Homework #11

Due **Wednesday, April 22** in Gradescope by **11:59 pm ET**

READ Sections 8.1, 8.2, 8.3 in Cox

WATCH
1. Video 28: Finitely Many Intermediate Fields (4:43)
2. Video 29: A Function Field Fact (13:09)
3. Optional Video 30: Some Advanced Group Theory Results (9:31)
4. Video 31: Radical vs Solvable (12:09)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (10 points) Cox, Section 8.1, Exercise 7, variant:
Prove that if n is an integer with $1 \leq n < 60$ that is divisible by at least three distinct primes, then either $n = 30$ or $n = 42$.
Then use Use Burnside's $p^a q^b$ Theorem (Theorem 8.1.8) to prove that any group G with $|G| < 60$ and with $|G| \neq 30, 42$ must be solvable.

[**FYI**: It turns out that all groups of order 30 and 42 are also solvable, but more work is required to prove that. See Cox Example 8.1.11 and Exercise 8.1.6 if you're curious.]

Problem 2. (15 points) Cox, Section 8.1, Exercise 8, slightly rephrased:
Let G be a finite group, and suppose that there are subgroups

$$\{e\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

such that $G_{i-1} \lhd G_i$ for each $1 \leq i \leq n$.

(a) Suppose that G_i/G_{i-1} is abelian for each $1 \leq i \leq n$. Prove that G is solvable.

(b) Suppose that G_i/G_{i-1} is solvable for each $1 \leq i \leq n$. Prove that G is solvable.

[**Note**: these facts were stated but not proven in lectures and/or videos. Of course, you may not just quote those facts. However, you **may** use facts that were not just stated but actually proven in lectures and/or videos and/or Section 8.1]

Problem 3. (10 points) Cox, Section 8.1, Exercise 5, variant:
The center of a group G is the set $Z(G) = \{g \in G \,|\, xg = gx \text{ for all } x \in G\}$. In this problem, you may assume the following two facts from Math 350:

Fact 1. $Z(G)$ is a normal subgroup of G.

Fact 2. If $|G| = p^n$ for a prime p and integer $n \geq 1$, then $|Z(G)| > 1$.

Use the above facts to prove that for any group G whose order $|G|$ is a power of a prime, G is solvable.

[**Suggestion**: Write $|G| = p^n$ and proceed by induction on $n \geq 0$. Feel free to use use facts that were not just stated but actually proven in lectures and/or videos and/or Section 8.1, and/or the results of the previous problem.]

[**FYI**: Fact 1 is easy to prove from the definitions, whereas Fact 2 requires the use of the Class Equation. But again, you may assume both facts without proof.]

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Problem 4. (10 points) Cox, Section 8.2, Exercise 6:
Let $M/L/F$ be finite extensions of fields, and let $\sigma \in \mathrm{Gal}(M/F)$. Assume that L/F is a radical extension. Prove that $(\sigma L)/F$ is also radical.

Problem 5. (8 points) Cox, Section 8.3, Exercise 3, slight variant:
Let p be a prime, let K be a field, and let $\zeta \in K$ be a primitive p-th root of unity. [That is, $\zeta^p = 1$ but $\zeta \neq 1$.] For any integer $n \in \mathbb{Z}$ with $p \nmid n$, prove that

$$1 + \zeta^n + \zeta^{2n} + \cdots + \zeta^{(p-1)n} = 0.$$

[**Note**: we are in an abstract field K, not a subfield of $\mathbb{C}$. So you can't write $\zeta = e^{2\pi i/p}$!!! All you know is that $\zeta \in K$, that $\zeta^p = 1$, and that $\zeta \neq 1$. Also remember that n might be negative, so don't implicitly assume $n > 0$.]

Problem 6. (6 points) Cox, Section 8.3, Exercise 6, slight variant:
Let L be a field, let $m \geq n \geq 1$ be integers with $n|m$, and let $\zeta \in L$ be a primitive m-th root of unity. [So $\zeta^m = 1$ but for any integer d with $1 \leq d < m$, we have $\zeta^d \neq 1$.] Prove that $\zeta^{m/n}$ is a primitive n-th root of unity.

Optional Challenges (do NOT hand in): Cox Problems 8.1 #3, 6; 8.2 #3

Questions? You can ask in:

Class: MWF 9:00am – 9:50am, SCCE C101

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

Also, you may email me any time at `rlbenedetto@amherst.edu`