Math 410, Spring 2026 Professor Rob Benedetto

Homework #6

Due **Wednesday, March 4** in Gradescope by **11:59 pm ET**

READ Sections 5.2, 5.3 in Cox

WATCH
1. Video 13: Mapping One Root to Another (4:53)
2. Video 14: Normal and Splitting (8:30)
3. Optional Video 15: An Alternative Normality Proof (14:29)
4. Video 16: Defining Separability (13:47)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (8 points) Cox, Section 5.2, Exercise 1:
Prove that $\mathbb{Q}(\sqrt[4]{2})$ is not the splitting field (over $\mathbb{Q}$) of any polynomial in $\mathbb{Q}[x]$.

Problem 2. (20 points) Cox, Section 5.2, Exercise 3:
For each of the following field extensions, determine whether or not it is normal. (And, of course, prove your answers.)

(a) $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, where $n \geq 1$ and $\zeta_n = e^{2\pi i/n}$, a primitive n-th root of unity.

(b) $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})/\mathbb{Q}$

(c) $F(\alpha)/F$, where $F = \mathbb{F}_3(t)$ and α is a root of $x^3 - t \in F[x]$.

Problem 3. (5 points) Cox, Section 5.2, Exercise 4:
Give an example of a normal extension of fields that is not finite. (And, of course, prove your answer.)

Problem 4. (10 points) Cox, Section 5.3, Exercise 1:
Prove equations (5.6). That is, for any $g, h \in F[x]$ and any $a, b \in F$, prove that:

(a) $(ag + bh)' = ag' + bh'$

(b) $(gh)' = g'h + gh'$

Here, of course, f' denotes the **formal derivative** of $f \in F[x]$.

Problem 5. (7 points) Cox, Section 5.3, Exercise 2:
Let F be a field of characteristic $p \geq 2$. Recall (from Lemma 5.3.10) that for all $\alpha, \beta \in F$, we have $(\alpha + \beta)^p = \alpha^p + \beta^p$. Use this to prove the following identities for all $\alpha, \beta \in F$:

(a) $(\alpha - \beta)^p = \alpha^p - \beta^p$

(b) $(\alpha + \beta)^{p^e} = \alpha^{p^e} + \beta^{p^e}$, for any integer $e \geq 1$.

Problem 6. (8 points) Cox, Section 5.3, Exercise 3:
Let F be a field of characteristic $p \geq 2$, let $n \geq 1$, and define L to be the splitting field of $x^n - 1$ over F. The n-th roots of unity are defined to be the roots of $x^n - 1$ in L.

(a) If $p \nmid n$, prove that there are n distinct n-th roots of unity in L.

(b) Prove that there is only one p-th root of unity, namely $1 \in F$.

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Problem 7. (20 points) Cox, Section 5.3, Exercise 7:
Let F be a field of characteristic $p \geq 2$, and let $f \in F[x]$ be irreducible. In this problem you'll prove Proposition 5.3.16.

(a) Suppose f' is **not** the zero polynomial. Prove that f is separable.
[Cox suggests using the argument in the proof of Lemma 5.3.5.]

(b) Suppose f' **is** the zero polynomial.
Prove that there is a polynomial $g_1 \in F[x]$ such that $f(x) = g_1(x^p)$.

(c) In the situation of part (b), prove that the polynomial g_1 is irreducible.

(d) Prove Proposition 5.3.16: For any $f \in F[x]$ irreducible, there is an integer $e \geq 0$ and a separable, irreducible $g \in F[x]$ such that $f(x) = g\big(x^{p^e}\big)$.
[**Suggestion**: Cox says to "apply parts (a)–(c) repeatedly".]

Problem 8. (15 points) Cox, Section 5.3, Exercise 9:
Let F be a field of characteristic $p \geq 2$, let $a \in F$, and define $f(x) = x^p - a$. Suppose that f has no roots in F (and hence is irreducible over F, by Proposition 4.2.6). Let α be a root of f in some extension L/F.

(a) Prove that $F(\alpha)$ is the splitting field of f over F and that $[F(\alpha) : F] = p$.
[Cox suggests using the argument in Example 5.3.11.]

(b) Let $\beta \in F(\alpha)$ with $\beta \notin F$. Use Lemma 5.3.10 to prove that $\beta^p \in F$.

(c) For β as in part (b), use parts (a) and (b) to prove that the minimal polynomial of β over F is $x^p - \beta^p$.

(d) Conclude by proving that the extension $F(\alpha)/F$ is purely inseparable.

[**FYI**: Problems 6–8 together show that there is something truly different about taking p-th roots in characteristic p. The resulting inseparability poses an obstacle to Galois Theory in that case. This issue can be partially remedied via *Artin-Schreier* theory, the basics of which will appear in Homework 7, Problem 6.]

Optional Challenges (do NOT hand in): Cox Problems 5.3 #4,5,17

Questions? You can ask in:

Class: MWF 9:00am – 9:50am, SCCE C101

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

Also, you may email me any time at `rlbenedetto@amherst.edu`