Homework #4

Due **Wednesday, February 18** in Gradescope by **11:59 pm ET**

READ Sections 4.1, 4.2, 4.3 in Cox

WATCH 1. Video 7: Some Proofs on Adjoining Elements (17:20)

2. Optional Video 8: Proof of Eisenstein's Criterion (24:38)
3. Optional Video 9: Straightedge and Compass (28:30)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (6 points) Cox, Section 4.1, Exercise 1:
Let $\alpha \in L \setminus \{0\}$ be algebraic over a subfield F. Prove that $1/\alpha$ is also algebraic over F.

Problem 2. (12 points) Cox, Section 4.1, Exercise 8:
If $f(x) \in F[x]$ is irreducible, it may or may not be irreducible over a particular extension field L/F, as you will show in this problem.

(a) Prove that $f(x) = x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$.

(b) In Example 4.1.7, it was shown that $g(x) = x^4 - 10x^2 + 1$ is irreducible over $\mathbb{Q}$ (and it is the minimal polynomial of $\alpha = \sqrt{2} + \sqrt{3}$).
Prove that g is *reducible* over $\mathbb{Q}(\sqrt{3})$, by finding an explicit factorization.

Problem 3. (12 points) Cox, Section 4.2, Exercise 5, variant:
Find the cyclotomic polynomial Φ_{24}, i.e., the minimal polynomial of ζ_{24} over $\mathbb{Q}$, as follows.

(a) Factor $x^{24} - 1$ over $\mathbb{Q}$.

(b) Remembering that the factors of $x^{24} - 1$ must be Φ_n for each $n \geq 1$ with $n|24$, identify which factor is Φ_{24}.

[**Note**: You may assume without proof that each Φ_n is irreducible over $\mathbb{Q}$, and that $\deg(\Phi_n) = \phi(n)$, where $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$. FYI: ϕ is known as the Euler totient function, or the Euler-ϕ function.]

Problem 4. (12 points) Cox, Section 4.2, Exercise 7:
For each of the following polynomials, determine (and prove) whether or not it is irreducible over the given field, without using a computer.

(a) (4 points) $x^3 + x + 1$ over $\mathbb{F}_5$.

(b) (8 points) $x^4 + x + 1$ over $\mathbb{F}_2$.

[**Note**: For this class, unless I specifically say otherwise, you should **never** use a computer to solve a problem. But Cox includes the specific restriction against computers because sometimes he talks a bit about implementing these sorts of algorithms on a computer. But the point here is that for such relatively low degrees, you really can do this by hand without much trouble.]

(continued next page)

Problem 5. (5 points) Cox, Section 4.2, Exercise 8:
Let $a \in \mathbb{Z}$ be a product of (a positive number of) distinct primes, and let $n \geq 1$.
Prove that $x^n - a$ is irreducible over $\mathbb{Q}$.

Problem 6. (20 points) Cox, Section 4.3, Exercise 2:
Compute the degrees of the following field extensions.

(a) $\mathbb{Q}(i, \sqrt[4]{2}) \,/\, \mathbb{Q}$

(b) $\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}) \,/\, \mathbb{Q}$

(c) $\mathbb{Q}(\sqrt{2+\sqrt{2}}) \,/\, \mathbb{Q}$

(d) $\mathbb{Q}(i, \sqrt{2+\sqrt{2}}) \,/\, \mathbb{Q}$

Problem 7. (8 points) Cox, Section 4.3, Exercise 4:
Let L/F be a finite extension with $[L : F]$ prime.

(a) Prove that the only intermediate fields K (i.e., fields K with $L/K/F$) are F and L.

(b) For any $\alpha \in L \setminus F$, prove that $L = F(\alpha)$.

Problem 8. (15 points) Cox, Section 4.3, Exercise 5:
Let $L = \mathbb{Q}(\sqrt[4]{2}, \sqrt[3]{3})$. In this problem, you will compute $[L : \mathbb{Q}]$.

(a) Prove that both $x^4 - 2$ and $x^3 - 3$ are irreducible over $\mathbb{Q}$.

(b) Let $K_1 = \mathbb{Q}(\sqrt[4]{2})$, so that $L/K_1/\mathbb{Q}$. Use K_1 to prove that $4 | [L : \mathbb{Q}]$ and that $[L : \mathbb{Q}] \leq 12$.

(c) Let $K_2 = \mathbb{Q}(\sqrt[3]{3})$, so that $L/K_2/\mathbb{Q}$. Use K_2 to prove that $3 | [L : \mathbb{Q}]$.

(d) Use parts (b) and (c) to prove that $[L : \mathbb{Q}] = 12$.

Optional Challenges (do NOT hand in): Cox Problems 4.1 #7, 4.2 #1, 4.3 #3,6

Questions? You can ask in:

Class: MWF 9:00am – 9:50am, SCCE C101

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

Also, you may email me any time at `rlbenedetto@amherst.edu`