Math 410, Spring 2026 Professor Rob Benedetto

Homework #12

Due **Wednesday, April 29** in Gradescope by **11:59 pm ET**

READ Sections 8.3, 8.4, 9.1 in Cox

WATCH
1. Video 32: Galois and Radical (12:15)
2. Video 33: A Solvability Lemma (10:36)
3. Video 34: A Galois Degree Lemma (8:09)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (15 points) Cox, Section 8.3, Exercise 4, rephrased:
Let $m \geq 1$ be an integer, and let F be a field containing a primitive m-th root of unity $\zeta \in F$. Let $a \in F^\times$, and let $K = F(\gamma)$ where γ is a root of $x^m - a$. [That is, $\gamma = \sqrt[m]{a}$.]

(a) For each $\sigma \in \mathrm{Gal}(K/F)$, prove that there is a unique integer n such that $0 \leq n \leq m-1$ and $\sigma(\gamma) = \zeta^n \gamma$.

(b) Define $\varphi : \mathrm{Gal}(K/F) \to \mathbb{Z}/m\mathbb{Z}$ by $\varphi(\sigma) \equiv n \pmod{m}$, where n is the integer from part (a) for which $\sigma(\gamma) = \zeta^n \gamma$. Prove that φ is an injective homomorphism.

(c) Conclude that $\mathrm{Gal}(K/F)$ is cyclic, of order dividing m.

Problem 2. (5 points) Cox, Section 8.3, Exercise 5, slightly rephrased:
Let $M/L/K/F$ be finite extensions such that M/F and L/K are both Galois. Prove that $|\mathrm{Gal}(L/K)|$ divides $|\mathrm{Gal}(M/F)|$.

Problem 3. (10 points) Cox, Section 8.4, Exercise 1:
Let G be a nontrivial finite abelian group. Prove that G is simple if and only if $G \cong \mathbb{Z}/p\mathbb{Z}$ for some prime number p.

Problem 4. (16 points) Cox, Section 8.4, Exercise 6, slightly rephrased:
Let G be a finite group of order $n \geq 1$.

(a) Consider the set S of all proper normal subgroups of G. (I.e., $S = \{N \lhd G \mid N \neq G\}$.) If $n \geq 2$, prove that there exists $H \in S$ of maximal order, i.e., such that $|N| \leq |H|$ for all $N \in S$.

(b) If $n \geq 2$, for the normal subgroup H of part (a), prove that G/H is simple.

(c) Use strong induction on n to prove that every finite group has a composition series.

[**Suggestion**: For part (b), the Correspondence Theorem (between subgroups of G/H and subgroups of G that contain H) may come in handy.]

Problem 5. (12 points) Cox, Section 8.4, Exercise 8:
Prove that $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are nonisomorphic groups whose composition series consist of the same list of simple groups (up to isomorphism).

(continued next page)

In the next three problems, for positive integers $n \geq 1$, you'll prove properties of the cyclotomic polynomial

$$\boxed{\Phi_n(x) = \prod_{\substack{0 \leq i < n \\ \gcd(i,n)=1}} (x - \zeta_n^i)}$$

where ζ_n is a primitive n-th root of unity. Observe that the product is over all integers $0 \leq i < n$ with $\gcd(i, n) = 1$. (This is equation (9.3) in Cox's book, on page 231.)

Problem 6. (14 points) Cox, Section 9.1, Exercise 12a:
Let $n \geq 1$ be an integer, and define $m = \prod_{p|n} p$ be the product of all primes dividing n, each only to the first power.
Prove that $\Phi_n(x) = \Phi_m(x^{n/m})$.

[**Note**: This reduces the computation of Φ_n to the case that n is squarefree.]

Problem 7. (12 points) Cox, Section 9.1, Exercise 12b:
If $n \geq 3$ is odd, prove that $\Phi_{2n}(x) = \Phi_n(-x)$.

[**Note**: Together with the previous problem, this reduces the computation of Φ_n to the case that n is odd and squarefree.]

Problem 8. (12 points) Cox, Section 9.1, Exercise 12c:
If $n \geq 1$ and p is a prime *not* dividing n, prove that $\Phi_{pn}(x) = \Phi_n(x^p)/\Phi_n(x)$.

[**Note**: This gives a strategy for computing Φ_n for n odd and squarefree, although the computations can get quite messy in practice because of the quotient.]

Optional Challenges (do NOT hand in): Cox Problems 8.3 #7; 8.4 #2,3; 9.1 #11,13

Questions? You can ask in:

Class: MWF 9:00am – 9:50am, SCCE C101

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

Also, you may email me any time at `rlbenedetto@amherst.edu`