

What you need to know for the Midterm Exam

The exam on Wednesday, March 25 (in class) will cover Chapters 2–5 of Cox, up to and including Section 5.3. The following is a list of most of the topics covered.

THIS IS NOT A COMPREHENSIVE LIST, BUT MERELY AN AID.

- 2.1: Polynomial rings $F[x_1, \dots, x_n]$. The evaluation homomorphism. The elementary symmetric polynomials σ_r . The coefficients of a polynomial as $a_r = (-1)^r \sigma_r(\text{roots})$.
- 2.2: The Fundamental Theorem of Symmetric Polynomials (Theorem 2.2.2) and the associated uniqueness result (Theorem 2.2.7); know how to find the representation of a given symmetric polynomial in terms of σ_r 's. (This uses the graded lexicographic ordering.) Corollary 2.2.5: a symmetric polynomial evaluated at the roots of $f \in F[x]$ lies in F .
- 2.4: The (universal) discriminant $\Delta \in F[x_1, \dots, x_n]$; by symmetry Δ can be written as a polynomial in the σ_r 's. The discriminant $\Delta(f)$ of a polynomial $f \in F[x]$. Proposition 2.4.3, that if f splits completely over L , then the discriminant can be written as the product of the squares of differences of the various roots.
- 3.1: Basically, a review of some concepts from the end of Math 350: $F[x]$ is a PID, and a polynomial $f \in F[x]$ is irreducible iff $\langle f \rangle$ is a maximal ideal, iff the quotient $L = F[x]/\langle f \rangle$ is a field. In fact, this new field L can be thought of as containing both F and at least one root of f , namely $x + \langle f \rangle$, which we've often called α . By adjoining any still missing roots of f , we can make a field extension of F over which F splits completely.
- 3.2: Know what the Fundamental Theorem of Algebra says. Definition 3.2.5 (of algebraically closed).
- 4.1: Algebraic vs. transcendental elements (of L over a smaller field F). The minimal polynomial over F of an algebraic element $\alpha \in L$ (and its existence and uniqueness). Adjoining elements. Lemma 4.1.9; Corollary 4.1.11; Propositions 4.1.14 and 4.1.15, which say basic stuff about how $F(\alpha_1, \dots, \alpha_n)$ behaves. Know cyclotomic polynomials and roots of unity. (You should know that the degree of Φ_n is $\phi(n)$, and that $\Phi_n \in \mathbb{Q}[x]$, and you should know that Φ_n is irreducible over \mathbb{Q} , though you do not need to know why.)
- 4.2: Gauss' Lemma (or its corollary: Corollary 4.2.1). The Schönemann-Eisenstein Criterion. Proposition 4.2.6, that $x^p - a \in F[x]$ is irreducible over F if it has no roots in F .
- 4.3: The degree $[L : F]$. Finite extensions. Lemma 4.3.3 and Proposition 4.3.4. The Tower Theorem.
- 4.4: Algebraic extensions. All the Lemmas and Propositions and Theorems and Corollaries in this section.
- 5.1: Splitting fields (don't forget the minimality half of the definition). Theorem 5.1.5, bounding $[L : F]$ by $n!$. Theorem 5.1.6 and the uniqueness of the splitting field (up to isomorphism) that follows. Proposition 5.1.8 (moving one root of an irreducible f to another).
- 5.2: Normal (algebraic) extensions. Proposition 5.2.1 and Theorem 5.2.4, that L/F is a splitting field iff it is normal and finite.

- 5.3: Separable polynomials; separable (algebraic) elements; separable (algebraic) extensions. Proposition 5.3.2, giving equivalent conditions for a polynomial to be separable, including conditions that don't refer to a splitting field. All the various results in this section (except Proposition 5.3.8 and subsection C), including the results (with no proofs given) in the “Mathematical Notes” subsection.

Some things you don't need to know

- The precise formulas/methods for solving cubics and quartics.
- Chapter 1 was essentially just intro. Everything there that you'll need to know (like discriminants and symmetric polynomials) appears in greater generality elsewhere, so I can't exactly say you don't need to know it. But you don't need to know the presentation of it from Chapter 1.
- 2.4: The relationship among $\sqrt{\Delta}$, $\text{sgn}(\sigma)$, and the Alternating Group (Proposition 2.4.1 and Theorem 2.4.4).
- 2.4: Discriminant formulas involving matrices (Proposition 2.4.5 and following)
- 4.2: Maple and Mathematica; Kronecker's algorithm (Proposition 4.2.2) for factoring polynomials over \mathbb{Q} ; tangential results like Theorem 4.2.8 that we didn't discuss in class.
- 4.3: Algebras over a field.
- 4.4: Algebraic integers (and any stuff I said in videos about algebraic number theory).
- 5.3: Proposition 5.3.8; Mathematica and Maple computations in subsection C.
- 5.4: You don't need to know anything from this section (on the Theorem of the Primitive Element).
- Precise proofs of technical results like Cox's proof of the Fundamental Theorem of Algebra; the proof of Theorem 5.1.6; the proof of the results in Section 5.2; all the separability proofs involving separable degree; and so on.

Tips

- Don't memorize theorem numbers. Just say “by a theorem” or something like that, but say enough that it's clear that I can tell what theorem you're using. For example, if you use the Tower Theorem, a quick arrow with “Tower” pointing to the equality is plenty. Or if you're using Proposition 4.1.14, then just writing $F[\alpha] = F(\alpha)$ is enough to be crystal clear about what you're doing. (Of course, make sure, if you do that, that you have actually written somewhere that α is algebraic over F . After all, you *always* need to verify the hypotheses of any theorem you invoke.)
- Be familiar with lots of examples: \mathbb{Q} , $\mathbb{Q}(\sqrt{D})$, $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$, $\mathbb{Q}(\zeta_n)$, finite fields \mathbb{F}_q (where q is a power of a prime p , but mainly in the case $q = p$), infinite fields of positive characteristic (like $\mathbb{F}_p(t)$), and so on.
- Get a good night's sleep the night before. Be ready to think.