Math 410, Spring 2026 — Professor Rob Benedetto

Homework #5

Due **Wednesday, February 25** in Gradescope by **11:59 pm ET**

READ Sections 4.4, 5.1 in Cox

WATCH 1. Video 10: Algebraic over Algebraic (5:54)
2. Optional Video 11: Algebraic Integers (26:54)
3. Video 12: Splitting Field Degree (4:51)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (9 points) Cox, Section 4.4, Exercise 1:
Recall that $\overline{\mathbb{Q}}$ is the field of algebraic numbers, i.e., $\{\alpha \in \mathbb{C} | \alpha \text{ is algebraic over } \mathbb{Q}\}$.

(a) For each integer $n \geq 2$, prove that $\overline{\mathbb{Q}}$ has a subfield L such that $[L : \mathbb{Q}] = n$.
[**Suggestion**: Use Example 4.2.4.]

(b) Use part (a) to prove that $[\overline{\mathbb{Q}} : \mathbb{Q}] = \infty$.

[**Note**: Recall that Lemma 4.4.2 says that if $[L : K] < \infty$, then L is algebraic over $\mathbb{Q}$. This exercise shows that the converse is false.]

Problem 2. (14 points) Cox, Section 4.4, Exercise 3:
We say $\alpha \in \mathbb{C}$ is an *algebraic integer* if α is a root of a monic polynomial in $\mathbb{Z}[x]$ (i.e., monic and with *integer* coefficients).

(a) Prove that $\alpha \in \mathbb{C}$ is an algebraic integer if and only if α is algebraic over $\mathbb{Q}$ and its minimal polynomial $f \in \mathbb{Q}[x]$ has integer coefficients. [**Hint**: Use Gauss's Lemma.]

(b) Prove that $\omega/2$ is *not* an algebraic integer, where $\omega = \zeta_3$ is a root of $x^2 + x + 1$.

Problem 3. (15 points) Cox, Section 4.4, Exercise 6:
Let F be a field, and let $M = \{\alpha \in F(x) | \alpha \text{ is algebraic over } F\}$. Prove that $M = F$.

[**Note 1**: Example 4.3.7 shows that $[F(x) : F] = \infty$, but still, this exercise shows that the field M above (of Corollary 4.4.5) is as small as possible, i.e., M is F itself.]
[**Note 2**: When doing this problem, I'd recommend using a different letter, like t, for the variable in the ring $F[t]$ where polynomials $h(t)$ live that (might) have elements of M as their roots.]

Problem 4. (10 points) Cox, Section 5.1, Exercise 1:
Prove that the splitting field of $x^3 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\omega, \sqrt[3]{2})$.
[As before, $\omega = \zeta_3$ is a root of $x^2 + x + 1$.]

Problem 5. (8 points) Cox, Section 5.1, Exercise 3:
Let L/F be an extension of fields with $[L : F] = 2$.
Prove that L is a splitting field of some $f \in F[x]$.

Problem 6. (10 points) Cox, Section 5.1, Exercise 4, variant:
Consider the following three subfields of $\mathbb{C}$ [where as before, $\omega = \zeta_3$ is a root of $x^2 + x + 1$]:
$K_1 = \mathbb{Q}(\omega)$, $K_2 = \mathbb{Q}(\sqrt{-3})$, and K_3 is the splitting field of $x^6 - 1 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.
Prove that $K_1 = K_2 = K_3$.

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Problem 7. (10 points) Cox, Section 5.1, Exercise 6:
Let $f \in \mathbb{Q}[x]$ be the minimal polynomial of $\alpha = \sqrt{2+\sqrt{2}}$ over $\mathbb{Q}$.

(a) Prove that $f = x^4 - 4x^2 + 2$, and hence that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$.

(b) Prove that $\mathbb{Q}(\alpha)$ is the splitting field of f over $\mathbb{Q}$.

Problem 8. (14 points) Cox, Section 5.1, Exercise 7:
Let $f = x^3 - x + 1 \in \mathbb{F}_3[x]$.

(a) Prove that f is irreducible over $\mathbb{F}_3$.

(b) Let L be the splitting field of f over $\mathbb{F}_3$. Prove that $[L : F] = 3$.

(c) Prove that $|L| = 27$.

Problem 9. (10 points) Cox, Section 5.1, Exercise 11:
Let F be a field, let $f \in F[x]$ be irreducible over F of degree $n \geq 1$, and let L be the splitting field of f over F.

(a) Prove that $n | [L : F]$.

(b) Give an example with $n \geq 4$ to show that $n = [L : F]$ can occur.

[**Note**: In fact, for any $n \geq 1$, there are examples where $n = [L : F]$. Can you prove this?]

Optional Challenges (do NOT hand in): Cox Problems 4.4 #2,5,8,9; 5.1 #8,9,13

Questions? You can ask in:

Class: MWF 9:00am – 9:50am, SCCE C101

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

Also, you may email me any time at `rlbenedetto@amherst.edu`