{-1}$ and $d = -a^{-1}b$.
 (c) (7 points) Define $\text{AGL}(1, F) = \{\gamma_{a,b} \mid (a, b) \in F^\times \times F\}$.
 Prove that $\text{AGL}(1, F)$ is a group under composition.

Problem 2. (10 points) Cox, Section 6.4, Exercise 2, variant:With notation as in the previous problem, define $\varphi : \text{AGL}(1, F) \rightarrow F^\times$ by $\varphi(\gamma_{a,b}) = a$.

- (a) Prove that φ is a surjective group homomorphism.
 (b) Let $T = \ker(\varphi) = \{\gamma_{1,b} \mid b \in F\}$. Prove that T is isomorphic to the group $(F, +)$

[**Note:** It follows that $\text{AGL}(1, F)/T \cong F^\times$, and hence $\text{AGL}(1, F) \cong F \rtimes F^\times$.]**Problem 3.** (10 points) (not from Cox):Let $p \geq 2$ be prime, and let $H \subseteq S_p$ be a subgroup. Suppose that $p \mid |H|$ and that H contains a transposition. Prove that $H = S_p$.[**Suggestion:** Use HW 8 Problem 8 and ideas from Section 6.4.C, page 139.]**Problem 4.** (15 points) Cox, Section 6.4, Exercise 13:Let L be the splitting field of $f(x) = 2x^5 - 10x + 5$ over \mathbb{Q} . Prove that $\text{Gal}(L/\mathbb{Q}) \cong S_5$.[**Suggestion:** Use the previous problem and ideas from Section 6.4.C, page 139.]**Problem 5.** (7 points) Cox, Section 7.1, Exercise 1:Let L/F be a finite extension, and let $H \subseteq \text{Gal}(L/F)$ be a subgroup. Define

$$L_H = \{\alpha \in L \mid \sigma(\alpha) = \alpha \text{ for all } \sigma \in H\}.$$

Prove that L_H is a subfield of L that contains F .

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Problem 6. (22 points) Cox, Section 7.1, Exercise 9:

For each of the following extensions, determine whether it is a Galois extension. Of course, justify your answers, usually using one of the criteria in Theorem 7.1.1 or Theorem 7.1.5.

- (a) $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})/\mathbb{Q}$
- (b) $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$, where α, β are distinct roots of $x^3 + x^2 + 2x + 1$.
- (c) $\mathbb{F}_p(t)/\mathbb{F}_p(t^p)$, where $p \geq 2$ is prime and t is a (formal) variable.
- (d) $\mathbb{C}(t)/\mathbb{C}(t + t^{-1})$, where t is a (formal) variable.
- (e) $\mathbb{C}(t)/\mathbb{C}(t^n)$, where $n \geq 1$ is an integer and t is a (formal) variable.

Problem 7. (12 points) Cox, Section 7.2, Exercise 1:

Consider the extension L/\mathbb{Q} , where $L = \mathbb{Q}(\omega, \sqrt[3]{2})$, as in Example 7.3.2 and diagram (7.3).

- (a) Prove that the conjugate fields of $\mathbb{Q}(\sqrt[3]{2})$ are itself, $\mathbb{Q}(\omega\sqrt[3]{2})$, and $\mathbb{Q}(\omega^2\sqrt[3]{2})$.
- (b) Prove that the only conjugate field of $\mathbb{Q}(\omega)$ is itself.

Problem 8. (5 points) Cox, Section 7.2, Exercise 2:

Let $L/K/F$ be finite extensions, and let $\sigma \in \text{Gal}(L/F)$. Finish the proof of Lemma 7.2.4 by proving that $\text{Gal}(L/\sigma K) \subseteq \sigma \text{Gal}(L/K)\sigma^{-1}$.

Problem 9. (4 points) Cox, Section 7.2, Exercise 3:

Let $L/K_2/K_1$ be extensions of fields. Prove that $\text{Gal}(L/K_2) \subseteq \text{Gal}(L/K_1)$.

Optional Challenges (do NOT hand in): Cox Problems 6.4 #3,10, 7.1 #6,7

Questions? You can ask in:

Class: MWF 9:00am – 9:50am, SCCE C101

My office hours: in my office (SMUD 406):

Mon 2:00–3:30pm

Tue 1:30–3:15pm

Fri 1:00–2:00pm

Also, you may email me any time at rlbenedetto@amherst.edu