Homework #2

Due **Wednesday, February 4** in Gradescope by **11:59 pm ET**

READ Sections 2.1–2.2 in Cox

WATCH Video 3: Monomial Ordering and Symmetric Polynomials (18:27)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (16 points) Cox, Section 2.2, Problem 2:
Let F be a field and $n \geq 1$ an integer. Consider the ring $F[x_1, \ldots, x_n]$. The leading term (with respect to the graded lexicographic ordering $>$) of $f \in F[x_1, \ldots, x_n]$ will be denoted by $\mathrm{LT}(f)$. For an *exponent vector* $\alpha = (a_1, \ldots, a_n)$ of nonnegative integers a_i, let x^α denote the monomial

$$x^\alpha = x_1^{a_1} \cdots x_n^{a_n}.$$

Through the rest of this problem, $\alpha, \beta, \gamma, \delta$ will denote arbitrary exponent vectors. Clearly $x^\alpha x^\beta = x^{\alpha+\beta}$, a fact which you may use without proof.

- (a): If $x^\alpha > x^\beta$, prove that $x^{\alpha+\gamma} > x^{\beta+\gamma}$.
- (b): If $x^\alpha > x^\beta$ and $x^\gamma > x^\delta$, prove that $x^{\alpha+\gamma} > x^{\beta+\delta}$.
- (c): For any nonzero $f, g \in F[x_1, \ldots, x_n]$, prove that $\mathrm{LT}(fg) = \mathrm{LT}(f)\mathrm{LT}(g)$

Problem 2. (8 points) Cox, Section 2.2, Problem 7:
Let F be a field, and let $f \in F[x_1, \ldots, x_n]$. For any permutation $\sigma \in S_n$, denote by $\sigma \cdot f$ the polynomial obtained from f by permuting the variables according to σ.
Prove that both $\prod_{\sigma \in S_n} \sigma \cdot f$ and $\sum_{\sigma \in S_n} \sigma \cdot f$ are symmetric polynomials.

Problem 3. (8 points) Cox, Section 2.2, Problem 10:
Apply the proof method of Theorem 2.2.2 to express

$$\textstyle\sum_3 x_1^2 x_2 = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$$

in terms of σ_1, σ_2, σ_3.

Problem 4. (12 points) Cox, Section 2.2, Problem 11a,b:
Let $\alpha, \beta, \gamma \in \mathbb{C}$ be the roots of $y^3 + 2y^2 - 3y + 5$. Find the monic polynomials of degree three with integer coefficients that have the following roots:

- (a): $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$
- (b): $\alpha + 1$, $\beta + 1$, $\gamma + 1$

Problem 5. (10 points) Cox, Section 2.2, Problem 11c:
Let $\alpha, \beta, \gamma \in \mathbb{C}$ be the roots of $y^3 + 2y^2 - 3y + 5$. Find the monic polynomial of degree three with integer coefficients that has roots α^2, β^2, γ^2.

(Office Hours and optional challenge problems on next page)

Optional Challenges (do NOT hand in): Cox Problems 2.2 #6, 8

Questions? You can ask in:

Class: MWF 9:00am – 9:50am, SCCE C101

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

Also, you may email me any time at `rlbenedetto@amherst.edu`