Homework #10

Due **Wednesday, April 15** in Gradescope by **11:59 pm ET**

READ Sections 7.2, 7.3, 7.4 in Cox

WATCH 1. Optional Video 25: Counting Conjugate Fields (8:44)
2. Optional Video 26: Infinite Galois Extensions (22:29)
3. Video 27: Another Proof of a Key Lemma (17:36)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (12 points) Cox, Section 7.2, Exercise 8:
Let G be a group, let $H \subseteq G$ be a subgroup, and define $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$.

(a) Prove that $N_G(H)$ is a subgroup of G containing H.

(b) Prove that H is normal in $N_G(H)$.

(c) Let $N \subseteq G$ be a subgroup of G containing H.
Prove that H is normal in N if and only if $N \subseteq N_G(H)$.

(d) Prove that H is normal in G if and only if $N_G(H) = G$.

Problem 2. (15 points) Cox, Section 7.3, Exercise 3, variant:
Let $L = \mathbb{Q}(i, \sqrt[4]{2})$ and $G = \mathrm{Gal}(L/\mathbb{Q})$. We already know (cf. Homework 8, Problem 6b) that $L/\mathbb{Q}$ is Galois, with $|G| = [L : \mathbb{Q}] = 8$, and in fact $G \cong D_4$, with elements $\sigma, \tau \in G$ such that

$$\sigma(i) = i,\ \sigma(\sqrt[4]{2}) = i\sqrt[4]{2},\ \tau(i) = -i,\ \tau(\sqrt[4]{2}) = \sqrt[4]{2}$$

which satisfy $o(\sigma) = 4$, $o(\tau) = 2$, and $\tau\sigma = \sigma^{-1}\tau$. So $G = \{\sigma^j\tau^k \mid j \in \{0,1,2,3\} \text{ and } k \in \{0,1\}\}$.

(a) Let $K_1 = \mathbb{Q}(\sqrt[4]{2})$. Let $H_1 = \mathrm{Gal}(L/K_1)$. Determine H_1 as an explicit set of elements of the form $\sigma^j\tau^k$ with $j \in \{0,1,2,3\}$ and $k \in \{0,1\}$. Then verify that $H_1 \ntriangleleft G$ by finding $g \in G$ and $h \in H_1$ such that $ghg^{-1} \notin H_1$.

(b) Let $K_2 = \mathbb{Q}(i)$. Let $H_2 = \mathrm{Gal}(L/K_2)$. Determine H_2 as an explicit set of elements of the form $\sigma^j\tau^k$ with $j \in \{0,1,2,3\}$ and $k \in \{0,1\}$. Then verify that $H_2 \triangleleft G$ by proving $ghg^{-1} \in H_2$ for all $g \in G$ and $h \in H_2$.

(c) Let $H_3 = \{e, \sigma^2\} = \langle\sigma^2\rangle$, which is a normal subgroup of G. Determine the fixed field $K_3 = L_{H_3}$, and find a polynomial $f \in \mathbb{Q}[x]$ such that K_3 is the splitting field of f over $\mathbb{Q}$.

[**Note**: In all three cases above, this confirms the normality portion of the Fundamental Theorem: $K_1/\mathbb{Q}$ is *not* a normal extension (cf. Homework 6, Problem 1); $K_2/\mathbb{Q}$ *is* a splitting field and hence a normal extension; and H_3 *is* a normal subgroup of G.]

Problem 3. (16 points) Cox, Section 7.3, Exercise 5, variant:
Let $F = \mathbb{C}(t^4)$, let $L = \mathbb{C}(t)$, and let $f(x) = x^4 - t^4 \in F[x]$.

(a) Prove that L is the splitting field of f over F, and hence L/F is Galois.

(b) Prove that f is irreducible over F, and that $[L : F] = 4$.

(c) Prove that there exists $\sigma \in \mathrm{Gal}(L/F)$ such that $\sigma(t) = it$.

(d) Prove that $\mathrm{Gal}(L/F)$ is cyclic of order 4, generated by σ.

(e) Determine all the subgroups H of $\mathrm{Gal}(L/F)$, and for each one, determine the corresponding intermediate field L_H.

(continued next page)

Problem 4. (15 points) Cox, Section 7.3, Exercise 7, variant:
Let $p \geq 3$ be an odd prime. Recall that the cyclotomic polynomial $\Phi_p = x^{p-1} + \cdots + 1$, which we already know is irreducible over $\mathbb{Q}$, has root ζ_p. Let $L = \mathbb{Q}(\zeta_p)$, which we already know is the splitting field of Φ_p over $\mathbb{Q}$. We also already know that $\mathrm{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$, with $j \in (\mathbb{Z}/p\mathbb{Z})^\times$ corresponding to $\sigma_j \in \mathrm{Gal}(L/\mathbb{Q})$ satisfying $\sigma_j(\zeta_p) = \zeta_p^j$.
Let $\tau = \sigma_{-1}$, and let $H = \langle \tau \rangle \subseteq \mathrm{Gal}(L/\mathbb{Q})$.

(a) Let $K = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$. Prove that K is the fixed field L_H corresponding to H.

(b) Prove that $K \subseteq \mathbb{R}$.
[**Suggestion**: Recall that $\zeta_p = e^{2\pi i/p} = \cos(2\pi/p) + i\sin(2\pi/p)$.]

(c) Prove that is $K/\mathbb{Q}$ is a Galois extension with $\mathrm{Gal}(K/\mathbb{Q})$ cyclic of order $(p-1)/2$.

(d) Prove that the minimal polynomial of $\zeta_p + \zeta_p^{-1}$ over $\mathbb{Q}$ has degree $(p-1)/2$.

Problem 5. (5 points) (Not from Cox.)
Let F be a field with $\mathrm{char}\, F \neq 2$, and let K/F be an extension with $[K : F] = 2$.
Prove that there exists $a \in F$ such that $K = F(\sqrt{a})$.
[**Suggestion**: Use the quadratic formula.]

Problem 6. (15 points) Cox, Section 7.3, Exercise 9:
Let F be a field with $\mathrm{char}\, F \neq 2$, and let L/F be a finite extension. Prove that the following are equivalent:

(i) L/F is Galois with $\mathrm{Gal}(L/F) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

(ii) There exist $a, b \in F$ for which none of a, b, or ab is a square in F, such that L is the splitting field of $(x^2 - a)(x^2 - b)$ over F.

[**Suggestion**: Problem 5 may be useful.]

Problem 7. (10 points) Cox, Section 7.3, Exercise 12, variant:
Let G be a group, and let $H \subseteq G$ be a subgroup. Define $N = \bigcap_{g \in G} gHg^{-1}$.

(a) Prove that N is a normal subgroup of G. [**Note**: Don't forget the "subgroup" part!]

(b) Let K be a normal subgroup of G contained in H. Prove that $K \subseteq N$.

[**Note**: This problem shows that N is the largest normal subgroup of G contained in H.]

Problem 8. (12 points) Cox, Section 7.4, Exercise 2, variant:
Use Proposition 7.4.2 and the formula $\Delta(x^3 + px + q) = -4p^3 - 27q^2$ from equation (1.22) to determine the Galois groups of the following cubic polynomials:

(a) $x^3 - 4x + 2$ over $\mathbb{Q}$
(b) $x^3 - 4x + 2$ over $\mathbb{Q}(\sqrt{37})$
(c) $x^3 - t$ over $\mathbb{C}(t)$
(d) $x^3 - t$ over $\mathbb{Q}(t)$

Optional Challenges (do NOT hand in): Cox Problems 7.2 # 7, 9; 7.3 # 3, 10, 13

Questions? You can ask in:

Class: MWF 9:00am – 9:50am, SCCE C101

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

Also, you may email me any time at `rlbenedetto@amherst.edu`