Math 410, Spring 2026 | Professor Rob Benedetto

Final Exam, Take-Home
Due **Tuesday, May 12** in Gradescope by **11:59 pm ET**
(You are welcome to submit it before that, if you want.)

Instructions: Do all **six numbered problems** (totalling 200 points), and in addition, as a seventh "problem," **write out and sign the academic honesty pledge found later in this document**.

Answers must be written neatly and legibly, and matched to the correct problem numbers on Gradescope.

You must fully justify your answers. (So on computational problems, show all steps and explain your thinking; on proof problems, give rigorous proofs.)

You may quote theorems from class or from the sections of the book that we covered, as well as standard theorems from Math 350 (or an equivalent course). You may also quote the results of any assigned (**non-challenge**) homework problems or midterm exam problems, whether or not you correctly solved those problems yourself.

However, you must clearly verify all the hypotheses of any theorem you use, and you must clearly name or reference any such theorem. (E.g., "By Proposition 5.1.6..., or "By Lagrange's Theorem...," or "By Problem 4 of Homework 7....") If you are not sure how to reference a particular result, or whether or not some argument or statement requires further justification, please ask me about it.

You may use Cox's textbook (Chapters 1–11), your notes, your own old homework, and any materials from the **course** websites, including handouts, problem solutions, and videos from the course. You may also consult a textbook Groups, Rings, and Fields, (like Saracino's), provided it does not cover material beyond Math 350.

But of course, as a matter of Academic Honesty, until after the exam deadline has passed:

> **You may NOT use other books, online information, AI tools, calculators, or any other outside sources.**
> You also may NOT discuss the problems with anyone other than me.

But you **should** feel free to talk to me about anything on the exam.

The exam is due at **11:59 PM ET** on the Tuesday of exam period, on Gradescope. You may submit it early, even days early, but **no extensions will be granted**. Outside of truly exceptional circumstances, any exam not submitted on time will be graded as a zero.

I strongly recommend that you

> **plan to submit the exam before 8pm ET, Tuesday, May 12**

so that you have a four-hour grace period in case any unexpected snags arise.

1. (**20 points**). Let L/F be an extension of fields, and let $\alpha, \beta \in L$ be algebraic over F. Denote by $f, g \in F[x]$ the minimal polynomials of α, β (respectively) over F.

Suppose that f is irreducible over $F(\beta)$. Prove that g is irreducible over $F(\alpha)$.

2. (**30 points**). Let $f(x) = x^6 + 12 \in \mathbb{Q}[x]$, and let L be the splitting field of f over $\mathbb{Q}$. Prove that:

(a) L contains a primitive cube root of unity; call it ω.

(b) f factors over $\mathbb{Q}(\omega)$ as a product of two cubic polynomials.

(c) $\mathrm{Gal}(L/\mathbb{Q}(\omega))$ is cyclic of order 3.

3. (**30 points**). Let $f(x) = x^6 - 12 \in \mathbb{Q}[x]$, let L be the splitting field of f over $\mathbb{Q}$, and let $G = \mathrm{Gal}(L/\mathbb{Q})$. Prove that G is a nonabelian group of order 12.

4. (**35 points**). For each of the five integers $t = \pm 2, \pm 1, 0$, let $f_t(x) = x^3 + 3x^2 + t \in \mathbb{Q}[x]$, let L_t be the splitting field of f_t over $\mathbb{Q}$, and let $G_t = \mathrm{Gal}(L_t/\mathbb{Q})$.

In each of the five cases, compute G_t as isomorphic to some standard group from Math 350.

5. (**35 points**). Let L/F be a finite Galois extension, and let K_1 and K_2 be intermediate fields. Let $G = \mathrm{Gal}(L/F)$, $H_1 = \mathrm{Gal}(L/K_1)$, and $H_2 = \mathrm{Gal}(L/K_2)$. Let H be the subgroup of G generated by H_1 and H_2. (That is, H is the smallest subgroup of G containing both H_1 and H_2.) Prove that $\mathrm{Gal}(L/K_1 \cap K_2) = H$.

6. (**50 points**) Let M/F be an extension of fields, and let K and L be intermediate fields. Recall that KL denotes the compositum of K and L in M. (See Definition 8.2.5.) Suppose that K/F is a finite Galois extension.

Let $G_1 = \mathrm{Gal}(KL/L)$ and $G_2 = \mathrm{Gal}(K/(K \cap L))$. Define a function $\varphi : G_1 \to G_2$ by $\varphi(\sigma) = \sigma|_K$.

6a. Prove that KL/L and $K/(K \cap L)$ are finite Galois extensions.

6b. Prove that φ actually *is* a function that maps G_1 into G_2.

6c. Prove that φ is a homomorphism of groups.

6d. Prove that φ is one-to-one.

6e. Prove that φ is onto. (*Hint*: What's the fixed field of the image $\varphi(G_1)$?)

Thus, we have shown that if K/F is Galois, then $\mathrm{Gal}(KL/L) \cong \mathrm{Gal}(K/(K \cap L))$.

[**Note**: The following fact from class (on 4/13/26) might be useful at some point:
If $K = F(\alpha_1, \ldots, \alpha_n)$, then $KL = L(\alpha_1, \ldots, \alpha_n)$.]

7. (**Absolutely required**): Write out the following statement and sign your name to it:

I have read, understood, and followed the Academic Honesty instructions on the cover page of this Final Exam.